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Polynomial approximation of certain biharmonic mappings

S. Ponnusamy [a,b,*], J. Qiao [c]

[a] Indian Statistical Institute (ISI), Chennai Centre, SETS (Society for Electronic Transactions and security), MGR Knowledge City, CIT Campus, Taramani, Chennai 600 113, India

[b] Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India

[c] Department of Mathematics, Hebei University, Baoding, Hebei 071002, People's Republic of China

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ABSTRACT

In this paper we first determine conditions on the analytic polynomials p and q such that the family of univalent sense-preserving biharmonic polynomials of the form $P = |z|^2(p + \overline{q})$ is dense in the family of univalent sense-preserving biharmonic mappings F defined on the unit disk $\mathbb{D}$ of the form $F = |z|^2G$. Next, we consider the univalency of harmonic function G whenever $F = |z|^2G$ is univalent in $\mathbb{D}$. Finally, we give a partial answer to the problem raised by Muhanna about the radius of univalence of a family of sense-preserving biharmonic mappings with the form $F = |z|^2G + H$, where G is univalent in $\mathbb{D}$.

1. Introduction

The univalent harmonic mappings were first investigated in connection with minimal surfaces by E. Heinz [1]. They have gained much attention due to their function theoretic properties soon after the appearance of the paper by Clunie and Sheil-Small [2]. Biharmonic functions arise in many physical situations, particularly in fluid dynamics and elasticity problems and have many important applications in engineering and biology, see [3–5].

Let f be a complex-valued harmonic function defined on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then f can be written as $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. We say that f is locally univalent and sense-preserving if and only if its Jacobian $J_f(z)$ is positive, where

$$J_f(z) := |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2$$

(see Lewy [6]).

Clunie and Sheil-Small [2] considered the class $\mathcal{S}_H$ of univalent sense-preserving harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$ with the standard normalization so that

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{1}$$

* Corresponding author at: Indian Statistical Institute (ISI), Chennai Centre, SETS (Society for Electronic Transactions and security), MGR Knowledge City, CIT Campus, Taramani, Chennai 600 113, India. Tel.: +91 44 22574615; fax: +91 44 22574602.

E-mail addresses: samy@iitm.ac.in, samy@isichennai.res.in (S. Ponnusamy), qiaojinjing1982@yahoo.com.cn (J. Qiao).

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The class $\mathcal{S}_H$ corresponds to the classical univalent family of analytic functions: $\mathcal{S} = \{f = h + \overline{g} \in \mathcal{S}_H : g(z) \equiv 0\}$. The class $\mathcal{S}$ is fundamental in the study of univalent function theory [7] since Bieberbach raised the conjecture that $|a_n| \le n$ for $n \ge 2$. The affine mapping $\varphi(z) = z + \alpha\bar{z}$ ($|\alpha| < 1$) belongs to $\mathcal{S}_H$ but is not in $\mathcal{S}$. The subclass of $\mathcal{S}_H$ for which $f_{\bar{z}}(0) = b_1 = 0$ is denoted by $\mathcal{S}_H^0$. Then, we have the strict inclusions $\mathcal{S} \subsetneq \mathcal{S}_H^0 \subsetneq \mathcal{S}_H$, but the classes $\mathcal{S}_H$ and $\mathcal{S}_H^0$ are closely related. For example, if $f \in \mathcal{S}_H$ and $|b_1| < 1$, then the function

$$F = \frac{f - \overline{b_1 f}}{1 - |b_1|^2}$$

belongs to $\mathcal{S}_H^0$, so that $f = F + b_1\overline{F}$. Thus, we can limit our attention to the family $\mathcal{S}_H^0$ which is both compact and normal, whereas $\mathcal{S}_H$ is not a compact family although it is normal (see [2,8]).

A four times continuously differentiable function F defined on $\mathbb{D}$ is called *biharmonic* if the Laplacian of F is harmonic in $\mathbb{D}$, i.e., F satisfies the biharmonic equation $\Delta(\Delta F) = 0$, where Δ denotes the Laplacian operator. Clearly, every harmonic function is biharmonic but the converse is not necessarily true. Moreover, every biharmonic function F has the decomposition

$$F(z) = |z|^2 G(z) + H(z), \tag{2}$$

where G and H are complex-valued harmonic in $\mathbb{D}$, see [9,10].

We say that a biharmonic function F is sense-preserving if

$$J_F(z) = |F_z(z)|^2 - |F_{\bar{z}}(z)|^2 > 0$$

for $z \in \mathbb{D} \setminus \{0\}$.

In [9–11] the authors discussed properties, such as univalence and starlikeness, of biharmonic mappings of the form (2) for which $H(z) \equiv 0$ in $\mathbb{D}$. This case led to the family $\mathcal{S}_{BH}$ of functions F of the form $F = |z|^2 G$, where F is univalent sense-preserving in $\mathbb{D}$ and $G(z) = h(z) + \overline{g(z)}$, with $h(z)$ and $g(z)$ have the form (1).

Construction of analytic univalent polynomials is classical and is useful in testing conjectures in geometric function theory. Proving univalence is closely related with the problem of locating zeros of polynomials which is a fundamental question. On the other hand, very little is known about the construction of univalent harmonic polynomials, see [12–15].

Given a polynomial $Q(z) = \sum_{k=0}^{q} c_k z^k$ of degree q, where q is less than or equal to n, define the n-conjugate of Q to be

$$\hat{Q}(z) = z^n \overline{Q(1/\bar{z})} = \sum_{k=n-q}^{n} \overline{c_{n-k}} z^k. \tag{3}$$

The following result due to Suffridge [14, Theorem 1] helps to construct sense-preserving harmonic polynomials.

Theorem A *([14, Theorem 1]). Let $Q(z)$ be a polynomial of degree $q \le n-2$ with $Q(0) = 1$ and assume $Q(z) \neq 0$ when $z \in \mathbb{D}$. Let h and g be defined by $h(0) = g(0) = 0$ and $h'(z) = Q(z) + e^{i\phi}(1-t)z\hat{Q}(z)$, $g'(z) = e^{i\beta}tz\hat{Q}(z)$, where ϕ, β and t are real, $0 \le t \le 1$. Then the polynomial $f = h + \overline{g}$ has degree n and is sense-preserving with $g'(0) = 0$.*

A variation of this theorem was utilized by Thompson in [15]. These harmonic polynomials f were found to have the form $f = h + \overline{g}$, where $h(0) = g(0) = g'(0) = 0$, $h'(0) = 1$, and $g'(z) = B(z)h'(z)$, where $B(z)$ is a finite Blaschke product. Recently, Morgan [13] showed that the class of harmonic polynomials obtained in this way is dense in the class $\mathcal{S}_H^0$. In this paper, we generalize this result to the case of biharmonic mappings.

It is also natural to consider

$$\mathcal{S}_{BH}^0 = \{F = |z|^2 G \in \mathcal{S}_{BH} : G_{\bar{z}}(0) = b_1 = 0\}$$

and introduce

$$\mathcal{H}_n = \{f = h + \overline{g} : h \text{ and } g \text{ are polynomials of degree } n \text{ or less}, h(0) = g(0) = 0,$$
$$h'(0) = 1, g' \text{ is the } m\text{-conjugate of } h' \text{ for some } m \ge n\}.$$

Also, we let $\mathcal{B}_n = \{F = |z|^2 G : G \in \mathcal{H}_n\}$ and

$$\mathcal{B}_n^0 = \{F = |z|^2 G \in \mathcal{B}_n : G_{\bar{z}}(0) = 0\}.$$

Theorem 1. *The class $\mathcal{B}_n^0$ is dense in $\mathcal{S}_{BH}^0$.*

A continuously differentiable function $f : \mathbb{D} \to \mathbb{C}$, $f(0) = 0$, is called starlike in $\mathbb{D}$ if it is univalent and the range $f(\mathbb{D})$ is a starlike (with respect to the origin) domain.

In the case of analytic functions, $f(|z| = r)$ is a starlike curve for each $r \in (0, 1)$ if f is starlike in $\mathbb{D}$. This property is not true in general in the case of harmonic starlike mappings (see [16,17]).

Definition 1. A continuously differentiable function F on $\mathbb{D}$ is said to be *fully starlike* (see [18]) if it is sense-preserving, $F(0) = 0$, $F(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$ and the curve $F(re^{it})$ is starlike (with respect to the origin) for each $r \in (0, 1)$. The last condition is the same as saying that

$$\frac{\partial}{\partial t}\left(\arg F(re^{it})\right) = \mathrm{Re}\left(\frac{zF_z(z) - \overline{z}F_{\overline{z}}(z)}{F(z)}\right) > 0$$

for $z = re^{it} \in \mathbb{D}\setminus\{0\}$ (see also [16,17] in order to distinguish the starlikeness property of harmonic mappings from conformal mappings).

The above definitions of starlike and fully starlike functions are clearly applicable both for harmonic and biharmonic functions. In general, if $G \in \mathcal{S}$, then $F = |z|^2G$ is not necessarily univalent in $\mathbb{D}$ (see [10, Example 2.2] and [11, Example 1]). In fact, F is not even sense-preserving in $\mathbb{D}$ if $G \in \mathcal{S}$. On the other hand, in [9], the authors proved that $F = |z|^2G$ is univalent in $\mathbb{D}$ whenever G is harmonic and starlike in $\mathbb{D}$. Moreover, Muhanna [11, Theorem 2] determined the radius of univalence of $|z|^2G$ when $G \in \mathcal{S}$. It is natural to ask for the converse of this implication in a slightly general setting. That is, if $F = |z|^2G$ is univalent, what can be said about the harmonic function G?

Theorem 2. *Let $F = |z|^2G \in \mathcal{S}^0_{BH}$, where the analytic part of G is locally univalent in $\mathbb{D}$. Then G is univalent in $\mathbb{D}$.*

Since $F = |z|^2G$, for $G \in \mathcal{S}$, is not even sense-preserving in $\mathbb{D}$, Muhanna [11] seeks a condition on H so that $|z|^2G + H$ is sense-preserving for each $G \in \mathcal{S}$. More precisely Muhanna [11, Theorem 3] proved that the biharmonic function W_0 defined by

$$W_0(z) = |z|^2G(z) + \left[-G(z) + 2\int_0^z \frac{G(\zeta)}{\zeta}\, d\zeta\right], \tag{4}$$

has positive Jacobian in $\mathbb{D} \setminus \{0\}$ and if, in addition, G is starlike, then W_0 becomes univalent in $\mathbb{D}$. Here it is worth pointing out that the mapping should rather be in the form

$$W(z) = |z|^2G(z) + \left[-G(z) + 4\int_0^z \frac{G(\zeta)}{\zeta}\, d\zeta\right]. \tag{5}$$

That is, the number 2 in front of the integral in (4) should be replaced by 4. In fact, a closer examination of the proof of Muhanna shows that one has to use, for $G \in \mathcal{S}$, the sharp inequality $(1 - |z|^2)\left|\frac{zG'(z)}{G(z)}\right| \le 4$ in $\mathbb{D}$. The number 4 in this inequality cannot be replaced by a smaller number. This observation implies that the conclusions of [11, Theorem 3 and Proposition 3] hold for $W(z)$, but not for $W_0(z)$. That is, for each $G \in \mathcal{S}$, the biharmonic function W has positive Jacobian except at $z = 0$, and if G is starlike, then W is univalent in $\mathbb{D}$. Nevertheless the paper of Muhanna [11, Problem 1] ends with an interesting question.

Problem 1. What is the radius of univalence of W if $G \in \mathcal{S}$?

We first solve this problem for (5) under a weaker hypothesis.

Theorem 3. *Let $G(z) = z + \sum_{k=2}^{\infty} a_kz^k$ be analytic in $\mathbb{D}$ such that $|a_k| \le k$ for $k \ge 2$. Then the biharmonic function W defined by (5) has Jacobian $J_W(z) > 0$ except at $z = 0$ and univalent in $\{z : |z| < r_S\}$, where $r_S \approx 0.34195$ is the root of the equation*

$$6r^5 - 10r^4 - 5r^3 + 15r^2 - 13r + 3 = 0$$

in the interval $(0, 1)$. Moreover, W is fully starlike for $|z| < r_S$ and also by all its sections

$$W_n(z) = |z|^2G_n(z) + \left[-G_n(z) + 4\int_0^z \frac{G_n(\zeta)}{\zeta}\, d\zeta\right].$$

We remark that

$$G_0(z) = 2z - \frac{z}{(1-z)^2} = z - \sum_{k=2}^{\infty} kz^k$$

is analytic in $\mathbb{D}$, but $G_0(z)$ is not univalent in $\mathbb{D}$, because

$$G_0'(z) = 2 - \frac{1+z}{(1-z)^3}$$

vanishes in $\mathbb{D}$. In fact in 1970, Gavrilov [19] showed that the radius of univalence of functions satisfying the inequality $|a_k| \le k$ is the real root $r_0 \approx 0.164878$ of the equation $2(1-r)^3 - (1+r) = 0$. In this case, it was shown later by Yamashita [20] that the number r_0 is also the radius of starlikeness. Thus, G in Theorem 3 is not necessarily univalent in $\mathbb{D}$.

On the other hand, a well-known theorem of Louis de Branges gives that if $G(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{S}$ then $|a_k| \le k$ for $k \ge 2$. Applying this result, Theorem 3 immediately gives the following result. Thus, the answer to the above problem is affirmative.

Corollary 1. *Let $G \in \mathcal{S}$. Then the biharmonic function W defined by (5) has Jacobian $J_W(z) > 0$ except at $z = 0$. Moreover, W (and each of its polynomial section) is univalent and fully starlike in $\{z : |z| < r_S\}$, where $r_S \approx 0.34195$.*

It would be interesting to determine the sharp value of r_S in Corollary 1.

Theorem 3 may be improved whenever $G''(0) = 0$. More precisely, one has the following.

Theorem 4. *Let $G(z) = z + \sum_{k=3}^{\infty} a_k z^k$ be analytic in $\mathbb{D}$ such that $|a_k| \le k$ for $k \ge 3$. Then the biharmonic function W defined by (5) has Jacobian $J_W(z) > 0$ except at $z = 0$, and is univalent in $\{z : |z| < r_S\}$, where $r_S \approx 0.424553$ is the root of the equation $\phi_0(r) = 0$ in the interval $(0, 1)$, where*

$$\phi_0(r) = -8r^6 + 30r^5 - 38r^4 + 15r^3 + 3r^2 - 9r + 3.$$

Moreover, W is fully starlike for $|z| < r_S$.

We first remark that there are non-univalent analytic functions $G(z) = z + \sum_{k=3}^{\infty} a_k z^k$ (i.e. vanishing second coefficient) in $\mathbb{D}$ with $|a_k| \le k$ for $k \ge 3$. In this case, Corollary 1 takes the form.

Corollary 2. *Let $G \in \mathcal{S}$ with $G''(0) = 0$. Then the biharmonic function W defined by (5) has Jacobian $J_W(z) > 0$ except at $z = 0$. Moreover, W is univalent and fully starlike in $\{z : |z| < r_S\}$, where $r_S \approx 0.424553$.*

The paper is organized as follows. In Section 2, we present the proofs of Theorems 1 and 2. The proof of Theorem 3, will be presented in Section 3. We remark that Theorem 3 is applicable even when G is not univalent in $\mathbb{D}$.

2. Proofs of Theorems 1 and 2

We begin this section with the following lemma.

Lemma 1 ([7, p. 195, Lemma]). *Let P be a polynomial of degree at most n, with $|P(z)| \le 1$ in $|z| \le 1$. Then $|P'(z)| \le n$ in $|z| \le 1$.*

The proof of Theorem 1 is established by the proofs of Lemmas 2 through 5.

Lemma 2. *It is sufficient to show the result of Theorem 1 for biharmonic polynomials is contained in $\mathcal{S}_{BH}^0$.*

Proof. Let $F(z) = |z|^2 G(z)$, where $F \in \mathcal{S}_{BH}^0$ and $G(z) = h(z) + \overline{g(z)}$ with $h(z)$ and $g(z)$ have the form (1). Further let

$$F_r(z) = r^{-3} F(rz) = |z|^2 r^{-1} G(rz) \quad \text{and} \quad G_r(z) = r^{-1} G(rz)$$

for fixed r with $0 < r < 1$. With $z = \rho e^{i\theta}$, we introduce

$$A_r(\rho, \theta, \phi) =: \left| \frac{F(r\rho e^{i\theta}) - F(r\rho e^{i\phi})}{\rho^3(\theta - \phi)} \right| = r^2 \left| \frac{G(r\rho e^{i\theta}) - G(r\rho e^{i\phi})}{\rho(\theta - \phi)} \right|$$

for $0 \le \rho \le 1$, $0 \le |\theta - \phi| \le \pi$, and $\theta, \phi \in [0, 2\pi]$. We observe that $A_r(\rho, \theta, \phi)$ is a continuous function of ρ for $0 < \rho \le 1$, and of θ and ϕ, $0 < |\theta - \phi| \le \pi$ for $\theta, \phi \in [0, 2\pi]$.

First we see that, if $\theta \ne \phi$, as $\rho \to 0$,

$$\begin{aligned} A_r(0, \theta, \phi) &= \lim_{\rho \to 0} A_r(\rho, \theta, \phi) = \lim_{\rho \to 0} r^2 \left| \frac{h(r\rho e^{i\theta}) - h(r\rho e^{i\phi}) + \overline{g(r\rho e^{i\theta}) - g(r\rho e^{i\phi})}}{\rho(\theta - \phi)} \right| \\ &= r^3 \left| \frac{e^{i\theta} - e^{i\phi}}{\theta - \phi} \right| \\ &= r^3 \left| \frac{\sin((\theta - \phi)/2)}{(\theta - \phi)/2} \right| > 0, \end{aligned}$$

and as $\theta \to \phi$,

$$A_r(0, \phi, \phi) = \lim_{\theta \to \phi} A_r(0, \theta, \phi) = r^3 > 0.$$

In the case $\rho \neq 0$, as $\theta \to \phi$, it follows that

$$\begin{aligned} A_r(\rho,\phi,\phi) &= \lim_{\theta\to\phi} A_r(\rho,\theta,\phi) \\ &= r^2 \lim_{\theta\to\phi}\left|\frac{ir\rho e^{i\theta}h'(r\rho e^{i\theta}) + \overline{ir\rho e^{i\theta}g'(r\rho e^{i\theta})}}{\rho}\right| \\ &\geq \frac{r^2}{\rho}\left(\left|h(r\rho e^{i\theta}) + \overline{g(r\rho e^{i\theta})} + r\rho e^{i\theta}h'(r\rho e^{i\theta})\right| - \left|h(r\rho e^{i\theta}) + \overline{g(r\rho e^{i\theta})} + \overline{r\rho e^{i\theta}g'(r\rho e^{i\theta})}\right|\right), \\ &> 0, \end{aligned}$$

since F is sense-preserving.

If $\rho \neq 0, \theta \neq \phi, A_r(\rho,\theta,\phi) > 0$, since F is univalent in $\mathbb{D}$. Hence for each $r \in (0,1)$, $A_r(\rho,\theta,\phi)$ is a positive and continuous function of ρ,θ,ϕ for $0 \leq \rho \leq 1$ and $0 \leq |\theta-\phi| \leq \pi, \theta,\phi \in [0,2\pi]$. Therefore, the function must assume a minimum value; that is, there exists a constant $\varepsilon_0 > 0$ such that $A_r(\rho,\theta,\phi) \geq \varepsilon_0$ for these values of ρ,θ,ϕ.

Since F is sense-preserving, it follows that

$$|h(z) + \overline{g(z)} + zh'(z)| - |h(z) + \overline{g(z)} + \overline{zg'(z)}| > 0$$

in $0 < |z| < r$. For $z = \rho e^{i\theta} \neq 0$, let

$$D(\rho,\theta) = \frac{\left|h(\rho e^{i\theta}) + \overline{g(\rho e^{i\theta})} + \rho e^{i\theta}h'(\rho e^{i\theta})\right| - \left|h(\rho e^{i\theta}) + \overline{g(\rho e^{i\theta})} + \overline{\rho e^{i\theta}g'(\rho e^{i\theta})}\right|}{\rho}.$$

Then we observe that

$$\begin{aligned} \lim_{\rho\to 0} D(\rho,\theta) &= \lim_{\rho\to 0}\left(\left|2h'(\rho e^{i\theta})e^{i\theta} + \overline{g(e^{i\theta})e^{i\theta}} + \rho e^{2i\theta}h''(\rho e^{i\theta})\right| - \left|h'(\rho e^{i\theta})e^{i\theta} + \overline{2g(e^{i\theta})e^{i\theta}} + \overline{\rho e^{2i\theta}g''(\rho e^{i\theta})}\right|\right) \\ &= 1. \end{aligned}$$

Hence $D(\rho,\theta) > 0$ for $0 \leq \rho \leq r, \theta \in [0,2\pi]$, and furthermore there exists a constant $\varepsilon_1 > 0$ such that $D(\rho,\theta) > \varepsilon_1 > 0$.

Now let $K \subset \mathbb{D}$ be a compact set and let $\varepsilon > 0$ be an arbitrary constant. Write

$$F_r(z) = |z|^2 G_r(z) = |z|^2\left(\sum_{k=1}^{\infty} a_k r^{k-1} z^k + \overline{\sum_{k=2}^{\infty} b_k r^{k-1} z^k}\right),$$

$$H_N(z) = |z|^2 G_N(z) = |z|^2\left(\sum_{k=1}^{N} a_k r^{k-1} z^k + \overline{\sum_{k=2}^{N} b_k r^{k-1} z^k}\right),$$

and

$$B(\rho,\theta,\phi) = \left|\frac{H_N(\rho e^{i\theta}) - H_N(\rho e^{i\phi})}{\rho(\theta-\phi)}\right|.$$

Choose $r (0 < r < 1)$ so that

$$|F_r(z) - F(z)| < \frac{\varepsilon}{2} \quad \text{and} \quad |G_r(z) - G(z)| < \frac{\varepsilon}{2} \quad \text{when } z \in K.$$

Choose N sufficiently large that

$$|H_N(z) - F_r(z)| < \frac{\varepsilon}{2} \quad \text{and} \quad |G_N(z) - G_r(z)| < \frac{\varepsilon}{2} \quad \text{when } z \in K$$

and $B(\rho,\theta,\phi) \geq \frac{\varepsilon_0}{2}$ for $0 \leq \rho \leq 1, |\theta-\phi| \leq \pi$, and $\theta,\phi \in [0,2\pi]$. Also,

$$\frac{|h(z) + \overline{g(z)} + zh'(z)| - |h(z) + \overline{g(z)} + \overline{zg'(z)}|}{|z|} > \frac{\varepsilon_1}{2}$$

for $|z| \leq r$. Thus we obtain a sequence $\{H_N = |z|^2 G_N\}$ of univalent polynomials in $\mathcal{S}_{BH}^0$ that converges to $F = |z|^2 G$ on compact subsets of $\mathbb{D}$, and also the sequence $\{G_N\}$ converges to G on compact subsets of $\mathbb{D}$. □

Remark 1. From the proof of the above lemma, we obtain a sequence of biharmonic polynomials $P_j(z) \in \mathcal{S}_{BH}^0$ of the form

$$P_j(z) = |z|^2\left(p(z) + \overline{q(z)}\right) = |z|^2\left(\sum_{k=1}^{N} a_k z^k + \overline{\sum_{k=2}^{N} b_k z^k}\right) \tag{6}$$

each of which satisfies

$$\frac{\left|p(z)+\overline{q(z)}+zp'(z)\right|-\left|p(z)+\overline{q(z)}+\overline{zq'(z)}\right|}{|z|}>\varepsilon_j, \tag{7}$$

where $\varepsilon_j > 0$.

Given a sequence of polynomials as in (6), the aim is to construct a sequence of biharmonic polynomials in $\mathcal{B}_n^0$. To this end, let $P(z) = |z|^2\left(p(z)+\overline{q(z)}\right)$ be a univalent and sense-preserving biharmonic polynomial which satisfies

$$\frac{\left|p(z)+\overline{q(z)}+zp'(z)\right|-\left|p(z)+\overline{q(z)}+\overline{zp'(z)}\right|}{|z|}>\varepsilon^*>0.$$

Let $\hat{p'}$ and $\hat{q'}$ be the $(M-1)$-conjugate polynomials of p' and q', respectively, where p and q are given by (6) and $M > n$. In this way, we obtain

$$\hat{p}(z) =: \sum_{k=M-n+1}^{M} \frac{M-k+1}{k}\overline{a}_{M-k+1}z^k,$$

$$\hat{q}(z) =: \sum_{k=M-n+1}^{M-1} \frac{M-k+1}{k}\overline{b}_{M-k+1}z^k,$$

and

$$\hat{P}(z) =: |z|^2\left(p(z)+\hat{q}(z)+\overline{q(z)+\hat{p}(z)}\right). \tag{8}$$

Lemma 3. *Given a sequence $\{P_j(z)\}$ of biharmonic polynomials of the form (6) that converges to a biharmonic mapping F uniformly on compact sets, the corresponding sequence given by (8) with the members M_j chosen sufficiently large also converges to F uniformly on compact sets.*

Proof. It suffices to prove that $\hat{p}_j$ and $\hat{q}_j$ tend to zero uniformly on compact sets if the constants M_j are appropriately chosen. Similar to the arguments as in the proof of [13, Lemma 2.6], we obtain

$$|\hat{p}(z)| < \frac{n^2}{M-n+1}\max_{1\le k\le n}|a_k| \quad \text{and} \quad |\hat{q}(z)| < \frac{n^2}{M-n+1}\max_{1\le k\le n}|b_k|.$$

The proof is now complete. □

Lemma 4. *The biharmonic polynomials given by (8) are sense-preserving and belong to the class $\mathcal{B}_n^0$.*

Proof. Let $h(z) = p(z) + \hat{q}(z)$ and $g(z) = q(z) + \hat{p}(z)$. Then it is easy to verify that h' and g' are $(M-1)$-conjugate polynomials. From the proof of Lemma 3, we have

$$|\hat{p}(z)| < \frac{n^2}{M-n+1}\max_{1\le k\le n}|a_k|\,|z| \quad \text{and} \quad |\hat{q}(z)| < \frac{n^2}{M-n+1}\max_{1\le k\le n}|b_k|\,|z|.$$

It follows from Lemma 1 that

$$|\hat{p}'(z)| < \frac{n^3}{M-n+1}\max_{1\le k\le n}|a_k| \quad \text{and} \quad |\hat{q}'(z)| < \frac{n^3}{M-n+1}\max_{1\le k\le n}|b_k|.$$

Choose M large enough such that

$$\frac{2n^2+n^3}{M-n+1}\max_{1\le k\le n}|a_k| + \frac{2n^2+n^3}{M-n+1}\max_{1\le k\le n}|b_k| < \frac{\varepsilon^*}{2}.$$

Then

$$\frac{\left|p(z)+\hat{q}(z)+\overline{q(z)+\hat{p}(z)}+zp'(z)+z\hat{q}'(z)\right|}{|z|} - \frac{\left|p(z)+\hat{q}(z)+\overline{q(z)+\hat{p}(z)}+\overline{zq'(z)+z\hat{p}'(z)}\right|}{|z|}$$

$$> \frac{\left|p(z)+\overline{q(z)}+zp'(z)\right|}{|z|} - \frac{\left|p(z)+\overline{q(z)}+\overline{zq'(z)}\right|}{|z|} - \frac{\left|\hat{q}(z)+\overline{\hat{p}(z)}+z\hat{q}'(z)\right|}{|z|} - \frac{\left|\hat{q}(z)+\overline{\hat{p}(z)}+\overline{z\hat{p}'(z)}\right|}{|z|}$$
$$> \frac{\left|p(z)+\overline{q(z)}+zp'(z)\right|}{|z|} - \frac{\left|p(z)+\overline{q(z)}+\overline{zq'(z)}\right|}{|z|} - \frac{2n^2+n^3}{M-n+1}\max_{1\le k\le n}|a_k| - \frac{2n^2+n^3}{M-n+1}\max_{1\le k\le n}|b_k|$$
$$> \frac{\varepsilon^*}{2} > 0,$$

which implies that the biharmonic polynomial of the form (8) is sense-preserving. □

Lemma 5. *The biharmonic polynomials given by (8) are univalent.*

Proof. Let $F(z) = \hat{P}(z) = |z|^2\left(h(z)+\overline{g(z)}\right) = |z|^2\left(p(z)+\hat{q}(z)+\overline{q(z)+\hat{p}(z)}\right)$. With $D(\theta) = (e^{i\theta}+e^{-iM\theta}/M)$, we define

$$B(\theta,\phi) = \left|\frac{F(e^{i\theta})-F(e^{i\phi})}{D(\theta)-D(\phi)}\right|.$$

The proof of this theorem is based on the work of [13, Lemma 2.7]. It follows that

$$\lim_{\theta\to\phi} B(\theta,\phi) = |p'(e^{i\phi}) + e^{i(M-1)\phi}\overline{q'(e^{i\phi})}|. \tag{9}$$

Because of the sense-preserving property, namely,

$$\left|p(z)+\hat{q}(z)+\overline{q(z)+\hat{p}(z)}+zp'(z)+z\hat{q}'(z)\right| - \left|p(z)+\hat{q}(z)+\overline{q(z)+\hat{p}(z)}+\overline{zq'(z)+z\hat{p}'(z)}\right| > 0$$

for $0<|z|\le 1$, it follows that

$$\begin{aligned}
&\left|e^{i\theta}p'(e^{i\theta}) + e^{i\theta}e^{i(M-1)\theta}\overline{q'(e^{i\theta})} + \overline{e^{i\theta}q'(e^{i\theta})} + e^{-i\theta}e^{-i(M-1)}p'(e^{i\theta})\right| \\
&= \left|\left(e^{i\theta}+e^{-iM\theta}\right)p'(e^{i\theta}) + \left(e^{-i\theta}+e^{iM\theta}\right)\overline{q'(e^{i\theta})}\right| \\
&= \left|e^{i\theta}+e^{-iM\theta}\ \|\ p'(e^{i\theta})+e^{i(M-1)\theta}\overline{q'(e^{i\theta})}\right| \\
&\ge \left|p(e^{i\theta})+\hat{q}(e^{i\theta})+\overline{q(e^{i\theta})+\hat{p}(e^{i\theta})}+e^{i\theta}p'(e^{i\theta})+e^{i\theta}\hat{q}'(e^{i\theta})\right| \\
&\quad - \left|p(e^{i\theta})+\hat{q}(e^{i\theta})+\overline{q(e^{i\theta})+\hat{p}(e^{i\theta})}+\overline{e^{i\theta}q'(e^{i\theta})+e^{i\theta}\hat{p}'(e^{i\theta})}\right| \\
&> 0.
\end{aligned}$$

Thus, the limit value $B(\phi,\phi)$ in (9) exists and is positive.

Let $c = \min_{0\le\phi\le 2\pi}\{B(\phi,\phi)\}$. Then there is a $\delta>0$ such that

$$B(\theta,\phi) > \frac{c}{2} \quad \text{when } |\theta-\phi|<\delta,\ \theta,\phi\in[0,2\pi].$$

It remains to show that $B(\theta,\phi)>0$ on the complement part $|\theta-\phi|\ge\delta$, $\theta,\phi\in[0,2\pi]$. However, for sufficiently large M, from the inequalities

$$|\hat{p}(z)| < \frac{n^2}{M-n+1}\max_{1\le k\le n}|a_k| \quad\text{and}\quad |\hat{q}(z)| < \frac{n^2}{M-n+1}\max_{1\le k\le n}|b_k|$$

(obtained from the proof of Lemma 3), it follows that

$$|e^{i\theta}p'(e^{i\theta}) - \overline{e^{i\theta}q'(e^{i\theta})}| > 0 \quad \text{for } \theta,\phi\in[0,2\pi],$$

since $|z|^2\left(p(z)+\overline{q(z)}\right)$ is sense-preserving. Hence, the curve $F(e^{it})(0\le t\le 2\pi)$ is a simple closed curve and we apply degree principle (see, for example, [21]) to complete the proof. □

Proof of Theorem 2. An argument analogous to that used in the proof of Theorem 1, the condition that $|(H_N)_z(z)| > \varepsilon_2 > 0$ in the compact set K in Lemma 2, and the condition that, for M sufficiently large, $|\hat{q}'(z)| < \varepsilon_2/2$ in Lemma 4, yield that $p'(z)+\hat{q}'(z)\ne 0$. Then, by Theorem A, we conclude that $p(z)+\hat{q}(z)+\overline{q(z)+\hat{p}(z)}$ is sense-preserving. Finally, applying Lemma 5 and the argument principle from [22], we obtain that $p(z)+\hat{q}(z)+\overline{q(z)+\hat{p}(z)}\in\mathcal{S}_H^0$. The desired conclusion of the theorem follows from the fact the $\mathcal{S}_H^0$ is compact. □

3. The proof of Theorem 3

Many interesting results follow simply from the coefficient results of the following type (see also [17,23]).

Lemma 6. *Let $F(z) = |z|^2G(z) + H(z)$ be a biharmonic mapping in $\mathbb{D}$, where $H(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $G(z) = \sum_{k=1}^{\infty} b_k z^k$ are analytic in $\mathbb{D}$, and satisfy the condition*

$$\sum_{k=1}^{\infty}(k+2)|b_k| + \sum_{k=2}^{\infty} k|a_k| \leq 1. \tag{10}$$

Then F is sense-preserving, univalent and fully starlike in $\mathbb{D}$. Moreover, $H + \overline{G}$ also has this property.

Proof. A computation gives

$$F_z(z) = |z|^2 \sum_{k=1}^{\infty}(k+1)b_k z^{k-1} + 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} \quad \text{and} \quad F_{\overline{z}}(z) = \sum_{k=1}^{\infty} b_k z^{k+1}.$$

Then $J_F(0) = |F_z(0)|^2 - |F_{\overline{z}}(0)|^2 = 1$ and for $z \neq 0$,

$$J_F(z) = \Big(|F_z(z)| + |F_{\overline{z}}(z)|\Big)\Big(|F_z(z)| - |F_{\overline{z}}(z)|\Big) > 0$$

because for $z \neq 0$,

$$\begin{aligned}|F_z(z)| - |F_{\overline{z}}(z)| &= \left||z|^2 \sum_{k=1}^{\infty}(k+1)b_k z^{k-1} + 1 + \sum_{k=2}^{\infty} k a_k z^{k-1}\right| - \left|\sum_{k=1}^{\infty} b_k z^{k+1}\right| \\ &> 1 - \sum_{k=1}^{\infty}(k+2)|b_k| - \sum_{k=2}^{\infty} k|a_k| \\ &\geq 0.\end{aligned}$$

Thus, F is sense-preserving in $\mathbb{D}$.

Moreover, for each fixed $r_0 \in (0, 1)$, we have

$$F_{r_0}(z) =: F(z) = r_0^2 G(z) + H(z) = \sum_{k=1}^{\infty} r_0^2 b_k z^k + z + \sum_{k=2}^{\infty} a_k z^k$$

which is harmonic in $\mathbb{D}$. Since

$$\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|r_0^2 b_k| \leq \sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty}(k+2)|b_k| \leq 1,$$

it follows from [17, Lemma 2.1] that F_{r_0} is univalent and fully starlike in $\mathbb{D}$, which implies that for each $z = re^{i\theta} \neq 0$,

$$\frac{d}{d\theta}(\arg F_{r_0}(re^{i\theta})) > 0.$$

Letting $r_0 = r$ yields that

$$\frac{d}{d\theta}(\arg F(re^{i\theta})) = \frac{d}{d\theta}(\arg F_r(re^{i\theta})) > 0 \tag{11}$$

for each $z = re^{i\theta} \neq 0$, and also F is univalent on each circle $|z| = r$.

From the sense-preserving property of F and the univalence of F on the each circle $|z| = r$, it follows that F is univalent in $\mathbb{D}$. By using (11), we have that the harmonic function F is indeed fully starlike in $\mathbb{D}$.

Finally, it follows from [17, Lemma 2.1] that the condition (10) implies that $H + \overline{G}$ is also univalent and fully starlike in $\mathbb{D}$. □

Proof of Theorem 3. Let $W_r(z) = \frac{1}{r}W(rz)$ for $0 < r < 1$, where W is defined by (5) and $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ (with $a_1 = 1$) such that $|a_k| \leq k$ for all $k \geq 2$. Then a computation gives that

$$W_r(z) = |z|^2 \sum_{k=1}^{\infty} r^{k+1} a_k z^k + \sum_{k=1}^{\infty} \left(\frac{4}{k} - 1\right) r^{k-1} a_k z^k$$

so that

$$\frac{W_r(z)}{3} = |z|^2 \sum_{k=1}^{\infty} B_k z^k + z + \sum_{k=2}^{\infty} A_k z^k$$

where

$$B_k = \frac{1}{3} r^{k+1} a_k \quad \text{and} \quad A_k = \frac{4-k}{3k} r^{k-1} a_k.$$

Note that $W(z)$ is univalent and fully starlike in $|z| < r$ if and only if $W_r(z)$ is univalent and fully starlike in the unit disk $|z| < 1$. Thus, by Lemma 6, it suffices to show that

$$S(r) := \sum_{k=1}^{\infty} (k+2)|B_k| + \sum_{k=2}^{\infty} k|A_k| \leq 1$$

for $0 < r \leq r_S$. Now, since $|a_k| \leq k$, it follows that $S(r) \leq \frac{1}{3} T(r)$, where

$$T(r) = \sum_{k=1}^{\infty} k(k+2) r^{k+1} + \sum_{k=2}^{\infty} k|k-4| r^{k-1}.$$

By the last inequality, $S(r) \leq 1$ holds if $T(r) \leq 3$. Again, since

$$\frac{r}{(1-r)^2} = \sum_{k=1}^{\infty} k r^k \quad \text{and} \quad \frac{r(1+r)}{(1-r)^3} = \sum_{k=1}^{\infty} k^2 r^k,$$

we find that

$$\begin{aligned} T(r) &= \sum_{k=1}^{\infty} (k^2+2k) r^{k+1} + 4r + 3r^3 + \sum_{k=4}^{\infty} k(k-4) r^{k-1} \\ &= \sum_{k=1}^{\infty} k^2 r^{k+1} + 2\sum_{k=1}^{\infty} k r^{k+1} + \sum_{k=1}^{\infty} k^2 r^{k-1} - 4\sum_{k=1}^{\infty} k r^{k-1} + 3 + 8r + 6r^2 \\ &= \frac{r^2(1+r)}{(1-r)^3} + \frac{2r^2}{(1-r)^2} + \frac{1+r}{(1-r)^3} - \frac{4}{(1-r)^2} + 3 + 8r + 6r^2 \end{aligned} \tag{12}$$

and therefore, by a computation, we see that $T(r) \leq 3$ is equivalent to the inequality $\phi(r) \geq 0$, where

$$\phi(r) = 6r^5 - 10r^4 - 5r^3 + 15r^2 - 13r + 3.$$

The inequality $\phi(r) \geq 0$ holds if $0 < r \leq r_S$, where $r_S \approx 0.34195$ is the root of the equation $\phi(r) = 0$ in the interval $(0, 1)$. Thus, $S(r) \leq 1$ for $0 < r \leq r_S$ and so, by Lemma 6, $W_r(z)$ is univalent sense-preserving and fully starlike in $\mathbb{D}$ with $0 < r \leq r_S$. The proof of the theorem is complete. □

Proof of Theorem 4. We proceed to obtain a proof exactly as in the proof of Theorem 3. We just include important steps. Thus, following the proof of Theorem 3, it suffices to show that $T_1(r) \leq 3$, where

$$T_1(r) = \sum_{k=1, k\neq 2}^{\infty} k(k+2) r^{k+1} + \sum_{k=3}^{\infty} k|k-4| r^{k-1} = T(r) - (8r^3 + 4r).$$

Here $T(r)$ is defined by (12). Thus, a simple calculation shows that the inequality, $T_1(r) \leq 3$ is equivalent to

$$\phi(r) + (8r^3 + 4r)(1-r)^3 = 6r^5 - 10r^4 - 5r^3 + 15r^2 - 13r + 3 + (8r^3 + 4r)(1-r)^3 \geq 0$$

which is the same as

$$\phi_0(r) = -8r^6 + 30r^5 - 38r^4 + 15r^3 + 3r^2 - 9r + 3 \geq 0.$$

We see that the inequality $\phi_0(r) \geq 0$ holds if $0 < r \leq r_S$, where $r_S \approx 0.424553$ is the root of the equation $\phi_0(r) = 0$ in the interval $(0, 1)$. The proof of the theorem is complete. □

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References

[1] E. Heinz, Über die Lösungen der Minimalflächengleichung, Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. 1952 (1952) 51–56 (German).
[2] J.G. Clunie, T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A. I. 9 (1984) 3–25.
[3] J. Happel, H. Brenner, Low Reynolds Numbers Hydrodynamics, Princeton-Hall, 1965.
[4] S.A. Khuri, Biorthogonal series solution of Stokes flow problems in sectorial regions, SIAM J. Appl. Math. 56 (1996) 19–39.
[5] W.E. Langlois, Slow Viscous Flow, Macmillan Company, 1964.
[6] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, Bull. Amer. Math. Soc. 42 (1936) 689–692.
[7] P. Duren, Univalent Functions, Springer-Verlag, New York, 1982.
[8] P. Duren, Harmonic Mappings in the Plane, Cambridge Univ. Press, 2004.
[9] Z. Abdulhadi, Y. Abu Muhanna, S. Khuri, On univalent solutions of the biharmonic equations, J. Inequal. Appl. 5 (2005) 469–478.
[10] Z. Abdulhadi, Y. Abu Muhanna, S. Khuri, On some properties of solutions of the biharmonic equation, Appl. Math. Comput. 177 (2006) 346–351.
[11] Y. Abu Muhanna, On univalence of biharmonic maps, Complex Var. Elliptic Equ. 53 (2008) 745–751.
[12] J.M. Jahangiri, C.J. Morgan, T.J. Suffridge, Construction of close-to-convex harmonic polynomials, Complex Variables Theory Appl. 45 (4) (2001) 319–326.
[13] C.J. Morgan, Polynomial approximation of harmonic mappings, Complex Var. Elliptic Equ. 1 (2011) 6. http://dx.doi.org/10.1080/17476933.2011.617008.
[14] T.J. Suffridge, Harmonic univalent polynomials, Complex Variables Theory Appl. 35 (1998) 93–107.
[15] J.W. Thompson, A family of meromorphic harmonic mappings, Complex Variables Theory Appl. 48 (2003) 627–648.
[16] M. Chuaqui, P. Duren, B. Osgood, Curvature properties of planar harmonic mappings, Comput. Methods Funct. Theory 4 (1) (2004) 127–142.
[17] D. Kalaj, S. Ponnusamy, M. Vuorinen, Radius of close-to-convexity of harmonic functions, Complex Var. Elliptic Equ. (revised).
[18] P.T. Mocanu, Starlikeness and convexity for nonanalytic functions in the unit disc, Mathematica (Cluj) 22 (45) (1980) 77–83.
[19] V.I. Gavrilov, Remarks on the radius of univalence of holomorphic functions, Mat. Zametki 7 (1970) 295–298.
[20] S. Yamashita, Radii of univalence, starlikeness, and convexity, Bull. Austral. Math. Soc. 25 (3) (1982) 453–457.
[21] T. Sheil-Small, Complex Polynomials, Cambridge University Press, NewYork, 2002.
[22] P. Duren, W. Hengartner, R.S. Laugesen, The argument principle for harmonic mappings, Amer. Math. Monthly 103 (1996) 411–415.
[23] J. Qiao, X. Wang, Univalent biharmonic mappings, Mathematica Acta Scientia (in Chinese) (in press).