Anal.Math.Phys.
DOI 10.1007/s13324-013-0066-5

Univalent harmonic mappings convex in one direction

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Received: 28 August 2013 / Revised: 25 November 2013 / Accepted: 11 December 2013

Abstract In this paper, we present a criterion for a harmonic function to be convex in one direction. Also, we discuss the class of harmonic functions starlike in one direction in the unit disk $\mathbb{D}$ and obtain a method to construct univalent harmonic functions convex in one direction. Although the converse of classical Alexander's theorem for harmonic functions was proved to be false, we obtain a version of converse of it under a suitable additional condition.

Keywords Univalent harmonic functions · Convex in one direction · Starlike in one direction · Close-to-convex · Fully starlike · Fully convex

Mathematics Subject Classification (2000) 30C45 · 31C05

1 Introduction, examples and preliminaries

Let $\mathcal{H}$ denote the class of all complex-valued harmonic functions $f = h + \overline{g}$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that

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Published online: 04 January 2014 Birkhäuser

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}.$$

A harmonic function $f \in \mathcal{H}$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if the Jacobian $J_f(z)$ of f is positive on $\mathbb{D}$, where

$$J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2$$

(see [9]). Note that $J_f(z) > 0$ if and only if there exists an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ such that $\omega(z) = g'(z)/h'(z)$. Here ω is referred to as the dilatation of f.

We are interested in studying functions $f \in \mathcal{H}$ which map $|z| = r$ onto a simple closed curve for some $r \leq 1$. When $f(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$, $J_f(z) > 0$ for all $z \in \mathbb{D}$ and satisfies condition such as $\text{Re}\,(Df(z)/f(z)) > 0$ for all $z \in \mathbb{D}\backslash\{0\}$ or $\text{Re}\left(D^2 f(z)/Df(z)\right) > 0$ for all $z \in \mathbb{D}\backslash\{0\}$, then f maps every circle $0 < |z| = r < 1$ onto a simple closed curve, where Df denotes the differential operator:

$$Df = zf_z - \overline{z} f_{\overline{z}} \tag{1}$$

and

$$D^2 f = D(Df) = z(f_z + zf_{zz}) + \overline{z}(f_{\overline{z}} + \overline{z} f_{\overline{z}\,\overline{z}}),$$

see [12]. However, one can easily see that there are conditions of this type which ensure that the corresponding function f has the property that the image $f(|z| = r)$ is a simple closed curve.

Definition 1 Let f be a harmonic function defined on $\mathbb{D}$ such that $f(0) = 0$. We say that f satisfies property P, if any one of the following conditions holds:

(a) There exists a real number $0 < \delta = \delta(f) \leq 1$ so that for every r in the open interval $1 - \delta < r < 1$, $f(z)$ maps $|z| = r$ into a simple closed curve.
(b) f is continuous on $\overline{\mathbb{D}}$, and $f(z)$ maps $|z| = 1$ onto a Jordan curve Γ such that $f(e^{it})$ runs once around Γ monotonically as e^{it} runs around $|z| = 1$.

We may now state our first result and include the proof of it together with some interesting results in Sect. 3.

Theorem 1 *Suppose that $f \in \mathcal{H}$ satisfies the following conditions:*

(i) $f(z) \neq 0$ *for all* $z \in \mathbb{D}\backslash\{0\}$, *and* $J_f(z) > 0$ *for all* $z \in \mathbb{D}$
(ii) f *has the property* P.

Then the harmonic function f is univalent in $\mathbb{D}$.

We recall that a harmonic function f defined on the unit disk $\mathbb{D}$ is called fully starlike (resp. fully convex) if f maps every sub disk $|z| < r < 1$ onto a starlike (resp. convex) domain. Clearly Theorem 1 gives the following result (but only with the conclusion that f is univalent in $\mathbb{D}$) due to Mocanu [12] for C^1-functions (i.e., continuously differentiable functions) f on $\mathbb{D}$, or briefly we write $f \in C^1(\mathbb{D})$.

Corollary 1 [12] *Suppose that f is a complex-valued function such that $f(0) = 0$, $f(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$ and $J_f(z) > 0$ for all $z \in \mathbb{D}$.*

(a) *If $f \in C^1(\mathbb{D})$ and satisfies the condition*

$$\frac{\partial}{\partial\theta}\left(\arg(f(re^{i\theta})\right) = \operatorname{Re}\left(\frac{Df(z)}{f(z)}\right) > 0 \text{ for all } z \in \mathbb{D}\backslash\{0\},$$

then f is univalent and fully starlike in $\mathbb{D}$.

(b) *If $f \in C^2(\mathbb{D})$ and satisfies the condition*

$$\frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta}(f(re^{i\theta})\right)\right) = \operatorname{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) > 0 \text{ for all } z \in \mathbb{D}\backslash\{0\},$$

then f is univalent and fully convex in $\mathbb{D}$.

We remark that if f is sense-preserving, then it is easy to see that $Df \neq 0$. A domain Ω is said to be close-to-convex if the complement of Ω can be written as a union of non-crossing half lines. An univalent function f in $\mathbb{D}$ is said to be close-to-convex if its range $f(\mathbb{D})$ is a close-to-convex domain. We refer to the monograph of Duren [7] and Dorff [5] (see also [15]) for important contributions on planar harmonic mappings. Many well known geometric subclasses of univalent harmonic functions such as starlike, convex and close-to-convex functions are studied systematically in the recent years (see [7,15]) and for the conformal case we refer to [6,8]. Moreover, starlikeness and convexity posses hereditary property for conformal mappings. That is, if f is analytic and univalent in $\mathbb{D}$ and maps $\mathbb{D}$ onto a starlike (resp. convex) domain, then the image of every subdisk $|z| < r < 1$ is also a starlike (resp. convex) domain. In a similar fashion if a univalent and sense-preserving harmonic function f maps the unit disk $\mathbb{D}$ onto a starlike (resp. convex) domain, then we say that f is starlike (resp. convex) in $\mathbb{D}$. On the other hand, the image of every subdisk $|z| < r < 1$ under a harmonic starlike (resp. convex) function f in $\mathbb{D}$ is not necessarily mapped onto a starlike (resp. convex) domain (see [2]).

The following result is a consequence of Kaplan characterization (see [6] [p. 48, Theorem 2.18]).

Theorem A *Let f be analytic, locally univalent in $\mathbb{D}$ and satisfy the following condition*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \text{ for all } z \in \mathbb{D}. \tag{2}$$

Then f is univalent and close-to-convex in $\mathbb{D}$.

There are a number of problems in connection with functions that are convex in one direction (see [3,4,11,17,19] and the references therein).

Definition 2 A domain $D \subset \mathbb{C}$ is called convex in the direction of α $(0 \leq \alpha < \pi)$ if the intersection of the domain $f(\mathbb{D})$ with every line parallel to the line through 0 and

$e^{i\alpha}$ is either empty or an interval. A univalent harmonic function f in $\mathbb{D}$ is said to be convex in the direction of α if $f(\mathbb{D})$ is convex in the direction of α. We say that f is *convex in one direction* if there exists an α such that f is convex in the direction of α.

According to the result of Umezawa [20] (see Theorem B), the hypothesis of Theorem A not only implies that the corresponding function is close-to-convex in $\mathbb{D}$ but is also convex in one direction. One of our aims is to address a harmonic analog of Theorem A as the analytic part of certain harmonic function satisfying the condition of Theorem A plays a crucial role in dealing with an open problem by Mocanu [13] and has been settled affirmatively by Bshouty and Lyzzaik [1]. A general situation has been handled by the authors in [16].

The best known method of constructing univalent harmonic mappings in the unit disk is the so called shearing construction due to Clunie and Sheil-Small [3, Theorem 5.3]. Using this method, in a recent paper [14], Ponnusamy and Qiao constructed a number of univalent harmonic functions that are also convex in the real direction or in the vertical direction, in particular. For example, the following functions are proved to be convex in one direction in $\mathbb{D}$:

$$\begin{gathered} f_1(z) = -2\log|1-z| - \overline{z},\ \ f_2(z) = -2i\arg(1-z) + \overline{z}, \\ f_3(z) = \mathrm{Re}\left(\frac{z}{(1-z)^2}\right) + i\,\mathrm{Im}\left(\frac{z}{1-z}\right), \\ f_4(z) = \frac{1}{2}\log\left|\frac{1+z}{1-z}\right| + i\,\mathrm{Im}\left(\frac{z}{1-z}\right), \\ f_5(z) = \mathrm{Re}\left(\frac{z}{1-z}\right) + i\,\mathrm{Im}\left(\frac{z}{(1-z)^2}\right), \\ f_6(z) = z + \overline{\frac{z^2}{2}}. \end{gathered} \tag{3}$$

Moreover it is a simple exercise to see that each of the above six functions satisfies the condition

$$\mathrm{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) > -\frac{1}{2}, \quad z \in \mathbb{D}, \tag{4}$$

which in particular implies (2) in the analytic case. Here Df is defined by (1). It is natural to ask whether the condition (4) on f implies that f is convex in some direction (and hence, close-to-convex).

In fact, these examples and Corollary 1 motivate us to establish a harmonic analog of Kaplan's Theorem (see [6, p. 48, Theorem 2.18]) for sense-preserving harmonic mappings (see Corollary 3, and Theorem 3).

The article is organized as follows. Proof of Theorem 1 is given in Sect. 3. In Sect. 2, we investigate functions that are convex in one direction. A generalization of the classical Alexander theorem for harmonic mappings is also presented in Sect. 2 (see Theorem 2). As a consequence of related investigation, we obtain a new half-plane mapping in Example 4. We conjecture that this function will be useful to derive new convolution results for planar harmonic mappings. Several corollaries and discussions provide us with a method of constructing examples of univalent harmonic functions convex in one direction.

2 Functions that are convex/starlike in one direction

In [20, Theorem 1], Umezawa obtained

Theorem B *Let $f(z)$ be meromorphic for $|z| < 1$. If the following relation holds in $|z| < 1$*

$$\alpha > \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{\alpha}{2\alpha - 3},$$

where α is an arbitrary number not less than $3/2$, then $f(z)$ is regular and univalent in $|z| < 1$. Moreover, $f(z)$ maps $|z| = r$ for every $r < 1$ into a curve which is convex in one direction.

Several special cases can be drawn by allowing $\alpha \to \infty$, $\alpha \to 3/2$ and $\alpha = 3$, respectively in Theorem B. In [18], Robertson discussed functions that are starlike in one direction and pointed out that such functions are not necessarily univalent in $\mathbb{D}$. Moreover he showed that $zf'(z)$ is starlike in one direction if and only if $f(z)$ is convex in one direction. The work of Robertson [18] for analytic functions starlike in one direction naturally motivates one to consider a harmonic analog of it and discuss properties of such functions.

Definition 3 Assume that $f \in \mathcal{H}$ and satisfies any one of the following conditions:

(A) There exists a positive real $\delta = \delta(f)$ so that for every r in the open interval $1 - \delta < r < 1$, $f(z)$ maps $|z| = r$ into a contour C_r which intersects with the real axis in two and not more than two points.

(B) f is continuous in $\overline{\mathbb{D}}$ and $f(z)$ maps $|z| = 1$ into a contour which intersects with the real axis in two and not more than two points.

Then f is said to be starlike in the direction of real axis. Denote by $\mathcal{S}^*_{H_0}$ the class of functions in $\mathcal{H}$ which are starlike in the direction of real axis. Similarly we can define functions starlike in an arbitrary direction. If $f(z)$ is starlike in some other direction, then $e^{i\alpha} f(e^{-i\alpha} z)$ belongs to $\mathcal{S}^*_{H_0}$ for a suitable choice of a real number α.

Suppose that $f \in \mathcal{S}^*_{H_0}$ satisfies (A). Then there exist exactly two points $z_1 = re^{i\theta_1(r,f)}$ and $z_2 = re^{i\theta_2(r,f)}$ at which the imaginary parts of $f(z)$ is zero. For simplicity we may use the notation $\theta_1(r)$ and $\theta_2(r)$ instead of writing $\theta_1(r, f)$ and $\theta_2(r, f)$, respectively. If $f(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta})$, then

$$v(re^{i\theta}) \begin{cases} > 0 & \text{when} \quad \theta_1(r) < \theta < \theta_2(r) \\ < 0 & \text{when} \quad \theta_2(r) < \theta < \theta_1(r) + 2\pi, \end{cases}$$

where $0 \leq \theta_1(r) < 2\pi$ and $0 < \theta_2(r) - \theta_1(r) < 2\pi$. Now let us discuss the relationship between the functions starlike in one direction and functions convex in one direction. Let $\phi(z)$ be a function convex in the vertical direction. Suppose that, there is a positive real number $\delta = \delta(\phi)$ such that for every r in the open interval $1-\delta < r < 1$, $\phi(z)$ maps $|z| = r$ onto a contour C_r which intersects every straight line parallel to the imaginary axis in two and not more than two points. This will happen if and only if there exist two real numbers $\theta_1(r)$ and $\theta_2(r)$, $0 \leq \theta_1(r) < 2\pi$ and $0 < \theta_2(r) - \theta_1(r) < 2\pi$, such that $\mathrm{Re}\,\{\phi(re^{i\theta})\}$ is a monotonic decreasing function of θ in the interval $(\theta_1(r), \theta_2(r))$ and monotonic increasing function of θ in the interval $(\theta_2(r), \theta_1(r) + 2\pi)$. It is clear that

$$\mathrm{Im}\,\{D(\phi(re^{i\theta}))\} = -\frac{\partial}{\partial\theta}\mathrm{Re}\,\{\phi(re^{i\theta})\}$$

where D is the operator defined by (1). From our earlier discussion, it is clear that $\phi(z)$ is convex in the vertical direction if and only if $D\phi(z)$ is starlike in the horizontal direction. In general, one has

Proposition 1 *The harmonic function $\phi(z)$ is convex in the direction of $\alpha + \pi/2$ if and only if $D\phi(z)$ is starlike in the direction of α.*

We recall that a function starlike in one direction in $\mathbb{D}$ is not necessarily univalent in $\mathbb{D}$ (see for eg. Examples 1 and 2). On the other hand, functions convex in one direction are always univalent and hence Proposition 1 provides a way to construct a univalent harmonic function convex in one direction from a given function starlike in one direction. This can be stated without proof as the proof is easy.

Theorem 2 *If $f = h + \overline{g}$ is a function starlike in one direction, and if H and G are analytic functions defined by*

$$zH'(z) = h(z), zG'(z) = -g(z), H(0) = G(0) = 0, \tag{5}$$

then $F = H + \overline{G}$ is univalent and convex in one direction in $\mathbb{D}$. In particular, if f is starlike in the direction of α, then F is convex in the direction of $\alpha + \pi/2$.

This theorem can be considered as a generalization of Alexander's Theorem (see [7, p. 108, Lemma]) which asserts that if $f = h + \overline{g}$ is a starlike mapping in $\mathbb{D}$, and if H and G are the analytic functions defined by the relations (5) then $F = H + \overline{G}$ is a convex mapping in $\mathbb{D}$. In [19], the author reviews known results on convex harmonic functions and their relation to analytic functions convex in one direction. As it is, the converse of Alexander's Theorem for harmonic functions is not true in general. At the end, we shall actually establish a version of converse of Alexander's theorem with some additional restriction.

Evidently a function convex in the direction of α is starlike in the direction of α. Therefore, we have the following result.

Corollary 2 *If $f = h + \overline{g}$ is a harmonic function convex in the direction of α, and if H and G are analytic functions defined by (5), then $F = H + \overline{G}$ is univalent and convex in the direction of $\alpha + \pi/2$ in $\mathbb{D}$.*

With the help of Theorem 2 and Corollary 2, one can construct examples of univalent harmonic functions convex in one direction. We discuss below three examples.

Example 1 Consider the harmonic function

$$f_7(z) = \frac{z}{1-z} - \overline{\frac{z^2}{1-z}}.$$

Then $f_7(z)$ is starlike in the direction of imaginary axis, but not univalent in $\mathbb{D}$. In fact, it can be shown that f_7 is not even locally univalent in $\mathbb{D}$. To do this, we compute the Jacobian,

$$J_{f_7}(z) = \left|\frac{1}{(1-z)^2}\right|^2 - \left|\frac{z(2-z)}{(1-z)^2}\right|^2 = \frac{1-|z(2-z)|^2}{|1-z|^4}.$$

We observe next that $J_{f_7}(0) > 0$ and $J_{f_7}(-1/2) < 0$. Therefore, $J_{f_7}(r) = 0$ for some $r \in (-1/2, 0)$. This proves that f_7 is not locally univalent in $\mathbb{D}$. To prove that $f_7(z)$ is starlike in the direction of imaginary axis, let us consider

$$if_7(-ire^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta}).$$

A simple calculation shows that $v(re^{i\theta}) = r\sin\theta$. It is obvious that

$$v(re^{i\theta}) \begin{cases} > 0 & \text{when} \quad 0 < \theta < \pi \\ < 0 & \text{when} \quad \pi < \theta < 2\pi. \end{cases}$$

Therefore, $if_7(-iz)$ is starlike in the direction of real axis, and hence $f_7(z)$ is starlike in the direction of imaginary axis. By taking $f(z) = f_7(z)$ in (5) we get

$$F_7(z) = -2\log|1-z| - \overline{z}$$

which is convex in the real direction.

Example 2 Consider the harmonic function

$$f_8(z) = z + \frac{5z^2}{4} + \frac{7z^3}{6} + \frac{15z^4}{16} + \frac{11z^5}{16} + \overline{\frac{z^3}{12} + \frac{5z^4}{32} + \frac{7z^5}{40} + \frac{5z^6}{32} + \frac{55z^7}{448}}.$$

The function $f_8(z)$ is starlike in the direction of imaginary axis, but not univalent in $\mathbb{D}$. The function H and G may be computed using the relation (5). This gives the univalent harmonic function

$$F_8(z) = z + \frac{5z^2}{8} + \frac{7z^3}{18} + \frac{15z^4}{64} + \frac{11z^5}{80} - \overline{\frac{z^3}{36} - \frac{5z^4}{128} - \frac{7z^5}{200} - \frac{5z^6}{192} - \frac{55z^7}{3136}}$$

which is convex in the direction of real axis in $\mathbb{D}$.

In fact, if we let $if_8(-ire^{i\theta}) = u_8(re^{i\theta}) + iv_8(re^{i\theta})$, then we see that

$$v_8(e^{i\theta}) = -\frac{5}{4}\cos 2\theta + \frac{35}{32}\cos 4\theta - \frac{5}{32}\cos 6\theta + \sin\theta - \frac{5}{4}\sin 3\theta + \frac{69}{80}\sin 5\theta - \frac{55}{448}\sin 7\theta$$

and $v_8(e^{i\theta}) = 0$ when $\theta = 1.05263$ and $\theta = 2.08896$.

$$v_8(e^{i\theta}) \begin{cases} > 0 & \text{when} \quad 1.05263 < \theta < 2.08896 \\ < 0 & \text{when} \quad 2.08896 < \theta < 7.33582. \end{cases}$$

From the definition of functions starlike in the direction of real axis, it is clear that the function $if_8(-iz)$ is starlike in the direction of real axis and hence $f_8(z)$ is starlike in the direction of imaginary axis. From Theorem 2, it is evident that the function $F_8(z)$ is convex in the direction of real axis.

Example 3 Applying Corollary 2 to $f_3(z)$ defined by (3) we get the function

$$F_3(z) = -\text{Re}\,(\log(1-z)) + i\text{Im}\left(\frac{z}{1-z}\right)$$

which is convex in the direction of imaginary axis. We see that $F_3(z)$ is not convex in $\mathbb{D}$ and $f_3(z)$ is not starlike in $\mathbb{D}$ (see [14]).

Next we derive a sufficient condition for a function $f \in \mathcal{H}$ to be starlike in one direction. This fact helps to obtain a sufficient condition for a function $f \in \mathcal{H}$ to be convex in one direction. These results are natural generalizations of Theorem 1 and Lemma 2 from Umezawa [20].

Lemma 1 *Suppose that $f = h + \overline{g} \in \mathcal{H}$ such that $f(z)$ is non-zero in $\mathbb{D}\backslash\{0\}$, $|h'(0)| > |g'(0)|$ and satisfies the condition*

$$\int_0^{2\pi} \left|\text{Re}\left(\frac{Df(z)}{f(z)}\right)\right| d\theta < 4\pi,\ z = re^{i\theta}, 0 < r < 1. \tag{6}$$

Then $f(z)$ maps $|z| = r(0 < r < 1)$ onto a curve which is starlike in one direction.

We can now state our next result.

Theorem 3 *Suppose that $f \in \mathcal{H}$ such that $Df(z) \neq 0$ in $\mathbb{D}\backslash\{0\}$, $|h'(0)| > |g'(0)|$ and satisfies the condition*

$$\alpha > \text{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) > -\frac{\alpha}{2\alpha - 3},\ z \in \mathbb{D}\backslash\{0\},$$

for some real number $\alpha > 3/2$. Then the harmonic function f is sense-preserving, univalent and convex in one direction in $\mathbb{D}$.

The proofs of Lemma 1, Theorem 3, and their consequences will be presented in Sect. 4.

3 Proof of Theorem 1 and its applications

Proof of Theorem 1 Let $f \in \mathcal{H}$ be sense-preserving in $\mathbb{D}$ and have the property P as in Definition 1(a). The proof for the other case follows easily by applying argument principle. For $0 < r < 1$, consider the circle $C_r = \{z : |z| = r\}$. From the hypothesis, it is evident that $f(z) \neq 0$ on C_r for each $r \in (0, 1)$. Let $z_1, z_2 \in \mathbb{D}\backslash\{0\}$ such that $z_1 \neq z_2$ and $f(z_1) = w_1$. Let $\rho_0 = \max\{|z_1|, |z_2|, 1 - \delta\}$, and ρ_1 be a real number such that $\rho_0 < \rho_1 < 1$. Consider the function $F(z) = f(z) - w_1$ which is locally univalent in $\mathbb{D}$. Suppose that for each $r \in (\rho_0, \rho_1)$ there exists at least one point z_r on C_r such that $F(z_r) = 0$. Then we get a sequence of distinct points $\{z_{r_j}\}$, where $\rho_0 < r_j < \rho_1$, $j \in \mathbb{N}$, such that $F(z_{r_j}) = 0$ for all $j \in \mathbb{N}$. By using sequentially compactness argument we obtain that F is not locally univalent in $\mathbb{D}$. Therefore, there exists at least one $\rho \in (\rho_0, \rho_1)$ such that $F(z) \neq 0$ for all $z \in C_\rho$. Using the argument principle for harmonic functions we get that ([7, p. 9, Theorem])

$$\triangle_{C_\rho} \arg F(z) = 2\pi N,$$

where N is the total number of zeros of F in $\mathbb{D}_\rho$. Since f satisfies the property P and $\rho > 1-\delta$, $f(C_\rho)$ is a simple closed curve and hence, $F(C_\rho)$ is a simple closed curve. Therefore N must be equal to 1 and thus, $F(z) = 0$ has a unique solution in the disk $|z| \leq \rho$. From the definition of F, it is clear that z_1 is a zero of F in $|z| \leq \rho$ which shows that $f(z_1) \neq f(z_2)$. This completes the proof. □

Remark 1 The assumption that f is sense-preserving cannot be removed from the hypotheses of Theorem 1. For example, consider the function

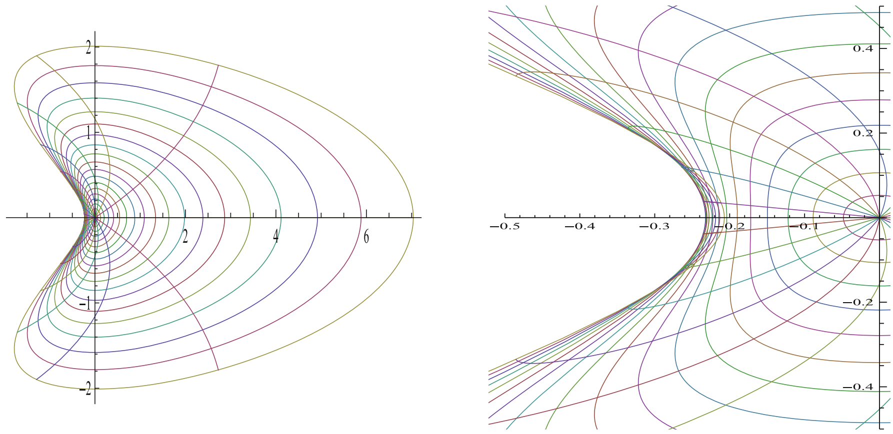
$$Q(z) = \frac{z}{(1-(\sqrt{2}-1)z)^3} + \overline{\frac{(\sqrt{2}-1)z^2}{(1-(\sqrt{2}-1)z)^3}}.$$

Then Q belongs to $\mathcal{H}$ and satisfies the conditions $Q(z) \neq 0$ and $\mathrm{Re}\left(\frac{DQ(z)}{Q(z)}\right) > 0$ for all $z \in \mathbb{D}\backslash\{0\}$ (see [2]), but it is not sense-preserving in $\mathbb{D}$. This function is fully starlike in $\mathbb{D}$ but is not univalent in any disk $|z| < r$ with $r > (2-\sqrt{3})/(\sqrt{2}-1) \approx 0.646\ldots$ The image of the unit disk $\mathbb{D}$ under $Q(z)$ is shown in Fig. 1a. The Fig. 1b explains the non-univalency of Q when r is greater than $0.646\ldots$

As a consequence of Theorem 1, we have

Corollary 3 *Suppose that $f \in \mathcal{H}$ is sense-preserving and $f(z) \neq 0$ for $z \in \mathbb{D}\backslash\{0\}$. If f satisfies*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(\frac{D^2 f(re^{i\theta})}{Df(re^{i\theta})}\right) d\theta > -\pi, 0 < r < 1, \theta_1 < \theta_2 < \theta_1 + 2\pi,$$



(A) The range $Q(\mathbb{D})$ **(B)** The range $Q(\mathbb{D})$ zoomed in

Fig. 1 The image of unit disk under $Q(z)$ and the zoomed in image $Q(\mathbb{D})$.

then f is univalent and close-to-convex in $\mathbb{D}$.

Recall that the converse of Alexander's Theorem is not true in general. This means that, if $F = H + \overline{G}$ is a normalized convex mapping in $\mathbb{D}$, then the function $f = h + \overline{g}$, where h and g are related by

$$h(z) = zH'(z), \text{ and } g(z) = -zG'(z), \tag{7}$$

need not be a starlike mapping in $\mathbb{D}$. An example demonstrated by Duren (see [7, p.110, Lemma]) shows that f is not even locally univalent in $\mathbb{D}$. So, it is natural to expect f to be univalent in $\mathbb{D}$ if it is locally univalent in $\mathbb{D}$, in other words DF is locally univalent in $\mathbb{D}$. As an application of Theorem 1, we prove the converse of Alexander's Theorem with an additional condition.

Theorem 4 *Suppose that $F = H + \overline{G}$ is a sense-preserving normalized convex mapping, and DF is sense-preserving in $\mathbb{D}$. If the functions $h(z)$ and $g(z)$ are related by (7), then the harmonic function $f = h + \overline{g}$ is univalent sense-preserving and starlike in $\mathbb{D}$.*

Proof Let $F \in \mathcal{H}$ be convex in $\mathbb{D}$. As an application of approximation theorem [7, p. 27, Approximation Theorem], we can assume that F has a smooth extension to $\overline{\mathbb{D}}$ and that the boundary function gives a one-to-one sense-preserving mapping of the unit circle onto a convex curve. From the convexity of F, the relation between F and f and the above arguments, it is clear that

$$\frac{d}{d\theta} \arg\left\{f(e^{i\theta})\right\} = \frac{d}{d\theta} \arg\left\{\frac{d}{d\theta} F(e^{i\theta})\right\} \geq 0.$$

This shows that f maps unit circle onto a starlike curve. Therefore, in order to prove that f is a starlike mapping, it is enough to prove that f is univalent in $\mathbb{D}$. Since F is sense-preserving, one has $f(z) = zH'(z) - \overline{zG'(z)} \neq 0$ in $\mathbb{D}\backslash\{0\}$. From the hypothesis, $f = DF$ is sense-preserving in $\mathbb{D}$. Applying Theorem 1 to the function f, we get that f is univalent in $\mathbb{D}$. This completes the proof. □

Remark 2 The sense-preserving condition on DF in Theorem 4 cannot be dropped as the example due to Duren (see [7, p.110, Lemma]) demonstrates. In Theorem 4 if we assume $F(z)$ is fully convex in $\mathbb{D}$, then the function $f = h + \overline{g}$ obtained using (7) will be fully starlike in $\mathbb{D}$. This result is indeed a consequence of Theorem 4 and Proposition 1. Similarly, if we assume f is fully starlike in the Alexander Theorem for harmonic mappings (see [7, p. 108, Lemma]), then the function $F = H + \overline{G}$ obtained using (7) will be fully convex in $\mathbb{D}$.

Example 4 Consider the harmonic function $F(z) = H(z) + \overline{G(z)}$, where

$$H(z) = \frac{(5-3z)z}{6(1-z)^2} - \frac{1}{6}\log(1-z) \text{ and } G(z) = \frac{z-3z^2}{6(1-z)^2} + \frac{1}{6}\log(1-z).$$

The dilatation ω of F is easy to compute and is in fact given by

$$\omega(z) = \frac{G'(z)}{H'(z)} = \frac{-z(3+z)}{z^2-3z+6}.$$

We see that F is sense-preserving in $\mathbb{D}$. The dilatation of DF is z and hence, DF is sense-preserving in $\mathbb{D}$. In order to discuss the mapping properties of $F(z)$, we may rewrite it as

$$F(z) = \mathrm{Re}\left(\frac{1+z}{1-z} - \frac{1}{1-z}\right) + \frac{i}{6}\mathrm{Im}\left\{\left(\frac{1+z}{1-z}\right)^2 - 2\log(1-z)\right\}.$$

Substituting

$$w = \frac{1+z}{1-z} = u + iv,$$

one obtains

$$F(z) = \frac{u-1}{2} + \frac{i}{3}\left\{uv - \arctan\left(\frac{-v}{1+u}\right)\right\}. \tag{8}$$

Consider the straight line $u = c$ for $c > 0$. For a fixed c, the function $t(v) = cv - \arctan((-v)/(1+c))$ is a monotonically increasing function of v on the real line $\mathbb{R}$. Therefore, under $F(z)$, the line $u = c$ will be mapped univalently onto the straight line defined by the set

$$\left\{W = \frac{c-1}{2} + iV : V = \frac{1}{3}\left(cv - \arctan\left(\frac{-v}{1+c}\right)\right), v \in \mathbb{R}\right\}.$$

This observation shows that $F(z)$ maps the unit disk $\mathbb{D}$ onto the right half-plane $\operatorname{Re} W > -1/2$. From (8), it is clear that $F(z)$ maps $\partial\mathbb{D}\backslash\{1\}$ univalently onto the line segment defined by the set

$$\left\{W_1 = -\frac{1}{2} - \frac{i}{6}t : t \in (-\pi, \pi)\right\}.$$

Using the system of two equations given by (7), we can easily obtain

$$f(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}}$$

which is the well known harmonic Koebe function constructed by Clunie and Sheil-Small [3]. Also, from Theorem 4, f is starlike and sense-preserving in $\mathbb{D}$ but is not fully starlike. Finally, if we let

$$F(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n}, \quad z \in \mathbb{D},$$

then it is a simple exercise to see that

$$a_n = \frac{(2n+1)(n+1)}{6n} \text{ and } b_n = \frac{-(2n-1)(n-1)}{6n}.$$

What is interesting here is that the function $F(z)$ is a new right half-plane mapping, and $\big||a_n| - |b_n|\big| = 1$ for $n \geq 2$. Recently, Li and Ponnusamy [10,11] obtained results involving convolution of right half-plane and slanted right half-plane mappings. Since $F(z)$ is a right half-plane mapping, we can expect that this function would be helpful in deriving convolution results (see also [4]).

4 Proofs of Lemma 1 and Theorem 3

4.1 Proof of Lemma 1

Let $f \in \mathcal{H}$ such that $f(z) \neq 0$ in $\mathbb{D}\backslash\{0\}$. Denote by Γ_r the image of the circle $|z| = r$ $(0 < r < 1)$ under the function $f(z)$. Since $|h'(0)| > |g'(0)|$, $f(z)$ is locally univalent at the origin. Therefore there exists a $\rho > 0$ such that Γ_ρ is a simple closed curve which has index 1 with respect to 0. Since f is continuous in $\mathbb{D}$, the curves Γ_r are homotopic to Γ_ρ for all $r \in (0, 1)$. Since $f(z) \neq 0$ in $\mathbb{D}\backslash\{0\}$, and the curves Γ_r are homotopic to Γ_ρ for all $r \in (0, 1)$, it is clear that each Γ_r has index 1 with respect to 0. The total variation of $\arg w$ on Γ_r is given by

$$\int_{|z|=r} |d \arg f(z)| = \int_0^{2\pi} \left|\operatorname{Re}\left(\frac{Df(z)}{f(z)}\right)\right| d\theta,$$

where $z = re^{i\theta}$.

The total variation of $\arg w$ on Γ_r may be calculated in some other way. For $0 \leq \psi < \pi$, define $\Gamma_r(\psi)$ to be the set whose elements are points of intersection of the curve Γ_r with the straight line passing through the origin which makes an angle ψ with the positive real axis. Let $n(\psi)$ denote the number of elements in the set $\Gamma_r(\psi)$. Then the total variation of $\arg w$ on Γ_r can also be given by $\int_0^\pi n(\psi)\,d\psi$. That is,

$$\int_0^\pi n(\psi)\,d\psi = \int_0^{2\pi} \left|\mathrm{Re}\left(\frac{Df(z)}{f(z)}\right)\right| d\theta \text{ where } z = re^{i\theta}.$$

From (6) we have

$$\int_0^\pi n(\psi)\,d\psi < 4\pi.$$

Evidently, $n(\psi) \geq 2$ for all values of ψ and it is an even integer for all but finite number of points. Therefore, there exists at least one ψ_0 such that $n(\psi_0) = 2$. Hence, $f(z)$ maps the circle $|z| = r (0 < r < 1)$ onto a curve which is starlike in one direction, and the proof is complete. □

4.2 Proof of Theorem 3

Let $f \in \mathcal{H}$ satisfy the hypothesis of Theorem 3, and set $F(z) = Df(z)$. Since $|h'(0)| > |g'(0)|$, it is a simple exercise to see that

$$J_F(0) = |h'(0)|^2 - |g'(0)|^2 > 0$$

and hence $F(z)$ is locally univalent at the origin. From the definition of F, it is clear that

$$\alpha > \mathrm{Re}\left(\frac{DF(z)}{F(z)}\right) > -\frac{\alpha}{2\alpha - 3} \text{ for all } z \text{ in } \mathbb{D}\backslash\{0\}. \tag{9}$$

Fix r, $0 < r < 1$, and let C_1 denote the part(s) of $|z| = r$ on which

$$\mathrm{Re}\left(\frac{DF(z)}{F(z)}\right) \geq 0 \text{ and put } \int_{C_1} d\arg z = x. \tag{10}$$

Denote by C_2 the part(s) of $|z| = r$ on which

$$\mathrm{Re}\left(\frac{DF(z)}{F(z)}\right) < 0 \text{ and put } \int_{C_2} d\arg z = 2\pi - x. \tag{11}$$

Let

$$y_1 = \int_{C_1} \text{Re}\left(\frac{DF(z)}{F(z)}\right) d\theta \text{ and } y_2 = -\int_{C_2} \text{Re}\left(\frac{DF(z)}{F(z)}\right) d\theta.$$

From (9), (10) and (11) it is clear that

$$y_1 < \alpha x \text{ and } y_2 < \frac{2\pi\alpha - \alpha x}{2\alpha - 3}. \tag{12}$$

Since $F(0) = 0$, $F(z) \neq 0$ in $\mathbb{D}\backslash\{0\}$ and F is locally univalent at the origin, the change in argument of $F(z)$ when z traverses over the circle $|z| = r$ once is 2π. Therefore,

$$\int_{|z|=r} \text{Re}\left(\frac{DF(z)}{F(z)}\right) d\theta = y_1 - y_2 = 2\pi \tag{13}$$

and

$$\int_{|z|=r} \left|\text{Re}\left(\frac{DF(z)}{F(z)}\right)\right| d\theta = y_1 + y_2 = 2y_1 - 2\pi. \tag{14}$$

Now, we claim that $y_1 < 3\pi$. Suppose on the contrary that $y_1 \geq 3\pi$. Then, by (13), we obtain that $y_2 \geq \pi$. On the other hand, applying (12) together with $y_1 \geq 3\pi$, we get $y_2 < \pi$, which is a contradiction. Therefore, $y_1 < 3\pi$ and applying this to (14) gives the inequality

$$\int_{|z|=r} \left|\text{Re}\left(\frac{DF(z)}{F(z)}\right)\right| d\theta < 4\pi.$$

Thus, by Lemma 1, we conclude that $F(z) = Df(z)$ is starlike in one direction. Hence, from Theorem 2, it is evident that $f(z)$ is univalent and convex in one direction on $|z| < r$ for every r such that $0 < r < 1$. This is same as saying that $f(z)$ is univalent and convex in one direction in $\mathbb{D}$. This completes the proof. □

As an application of Theorem 3, we derive the following.

Corollary 4 *Suppose that* $f \in \mathcal{H}$, $Df(z) \neq 0$ *in* $\mathbb{D}\backslash\{0\}$, $|h'(0)| > |g'(0)|$ *and satisfies any one of the following conditions for all* z *in* $\mathbb{D}\backslash\{0\}$*:*

(i) $\left|1 - \text{Re}\left(\frac{D^2 f(z)}{Df(z)}\right)\right| < 2,$

(ii) $\text{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) > -\frac{1}{2},$

(iii) $\text{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) < \frac{3}{2}.$

Then f *is univalent and convex in one direction in* $\mathbb{D}$.

Proof By taking $\alpha = 3$ in Theorem 3 we get (i). The proofs of (ii) and (iii) follow from Theorem 3 if we allow α tends to ∞, and $3/2$, respectively. □

Corollary 5 *Suppose that* $f = h + \overline{g} \in \mathcal{H}$, $f(0) = 0$, $f(z) \neq 0$ *for all* $z \in \mathbb{D}\backslash\{0\}$, $|h'(0)| > |g'(0)|$ *and satisfies any one of the following conditions for all* z *in* $\mathbb{D}\backslash\{0\}$*:*

(i) $\alpha > \mathrm{Re}\left(\dfrac{Df(z)}{f(z)}\right) > -\dfrac{\alpha}{2\alpha - 3}$ *for some* $\alpha > 3/2$,

(ii) $\left|1 - \mathrm{Re}\left(\dfrac{Df(z)}{f(z)}\right)\right| < 2.$

Then the function F *defined by*

$$F(z) = H(z) + \overline{G(z)} = \int_0^z \frac{h(t)}{t}\,dt - \overline{\int_0^z \frac{g(t)}{t}\,dt}$$

is univalent, convex in one direction (and hence close-to-convex) in $\mathbb{D}$.

Proof From the definition of $F(z)$, it is clear that

$$DF(z) = zH'(z) - \overline{zG'(z)} = f(z) \neq 0 \text{ for all } z \in \mathbb{D}\backslash\{0\},$$

and

$$\mathrm{Re}\left(\frac{D^2F(z)}{DF(z)}\right) = \mathrm{Re}\left(\frac{Df(z)}{f(z)}\right) \text{ for all } z \in \mathbb{D}\backslash\{0\}.$$

From Theorem 3 and Corollary 4, it is clear that F is convex in one direction in $\mathbb{D}$. This completes the proof. □

The limit cases α tend to ∞ and α tend to $3/2$ in the condition (i) of Corollary 5 give

(i)$'$ $\mathrm{Re}\left(\dfrac{Df(z)}{f(z)}\right) > -\dfrac{1}{2}$,

(ii)$'$ $\mathrm{Re}\left(\dfrac{Df(z)}{f(z)}\right) < \dfrac{3}{2}$.

Acknowledgments The second author thanks Council of Scientific and Industrial Research (CSIR), India, for providing financial support in the form of a Senior Research Fellowship to carry out this research. The authors thank the referee for valuable comments.

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