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Estimates for the constants of Landau and Lebesgue via some inequalities for the Wallis ratio

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ABSTRACT

We establish several new inequalities of the constants of Landau and Lebesgue.

1. Introduction and results

Let $A_n = \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot (2n)}$, $n = 1, 2, \ldots$, be the Wallis ratio directly related to the Wallis formula

$$\lim_{n\to\infty} \frac{1}{nA_n^2} = \pi \tag{1}$$

and the well known classical inequalities

$$\frac{2}{2n+1} < \pi A_n^2 < \frac{1}{n}, \quad n = 1, 2, \ldots .$$

(See also the results of Gurland [1], Mačys [2] and Chen, Qi, Alzer [3]).

G.M. Zhang has proved in [4] the following double inequality

$$\frac{1}{\sqrt{\pi n\left(1 + \frac{1}{4n\left(1-\frac{1}{8n+3}\right)}\right)}} < A_n < \frac{1}{\sqrt{\pi n\left(1 + \frac{1}{4n\left(1-\frac{1}{8n+4}\right)}\right)}}. \tag{2}$$

We can rewrite this double inequality (2) as follows:

$$\frac{32n+8}{32n^2+16n+3} < \pi A_n^2 < \frac{8n+3}{8n^2+5n+1}, \quad n = 1, 2, \ldots \tag{3}$$

and we will establish an improvement for the right inequality in (3).

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Theorem 1. *Let* $\lambda \in [0, \infty)$ *be a real number and set* $I_\lambda = \left[\frac{72\lambda^2+27\lambda+2}{12\lambda+3}, \infty\right) \cap \mathbb{N}$, *where* $\mathbb{N}$ *denotes the set of positive integers* $\{1, 2, \ldots\}$. *Then*

$$\pi A_n^2 < \frac{(32\lambda+8)n+8\lambda+3}{(32\lambda+8)n^2+(16\lambda+5)n+3\lambda+1}, \quad \forall\, n \in I_\lambda. \tag{4}$$

Remark 1. Since

$$\frac{(32\lambda+8)n+8\lambda+3}{(32\lambda+8)n^2+(16\lambda+5)n+3\lambda+1} \le \frac{8n+3}{8n^2+5n+1} < \frac{1}{n}, \quad n = 1, 2, \ldots,$$

this inequality improves Zhang's estimate which corresponds to $\lambda = 0$.

Remark 2. According to Theorem 1, some simple computations lead to the inequalities

$$\pi A_n^2 < \frac{(32\lambda+8)n+8\lambda+3}{(32\lambda+8)n^2+(16\lambda+5)n+3\lambda+1} < \frac{8n+3}{8n^2+5n+1}, \quad \forall\, \lambda > 0, \forall\, n \in I_\lambda,$$

which certainly represent an improvement of the right inequality in (3).

The constants of Landau and Lebesgue are defined for all integers $n \ge 0$ by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2 \quad \text{and} \quad L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin \frac{(2n+1)t}{2}}{\sin \frac{t}{2}} \right| \mathrm{d}t.$$

They play an important role in complex analysis and in the theory of Fourier series, respectively. There is a huge literature concerning the Landau and Lebesgue constants (see e.g. [5–9].

D. Zhao establishes in [10] several inequalities for G_n and L_n:

$$\begin{gathered} \frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} < G_n \\ G_n < \frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \frac{3}{128\pi(n+1)^3}, \end{gathered} \tag{5}$$

where $n \ge 1$, $c_0 = \frac{1}{\pi}(\gamma + 4\ln 2) = 1.06627\ldots$, and $\gamma = 0.57721\ldots$ denotes Euler's constant;

$$\begin{gathered} \frac{4}{\pi^2}\ln(n+1) + c_1 + \frac{d_0}{(n+1)^2} - \frac{d_1}{(n+1)^4} < L_{n/2} \\ L_{n/2} < \frac{4}{\pi^2}\ln(n+1) + c_1 + \frac{d_0}{(n+1)^2} - \frac{d_1}{(n+1)^4} + \frac{d_2}{(n+1)^6}, \end{gathered} \tag{6}$$

where

$$\begin{aligned} c_1 &= \frac{8}{\pi^2}\sum_{k=1}^{\infty} \frac{\ln k}{4k^2-1} + \frac{4}{\pi^2}(\gamma + 2\ln 2) = 0.98943\ldots, \\ d_0 &= \frac{2}{3\pi^2} - \frac{1}{18} = 0.01199190\ldots, \\ d_1 &= \frac{7}{120\pi^2}\left(8 - \frac{2\pi^2}{3} - \frac{\pi^4}{90}\right) = 0.00199736\ldots, \\ d_2 &= \frac{1}{16\pi^2}\left(32 - \frac{8\pi^2}{3} - \frac{2\pi^4}{45} - \frac{\pi^6}{945}\right) = 0.00211774\ldots. \end{aligned}$$

Therefore,

$$\begin{gathered} \frac{4-\pi}{\pi^2}\ln(n+1) + (c_1 - c_0) + \frac{1}{4\pi(n+1)} + \frac{e_1}{(n+1)^2} - \frac{e_2}{(n+1)^3} - \frac{d_1}{(n+1)^4} < L_{n/2} - G_n \\ L_{n/2} - G_n < \frac{4-\pi}{\pi^2}\ln(n+1) + (c_1 - c_0) + \frac{1}{4\pi(n+1)} + \frac{e_1}{(n+1)^2} - \frac{d_1}{(n+1)^4} + \frac{d_2}{(n+1)^6}, \end{gathered} \tag{7}$$

where $e_1 = d_0 - \frac{5}{192\pi}$, $e_2 = \frac{3}{128\pi}$.

Using the Riemann zeta function, we obtained in [11,12] some improvements of inequalities (5) and (7) in the following evaluations:

$$\begin{aligned} &\frac{1}{\pi}\ln(n+1)+c_0-\frac{1}{4\pi(n+1)}+\frac{5}{192\pi(n+1)^2}+\frac{17}{256\pi}\left(\zeta(4)-\sum_{k=1}^{n+1}\frac{1}{k^4}\right)<G_n \\ &G_n<\frac{1}{\pi}\ln(n+1)+c_0-\frac{1}{4\pi(n+1)}+\frac{5}{192\pi(n+1)^2}+\frac{9}{128\pi}\left(\zeta(4)-\sum_{k=1}^{n}\frac{1}{k^4}\right)-\frac{2263}{61440(n+1)^4}\end{aligned} \tag{8}$$

and

$$\begin{aligned} &\frac{4-\pi}{\pi^2}\ln(n+1)+(c_1-c_0)+\frac{1}{4\pi(n+1)}+\frac{e_1}{(n+1)^2}-\frac{9}{128\pi}\left(\zeta(4)-\sum_{k=1}^{n}\frac{1}{k^4}\right) \\ &\quad+\left(\frac{2263}{61440(n+1)^4}-d_1\right)\frac{1}{(n+1)^4}<L_{n/2}-G_n. \\ &L_{n/2}-G_n<\frac{4-\pi}{\pi^2}\ln(n+1)+(c_1-c_0)+\frac{1}{4\pi(n+1)}+\frac{e_1}{(n+1)^2}-\frac{d_1}{(n+1)^4} \\ &\quad+\frac{d_2}{(n+1)^6}-\frac{17}{256\pi}\left(\zeta(4)-\sum_{k=1}^{n+1}\frac{1}{k^4}\right),\end{aligned} \tag{9}$$

where $\zeta(s)=\sum_{k=1}^{\infty}k^{-s}$ is the Riemann zeta function.

In this paper we will establish some sharp inequalities for G_n and $L_{n/2}-G_n$ in order to improve the inequalities (8) and (9). Also, some estimates for $L_{n/2}-G_{n-1}$ are provided.

Theorem 2. *For every $\lambda\geq 0$ and $n\in I_\lambda$ we have*

$$\begin{aligned} &\frac{1}{\pi}\ln(n+2)+c_0-\frac{1}{4\pi(n+2)}+\frac{5}{192\pi(n+2)^2}+\frac{17}{256\pi}\left(\zeta(4)-\sum_{k=1}^{n+2}\frac{1}{k^4}\right) \\ &\quad-\frac{1}{\pi}\cdot\frac{(32\lambda+8)n+40\lambda+11}{(32\lambda+8)n^2+(80\lambda+21)n+51\lambda+14}<G_n, \\ &G_n<\frac{1}{\pi}\ln(n+2)+c_0-\frac{1}{4\pi(n+2)}+\frac{5}{192\pi(n+2)^2}+\frac{9}{128\pi}\left(\zeta(4)-\sum_{k=1}^{n+1}\frac{1}{k^4}\right) \\ &\quad-\frac{2263}{61440\pi(n+2)^4}-\frac{1}{\pi}\cdot\frac{32n+40}{32n^2+80n+51}.\end{aligned} \tag{10}$$

Theorem 3. *For every $\lambda\geq 0$ and $n\in I_\lambda$ we have*

$$\begin{aligned} &\frac{1}{\pi}\ln\frac{(n+1)^{\frac{4}{\pi}}}{n+2}+(c_1-c_0)+\frac{1}{4\pi(n+2)}+\frac{e_1}{(n+1)^2}+\frac{5}{192\pi}\left(\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\right) \\ &\quad-\frac{e_3}{(n+1)^4}-\frac{2263}{61440\pi}\left(\frac{1}{(n+1)^4}-\frac{1}{(n+2)^4}\right)-\frac{9}{128\pi}\left(\zeta(4)-\sum_{k=1}^{n+1}\frac{1}{k^4}\right) \\ &\quad+\frac{1}{\pi}\cdot\frac{32n+40}{32n^2+80n+51}<L_{n/2}-G_n, \\ &L_{n/2}-G_n<\frac{1}{\pi}\ln\frac{(n+1)^{\frac{4}{\pi}}}{n+2}+(c_1-c_0)+\frac{1}{4\pi(n+2)}+\frac{e_1}{(n+1)^2} \\ &\quad+\frac{5}{192\pi}\left(\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\right)-\frac{d_1}{(n+1)^4}+\frac{d_2}{(n+1)^6}-\frac{17}{256\pi}\left(\zeta(4)-\sum_{k=1}^{n+2}\frac{1}{k^4}\right) \\ &\quad+\frac{1}{\pi}\cdot\frac{(32\lambda+8)n+40\lambda+11}{(32\lambda+8)n^2+(80\lambda+21)n+51\lambda+14},\end{aligned} \tag{11}$$

where $e_3=d_1-\frac{2263}{61440\pi}$.

Theorem 4. *For every $\lambda \geq 0$ and $n \in I_\lambda$ we have*

$$\begin{aligned}&\frac{4-\pi}{\pi^2}\ln(n+1)+(c_1-c_0)+\frac{1}{4\pi(n+1)}+\frac{e_1}{(n+1)^2}+\frac{2263}{61440\pi(n+1)^4}\\&-\frac{9}{128\pi}\left(\zeta(4)-\sum_{k=1}^{n}\frac{1}{k^4}\right)+\frac{1}{\pi}\cdot\frac{32n+8}{32n^2+16n+3}<L_{n/2}-G_{n-1},\\&L_{n/2}-G_{n-1}<\frac{4-\pi}{\pi^2}\ln(n+1)+(c_1-c_0)+\frac{1}{4\pi(n+1)}+\frac{e_1}{(n+1)^2}-\frac{d_1}{(n+1)^4}\\&+\frac{d_2}{(n+1)^6}-\frac{17}{256\pi}\left(\zeta(4)-\sum_{k=1}^{n+1}\frac{1}{k^4}\right)+\frac{1}{\pi}\cdot\frac{(32\lambda+8)n+8\lambda+3}{(32\lambda+8)n^2+(16\lambda+5)n+3\lambda+1}.\end{aligned} \tag{12}$$

2. Proofs of the theorems

We are now able to prove our theorems.

Proof of Theorem 1. We denote

$$\alpha_n=A_n^2\frac{(32\lambda+8)n^2+(16\lambda+5)n+3\lambda+1}{(32\lambda+8)n+8\lambda+3}.$$

From the Wallis formula (1) it follows that

$$\lim_{n\to\infty}\alpha_n=\frac{1}{\pi}. \tag{13}$$

We have now

$$\begin{aligned}\alpha_{n+1}-\alpha_n&=A_{n+1}^2\frac{(32\lambda+8)n^2+(80\lambda+21)n+51\lambda+14}{(32\lambda+8)n+40\lambda+11}-A_n^2\frac{(32\lambda+8)n^2+(16\lambda+5)n+3\lambda+1}{(32\lambda+8)n+8\lambda+3}\\&=A_n^2\Big(\frac{4n^2+4n+1}{4n^2+8n+4}\cdot\frac{(32\lambda+8)n^2+(80\lambda+21)n+51\lambda+14}{(32\lambda+8)n+40\lambda+11}\\&\quad-\frac{(32\lambda+8)n^2+(16\lambda+5)n+3\lambda+1}{(32\lambda+8)n+8\lambda+3}\Big).\end{aligned}$$

After simplifying we obtain

$$\alpha_{n+1}-\alpha_n=\frac{(12\lambda+3)n-72\lambda^2-27\lambda-2}{(4n^2+8n+4)\big((32\lambda+8)+40\lambda+11\big)\big((32\lambda+8)n+8\lambda+3\big)}.$$

It is obvious that $\alpha_{n+1}>\alpha_n$ for any $n\in I_\lambda$.

Next, by using (13), we deduce that

$$\alpha_n<\frac{1}{\pi}$$

and hence

$$\pi A_n^2<\frac{(32\lambda+8)n+8\lambda+3}{(32\lambda+8)n^2+(16\lambda+5)n+3\lambda+1},$$

for all $n\in I_\lambda$. □

Proof of Theorem 2. It is easy to verify that

$$G_n=G_{n+1}-A_{n+1}^2. \tag{14}$$

From (3) to (4) we deduce

$$\frac{32n+8}{32n^2+16n+3}<\pi A_n^2<\frac{(32\lambda+8)n+8\lambda+3}{(32\lambda+8)n^2+(16\lambda+5)n+3\lambda+1},$$

for $\lambda\in[0,\infty)$ and $n\in I_\lambda$.

Hence

$$\frac{1}{\pi}\cdot\frac{32n+40}{32n^2+80n+51} < A_{n+1}^2 < \frac{1}{\pi}\cdot\frac{(32\lambda+8)n+40\lambda+11}{(32\lambda+8)n^2+(80\lambda+21)n+51\lambda+14}. \tag{15}$$

Using (8) and (15), we obtain from (14) both inequalities (10) from the statement. □

Proof of Theorem 3. The inequalities (11) are obtained easily from (6) and (10). □

Remark 3. For $\lambda \geq 0$ and $n \geq 0$ we have

$$\frac{(32\lambda+8)n+40\lambda+11}{(32\lambda+8)n^2+(80\lambda+21)n+51\lambda+14} < \ln\left(1+\frac{1}{n+1}\right)-\frac{1}{4}\left(\frac{1}{n+2}-\frac{1}{n+1}\right)+\frac{5}{192}\left(\frac{1}{(n+2)^2}-\frac{1}{(n+1)^2}\right)-\frac{17}{256}\cdot\frac{1}{(n+2)^2}. \tag{16}$$

Proof. Let us consider the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{8x+3}{8x^2+5x+1} - \ln\left(1+\frac{1}{x}\right)+\frac{1}{4}\left(\frac{1}{x+1}-\frac{1}{x}\right) - \frac{5}{192}\left(\frac{1}{(x+1)^2}-\frac{1}{x^2}\right)+\frac{17}{256}\cdot\frac{1}{(x+1)^4}.$$

One has $\lim_{x\to\infty} f(x) = 0$ and $f'(x) > 0$ for $x > 0$ (see [11, pp. 115–116]). Hence $f(x) < 0$ and, for $n \geq 1$, we have

$$\frac{8n+3}{8n^2+5n+1} < \ln\left(1+\frac{1}{n}\right)-\frac{1}{4}\left(\frac{1}{n+1}-\frac{1}{n}\right)+\frac{5}{192}\left(\frac{1}{(n+1)^2}-\frac{1}{n^2}\right)-\frac{17}{256(n+1)^4}.$$

Since

$$\frac{(32\lambda+8)(n+1)+8\lambda+3}{(32\lambda+8)(n+1)^2+(16\lambda+5)(n+1)+3\lambda+1} < \frac{8(n+1)+3}{8(n+1)^2+5(n+1)+1}$$

the inequality (16) follows. □

Notice that, according to Remark 3, the first inequality from (10) is an improvement of the first inequality from (8) and the second inequality from (11) is an improvement of the second inequality from (9).

Remark 4. For each $n \geq 0$ one has

$$\frac{32n+40}{32n^2+80n+51} > \ln\left(1+\frac{1}{n+1}\right)-\frac{1}{4}\left(\frac{1}{n+2}-\frac{1}{n+1}\right) + \frac{5}{192}\left(\frac{1}{(n+2)^2}-\frac{1}{(n+1)^2}\right)-\frac{9}{128(n+1)^4}-\frac{2263}{61440}\left(\frac{1}{(n+2)^4}-\frac{1}{(n+1)^4}\right). \tag{17}$$

Proof. We consider the function $f : (0, \infty) \to \mathbb{R}$ given by

$$f(x) = \frac{32x+8}{32x^2+16x+8} - \ln\left(1+\frac{1}{x}\right)+\frac{1}{4}\left(\frac{1}{x+1}-\frac{1}{x}\right) - \frac{5}{192}\left(\frac{1}{(x+1)^2}-\frac{1}{x^2}\right)+\frac{9}{128x^4}+\frac{2263}{61440}\left(\frac{1}{(x+1)^4}-\frac{1}{x^4}\right).$$

We have $\lim_{x\to\infty} f(x) = 0$ and $f'(x) < 0$ for all $x > 0$ (see [12, p. 1459]). So $f(x) \geq 0$ for every $x > 0$ and

$$\frac{32n+8}{32n^2+16n+8} > \ln\left(1+\frac{1}{n}\right)-\frac{1}{4}\left(\frac{1}{n+1}-\frac{1}{n}\right) - \frac{5}{192}\left(\frac{1}{(n+1)^2}-\frac{1}{n^2}\right)-\frac{9}{128n^4}-\frac{2263}{61440}\left(\frac{1}{(n+1)^4}-\frac{1}{n^4}\right).$$

Therefore the inequality (17) is verified. □

According to Remark 4, the second inequality from (10) is an improvement of the second inequality from (8) and the first inequality from (11) is an improvement of the first inequality from (9).

Proof of Theorem 4. From the inequalities (10) one obtains

$$\frac{1}{\pi}\ln(n+1)+c_0-\frac{1}{4\pi(n+1)}+\frac{5}{192\pi(n+1)^2}+\frac{17}{256\pi}\left(\zeta(4)-\sum_{k=1}^{n+1}\frac{1}{k^4}\right)$$
$$-\frac{1}{\pi}\cdot\frac{(32\lambda+8)n+8\lambda+3}{(32\lambda+8)n^2+(16\lambda+5)n+3\lambda+1}<G_{n-1},$$
$$G_{n-1} < \frac{1}{\pi}\ln(n+1)+c_0-\frac{1}{4\pi(n+1)}+\frac{5}{192\pi(n+1)^2}+\frac{9}{128\pi}\left(\zeta(4)-\sum_{k=1}^{n}\frac{1}{k^4}\right)$$
$$-\frac{2263}{61440\pi(n+1)^4}-\frac{1}{\pi}\cdot\frac{32n+8}{32n^2+16n+3}.$$

Next, using (6), we obtain the inequality (12).

3. Examples

In this section we will illustrate in some particular cases that our results provided in Theorems 2 and 3 improve Zhao's inequalities (5) and (7).

Example 1. For $\lambda = 0$ and $n = 1$ Zhao's inequalities (5) become

$$1.2491891944\ldots < G_1 = 1.25$$

and

$$G_1 < 1.2501217429\ldots.$$

Now, our results in (10) lead to

$$1.2498738687\ldots < G_1,$$
$$G_1 < 1.2500601481\ldots.$$

Example 2. For $\lambda = 0.125$ and $n = 2$ Zhao's inequalities (5) are

$$1.3903643643\ldots < G_2 = 1.390625,$$
$$G_2 < 1.3906406749.$$

Our results (10) are

$$1.3905806118\ldots < G_2,$$
$$G_2 < 1.3906338832\ldots.$$

Example 3. Putting $\lambda = 0.125$ and $n = 2$ in Zhao's inequality, we obtain

$$0.045350794438\ldots < L_1 - G_2 = 0.045366124\ldots,$$
$$L_1 - G_2 < 0.045627105103\ldots.$$

Our results (11) lead to

$$0.045354681220\ldots < L_1 - G_2,$$
$$L_1 - G_2 < 0.045410857554\ldots.$$

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