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On the Lower Order of Locally Univalent Functions

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Dedicated to Walter Hayman FRS on his eightieth birthday

Abstract. Let f be analytic and $f'(z) \neq 0$ in $\mathbb{D}$ and let

$$A_f(z) = \frac{1-|z|^2}{2}\frac{f''(z)}{f'(z)} - \overline{z} \qquad \text{for } z \in \mathbb{D}.$$

Many properties of f can be described by the (linear-invariant) order

$$\sup_{z\in\mathbb{D}} |A_f(z)|.$$

The work of Avkhadiev and Wirths led to the introduction of the lower order of f defined by $\inf_{z\in\mathbb{D}} |A_f(z)|$. It is perhaps a surprise that there are many (necessarily unbounded) functions of positive lower order. This paper studies some properties of these functions.

Keywords. Locally univalent, lower order, linear-invariant, Poincaré metric, trajectory.

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1. Introduction

Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$. We suppose that the function f is analytic and locally univalent in $\mathbb{D}$, that is $f'(z) \neq 0$ for $z \in \mathbb{D}$. The quantity

$$A_f(z) := \frac{1-|z|^2}{2}\,\frac{f''(z)}{f'(z)} - \overline{z}, \qquad z \in \mathbb{D}, \tag{1}$$

plays an important role in geometric function theory. It appears in the Koebe transform [10, p. 32]

$$\frac{f\left(\dfrac{z+z_0}{1+\overline{z_0}z}\right) - f(z_0)}{(1-|z_0|^2)\, f'(z_0)} = z + A_f(z_0)z^2 + \cdots, \qquad z \in \mathbb{D}, \tag{2}$$

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and in the logarithmic derivative of the Poincaré metric

$$\frac{\partial}{\partial z}\log\left|(1-|z|^2)f'(z)\right| = \frac{1}{1-|z|^2}\,\mathrm{Re}\,A_f(z). \tag{3}$$

If φ is analytic and locally univalent in $\mathbb{D}$ and satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$, then

$$A_{f\circ\varphi}(z) = \frac{(1-|z|^2)\varphi'(z)}{1-|\varphi(z)|^2}\left(A_f(\varphi(z)) + \overline{\varphi(z)}\right) + A_\varphi(z). \tag{4}$$

If φ is a Möbius transformation of $\mathbb{D}$ onto $\mathbb{D}$ then

$$A_{f\circ\varphi}(z) = \frac{\varphi'(z)}{|\varphi'(z)|}\,A_f(\varphi(z)) \qquad \text{for } z \in \mathbb{D}. \tag{5}$$

The classical theory of univalent functions [13, § 1.2] [10, § 2.3] led to the introduction [17] of the *order*

$$\alpha(f) := \sup\{|A_f(z)|\colon z \in \mathbb{D}\}. \tag{6}$$

The *lower order* is defined [9] by

$$\mu(f) := \inf\{|A_f(z)|\colon z \in \mathbb{D}\}. \tag{7}$$

Since $A_{af+b} = A_f$ for $a, b \in \mathbb{C}$, $a \neq 0$ we have

$$\mu(f\circ\varphi) = \mu(f), \qquad \mu(af+b) = \mu(f) \tag{8}$$

if φ is a Möbius transformation of $\mathbb{D}$ onto $\mathbb{D}$; see (5).

The name of order (of growth) is used in many contexts, e.g. [12, pp. 16–18] for meromorphic functions and [13, p. 42] for univalent functions. For the sake of clarity, the present concepts might be called the *linear-invariance order* and the *lower linear-invariance order* because of the invariance properties (8).

The concept of the lower order has its root in the work of Avkhadiev and Wirths [3, 4]. An unbounded univalent function is called *concave* if $\mathbb{C} \setminus f(\mathbb{D})$ is convex. They showed that there are lower bounds for the coefficients a_n of a concave function, for instance $|a_n| \geq |a_1|$; see e.g. [2].

Since an analytic function cannot tend uniformly to ∞ on an arc of $\mathbb{T}$, it follows from (1) that

$$0 \leq \mu(f) \leq 1 \leq \alpha(f) \leq \infty. \tag{9}$$

We have $\alpha(f) = 1$ exactly if f is a convex univalent function [17, Folg. 2.4], and if f is univalent then $\mu(f) = 1$ holds exactly if f is concave [9].

There are many functions with positive lower order that are not univalent in $\mathbb{D}$, see Example 3.1. In Sections 3 and 4, we consider two classical conditions that are sufficient for univalence and positive lower order.

2. Level lines and the target point

First we consider the level lines of the density of the Poincaré metric, the sets

$$C(t) = \{z \in \mathbb{D}\colon (1-|z|^2)|f'(z)| = t\} \qquad \text{for } 0 < t < \infty. \tag{10}$$

The proofs will be postponed to Section 6. These proofs will use the trajectories $A_f(z)dz > 0$ studied in Section 5. The lower order $\mu(f)$ was defined in (7).

Theorem 2.1. *If $\mu(f) > 0$ then $C(t)$ is an open Jordan arc for every t and there exists a unique point $\zeta_\infty \in \mathbb{T}$ such that*

$$\sup\{|z-\zeta_\infty|\colon z \in C(t)\} \to 0 \qquad \text{as } t \to \infty. \tag{11}$$

In particular it follows that f is unbounded if $\mu(f) > 0$. We call ζ_∞ the *target point* of f. We shall show that all trajectories end at ζ_∞; see (40). By (8) we can achieve the normalization

$$f(0) = 0, \qquad f'(0) = 1, \qquad \zeta_\infty = 1 \tag{12}$$

without changing the lower order or other essential properties of f.

Theorem 2.2. *Let $\mu(f) > 0$. Then f has a continuous extension to $\overline{\mathbb{D}} \setminus \{\zeta_\infty\}$. If $\zeta \in \mathbb{T}$ and $|\zeta - \zeta_\infty| \geq \delta > 0$ then*

$$|f'(r\zeta)| \leq c(\delta)(1-r)^{\mu(f)-1} \qquad \text{for } 0 \leq r < 1. \tag{13}$$

It follows from (13) that f is locally Hölder-continuous in $\overline{\mathbb{D}} \setminus \{\zeta_\infty\}$; see e.g. [18, p. 47]. Next we study the form of the level lines defined in (10).

Theorem 2.3. *If $\mu(f) > 0$ then*

$$\overline{C(t)} = C(t) \cup \{\zeta_\infty\} \qquad \text{for } 0 < t < \infty \tag{14}$$

and $\overline{C(t)}$ is a (closed) Jordan curve.

Now we assume that the function f is univalent in $\mathbb{D}$. It does not follow that f is univalent in $\mathbb{T} \setminus \{\zeta_\infty\}$. Indeed, the unbounded curve

$$\{f(\zeta)\colon \zeta \in \mathbb{T},\, \zeta \neq \zeta_\infty\} \subset \mathbb{C}$$

is a slit for the Koebe function (Example 3.1) and has exactly one double point for the function of Example 3.2.

Theorem 2.4. *Let f be univalent in $\mathbb{D}$ and let $\mu(f) > 0$. Then the Jordan curve $\overline{C(t)} \subset \mathbb{D} \cup \{\zeta_\infty\}$ is tangential to $\mathbb{T}$ at ζ_∞ and*

$$f(z) \to \infty \qquad \text{as } z \to \zeta_\infty,\, z \in \triangle \tag{15}$$

for every Stolz angle $\triangle$ at ζ_∞.

The last theorem gives rise to two questions:

Problem 1. Is $C(t)$ always tangential to $\mathbb{T}$ at ζ_∞ even if f is not univalent?

Problem 2. Does $f(z)$ converge to ∞ as $z \to \zeta_\infty$ unrestrictedly? Then f would be spherically continuous in $\overline{\mathbb{D}}$. Is this at least true if f is univalent in $\mathbb{D}$?

3. Examples of functions with positive lower order

Example 3.1. Let $a \in \mathbb{C}$, $a \neq 0$ and

$$f(z) = \left(\frac{1+z}{1-z}\right)^a = \left(\frac{1+\xi}{1-\xi}\right)^a \left(\frac{1+iy}{1-iy}\right)^a \qquad \text{for } z = \frac{\xi + iy}{1 + i\xi y} \in \mathbb{D},$$

where $-1 < \xi < 1$ and $-1 < y < 1$. We obtain from (1) and from (5) with $\varphi(w) = (\xi + w)/(1 + \xi w)$ that

$$|A_f(z)| = |A_f(iy)| = \left|\frac{a(1-y^2) + 2iy}{1+y^2}\right|.$$

If $\operatorname{Re} a \neq 0$ then the numerator is different from 0 for $-1 \leq y \leq 1$, if $\operatorname{Re} a = 0$ then it has a zero. Hence we have

$$\mu(f) > 0 \quad \Leftrightarrow \quad \operatorname{Re} a \neq 0.$$

In particular for $a \in \mathbb{R}$, we find by computation that

$$\mu(f) = \min(|a|, 1), \;\; \alpha(f) = \max(|a|, 1).$$

The function f is univalent if and only if $|a-1| \leq 1$ or $|a+1| \leq 1$. In this case, if $a \in \mathbb{R}$ then $f(\mathbb{D})$ is a sector and if $a \notin \mathbb{R}$ then $f(\mathbb{D})$ is bounded by logarithmic spirals. ■

Now we prove a simple sufficient condition for a function to be of positive lower order.

Proposition 3.1. *Let f be analytic and locally univalent in $\mathbb{D}$ and let $0 \leq \lambda \leq 1$. If*

$$\left|\frac{f''(z)}{f'(z)} - \frac{2}{1-z}\right| \leq \frac{2\lambda}{1-|z|^2} \qquad \text{for } z \in \mathbb{D} \tag{16}$$

then $\mu(f) \geq 1 - \lambda$.

We can write condition (16) in a different way. If $g(z) := \log[(1-z)^2 f'(z)]$ then condition (16) becomes

$$(1-|z|^2)|g'(z)| \leq 2\lambda \qquad \text{for } z \in \mathbb{D} \tag{17}$$

and this, by definition, means that the Bloch semi-norm [18, p. 72] satisfies $\|g\|_{\mathcal{B}} \leq 2\lambda$.

Proof. We see from (1) that

$$A_f(z) - \frac{1-\overline{z}}{1-z} = \frac{1-|z|^2}{2}\left(\frac{f''(z)}{f'(z)} - \frac{2}{1-z}\right) \tag{18}$$

and since $|1-\overline{z}| = |1-z|$ this implies $|A_f(z)| \geq 1 - \lambda$ by (16).

Example 3.2. We consider the function f defined [5, Satz 3] by

$$f(z) + \log(f(z) - 1) = \frac{1+z}{1-z} \qquad (z \in \mathbb{D}), \tag{19}$$

which maps $\mathbb{D}$ conformally onto an unbounded domain bounded by a curve with exactly one double point in 0. We obtain from (19) that

$$(1-|z|^2)\left|\frac{f''}{f'} - \frac{2}{1-z}\right| = \frac{2(1-|z|^2)}{|1-z|^2|f|^2} = \frac{2}{|f|^2}\operatorname{Re}\frac{1+z}{1-z}.$$

By (19) this expression equals

$$\frac{2\operatorname{Re} f + \log(|f-1|^2)}{|f|^2} \leq \frac{2\operatorname{Re} f + |f-1|^2 - 1}{|f|^2} = 1.$$

Hence it follows from Proposition 3.1 that $\mu(f) \geq 1/2$. ■

Proposition 3.2. *Let* $0 < \lambda \leq 1$ *and*

$$\left|\arg\left[(1-z)^2 f'(z)\right]\right| < \frac{\pi\lambda}{2} \qquad \text{for } z \in \mathbb{D}. \tag{20}$$

Then f *is univalent in* $\mathbb{D}$ *and* $\mu(f) \geq 1 - \lambda$. *If* $\lambda = 1$ *then*

$$A_f(z) \neq 0 \text{ for } z \in \mathbb{D}, \qquad \text{or} \qquad f(\mathbb{D}) \text{ is a parallel strip.} \tag{21}$$

If $\lambda = 1$ then condition (20) can be written as

$$\operatorname{Re}\big((1-z)^2 f'(z)\big) > 0 \qquad \text{for } z \in \mathbb{D}, \tag{22}$$

which holds if and only if [14]

$$w \in f(\mathbb{D}),\ 0 \leq t < \infty \qquad \Rightarrow \qquad w + t \in f(\mathbb{D}).$$

Hence these functions are called convex in the direction of the positive real axis. The condition (22) plays an important role in the theory of analytic semigroups and iteration in $\mathbb{D}$; see e.g. [6]. The alternative (21) will also appear in Proposition 4.1.

Proof. We may assume that $f(0) = 0$ and $f'(0) = 1$. The functions that satisfy (20) also satisfy condition (22) and are therefore close-to-convex [10, p. 46] and thus univalent. Using subordination [10, p. 190], we can write (20) as

$$(1-z)^2 f'(z) = \left(\frac{1+\varphi(z)}{1-\varphi(z)}\right)^\lambda, \qquad \varphi(\mathbb{D}) \subset \mathbb{D}, \qquad \varphi(0) = 0. \tag{23}$$

Since $(1-|z|^2)|\varphi'(z)| \leq 1 - |\varphi(z)|^2$, we see that

$$\frac{1-|z|^2}{2}\left|\frac{f''(z)}{f'(z)} - \frac{2}{1-z}\right| = \lambda(1-|z|^2)\left|\frac{\varphi'(z)}{1-\varphi(z)^2}\right| \leq \lambda. \tag{24}$$

Now let $\lambda = 1$ and suppose that $A(z_0) = 0$ for some $z_0 \in \mathbb{D}$. Then it follows from (18) that equality holds in (24) for $z = z_0$. Hence we have

$$(1-|z_0|^2)|\varphi'(z_0)| = 1 - |\varphi(z_0)|^2$$

so that φ is a Möbius transformation of $\mathbb{D}$ onto $\mathbb{D}$, and since $\varphi(0) = 0$ we conclude that $\varphi(z) \equiv cz$ with $|c| = 1$. Furthermore it follows from the equality in (24) that $\varphi(z_0)^2 = |\varphi(z_0)|^2$ and therefore that $z_0 = \pm\bar{c}r$ with $r = |z_0|$. Hence we obtain from (23) that

$$f'(z) = \frac{1}{(1-z)^2}\frac{1+cz}{1-cz}, \qquad \frac{f''(z)}{f'(z)} - \frac{2}{1-z} = \frac{2c}{1-c^2z^2}. \tag{25}$$

Since $A_f(z_0) = 0$ it thus follows from (18) that $(1 \mp cr)/(1 \mp \bar{c}r) = -\bar{c}$ which implies $c = -1$. Integrating (25) we thus obtain by our normalization that

$$f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$$

and this function maps $\mathbb{D}$ onto $\{|\operatorname{Im} w| < \pi/4\}$. ■

4. The Schwarzian derivative

The Schwarzian derivative of a locally univalent function f is defined by

$$S_f := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2;$$

see e.g. [10, p. 258] where the notation $\{f, z\}$ is used instead of $S_f(z)$. If φ and ψ are Möbius transformations and $\varphi(\mathbb{D}) = \mathbb{D}$ then $S_{\psi\circ f} = S_f$ and

$$\sup_{z\in\mathbb{D}}\left|(1-|z|^2)^2\, S_{f\circ\varphi}(z)\right| = \sup_{z\in\mathbb{D}}(1-|z|^2)^2|S_f(z)|. \tag{26}$$

Proposition 4.1. *Let f be analytic, locally univalent and unbounded in $\mathbb{D}$ and suppose that*

$$\big(1-|z|^2)^2|S_f(z)\big| \le 2\lambda \qquad \textit{for } z \in \mathbb{D}. \tag{27}$$

If $0 \le \lambda < 1$ then $\mu(f) \ge \sqrt{1-\lambda} > 0$, if $\lambda = 1$ then (21) holds.

Condition (27) with $\lambda = 1$ implies [16] that f is univalent in $\mathbb{D}$; see e.g. [11, 7, 8] for results on this Nehari class. If (27) holds with $\lambda < 1$ then f maps $\mathbb{D}$ onto a quasidisk in $\hat{\mathbb{C}}$ as Ahlfors and Weill [1] have shown.

Proof. Let $z_0 \in \mathbb{D}$ and let g be the Koebe transform of f defined in (2), furthermore let $h = g/(1 + A_f(z_0)g)$. Since $g''(0)/2 = A_f(z_0)$ by (2), we see that $h''(0) = 0$. It follows from (26) and (27) that

$$\big|(1-|z|^2)^2|S_h(z)\big| = \big(1-|z|^2)^2|S_g(z)\big| \le 2\lambda \qquad \text{for } z \in \mathbb{D}. \tag{28}$$

First let $0 \le \lambda < 1$. Chuaqui and Osgood [7] have shown that $h''(0) = 0$ and (28) imply $|h(z)| \le 1/\sqrt{1-\lambda}$ for $z \in \mathbb{D}$. Now f is unbounded by assumption. Hence

$$g(z) = \frac{h(z)}{1 - A_f(z_0)h(z)}$$

is also unbounded in view of (2). Hence we have $|A_f(z_0)| \geq \sqrt{1-\lambda}$. It follows that $\mu(f) \geq \sqrt{1-\lambda}$.

Now let $\lambda = 1$ and suppose that $A_f(z_0) = 0$ for some $z_0 \in \mathbb{D}$. Then $g''(0) = 0$ and it follows [11, Thm. 1] that either g is bounded or $g(\mathbb{D})$ is a parallel strip. The first alternative is excluded by our assumption. ∎

Example 4.1. We start with the lacunary series

$$h(z) = \sum_{k=0}^{\infty} z^{2^k}, \qquad z \in \mathbb{D}. \tag{29}$$

Let $c_0, c_1, \ldots$ denote suitable positive constants and let

$$f'(z) = (1-z)^{-2} e^{i\alpha h(z)}, \qquad \alpha > 0. \tag{30}$$

We obtain from (26) that

$$S_f(z) = i\alpha h''(z) + \frac{\alpha^2}{2} h'(z)^2 - \frac{2i\alpha}{1-z} h'(z). \tag{31}$$

Now we use a standard trick of the theory of lacunary series. We obtain from (29) that, for $|z| = r < 1$,

$$\begin{aligned} \frac{|h'(z)|}{1-r} &\leq \frac{1}{1-r} \sum_{k=0}^{\infty} 2^k r^{2^k - 1} = \sum_{n=1}^{\infty} \left(\sum_{2^k \leq n} 2^k \right) r^{n-1} \\ &\leq c_1 \sum_{n=1}^{\infty} n r^{n-1} = \frac{c_1}{(1-r)^2}. \end{aligned}$$

Estimating h and h'' in a similar way, we obtain

$$|h(z)| \leq c_0 \log \frac{3}{1-|z|}, \qquad |h'(z)| \leq \frac{c_1}{1-|z|}, \qquad |h''(z)| \leq \frac{c_2}{(1-|z|)^2}.$$

Hence it follows from (31) that

$$(1-|z|^2)^2 |S_f(z)| \leq 4\alpha c_2 + 2\alpha^2 {c_1}^2 + 8\alpha c_1 < 1 \tag{32}$$

if $\alpha < 1/(4c_0)$ is chosen sufficiently small. Furthermore we see from (30) that $|f'(x)| > c_4(1-x)^{-3/2}$ for $0 < x < 1$. Hence f is unbounded and we conclude from Proposition 4.1 that $\mu(f) > 1/\sqrt{2}$. Moreover it follows [1] from (32) that f maps $\mathbb{D}$ conformally onto the inner domain of a quasicircle through ∞.

The special form (29) of h can be used [18, p. 193] to show that $\operatorname{Re} h(r\zeta)$ does not have a finite limit as $r \to 1$ for any $\zeta \in \mathbb{T}$. Hence it follows from (30) that $\arg f'(r\zeta)$ does not have a finite limit for any $\zeta \neq 1$. Therefore f is not isogonal at any $\zeta \neq 1$ [18, p. 80] so that the Jordan curve $f(\mathbb{T})$ does not have a tangent at any finite point [18, p. 81]. ∎

5. Trajectories

We assume that f is analytic and locally univalent in $\mathbb{D}$ and that

$$A_f(z) \neq 0 \qquad \text{for } z \in \mathbb{D}. \tag{33}$$

This condition is satisfied if $\mu(f) > 0$ but also in some other cases, see Propositions 3.2 and 4.1.

We want to study the solutions of the differential equation $A_f(z)dz > 0$. We choose the parameter t such that

$$A_f(z)\dot{z} = \frac{1-|z|^2}{2t}. \tag{34}$$

Since $A_f(z) \neq 0$, the solutions form smooth Jordan arcs Γ, called the *trajectories* of f. There is precisely one trajectory through each point of $\mathbb{D}$, and every trajectory goes from $\mathbb{T}$ to $\mathbb{T}$. If φ is a Möbius transformation of $\mathbb{D}$ onto $\mathbb{D}$ then the trajectories of $f \circ \varphi$ are the arcs $\varphi(\Gamma)$.

We see from (3) that the trajectories are orthogonal to the level lines of the Poincaré metric defined in (10). It follows from (3) and (34) that

$$\frac{d}{dt}\log\bigl((1-|z|^2)\,|f'(z)|\bigr) = \frac{2A_f(z)\dot{z}}{1-|z|^2} = \frac{1}{t}. \tag{35}$$

Hence we can normalize the parameter such that

$$\bigl(1-|z(t)|^2\bigr)\,|f'(z(t))| = t. \tag{36}$$

Let $\lambda_{\mathbb{D}}(\cdot,\cdot)$ denote the hyperbolic distance in $\mathbb{D}$ corresponding to the hyperbolic metric $|dz|/(1-|z|^2)$.

Proposition 5.1. *Let $\mu = \mu(f) > 0$ and consider a trajectory Γ. If $t_1 < t_2$ and $z_j = z(t_j)$, $j = 1, 2$, then*

$$\frac{t_2}{t_1} = \frac{(1-|z_2|^2)\,|f'(z_2)|}{(1-|z_1|^2)\,|f'(z_1)|} \geq \exp(2\mu\lambda_{\mathbb{D}}(z_1,z_2))\,. \tag{37}$$

Each trajectory Γ is defined for all $t \in (0,\infty)$ under the normalization (36).

Proof. We obtain from (35) and (36) that

$$\log\frac{t_2}{t_1} = \int_{t_1}^{t_2} \frac{2|A_f(z)\dot{z}|}{1-|z|^2}\,dt \geq 2\mu\int_{t_1}^{t_2} \frac{|\dot{z}|}{1-|z|^2}\,dt.$$

The last integral is the hyperbolic length of Γ between z_1 and z_2 and is thus at least $\lambda_{\mathbb{D}}(z_1,z_2)$, and (37) follows.

If we keep $z_1 \in \Gamma$ fixed and let $|z_2| \to 1$, $z_2 \in \Gamma$ (or vice versa), we obtain from (37) that the parameter t runs through the entire range $(0,\infty)$. ∎

6. Proof of the theorems

Proof of Theorem 2.1. (a) Let $0 < t_0 < \infty$. First we show that $C(t_0)$ is the union of open Jordan arcs C_n in $\mathbb{D}$. Let C_n be a connected component of $C(t_0)$ and $w \in C_n$. We parametrize C_n near w by the euclidean arc length s. It follows from (10) or (3) that

$$0 = \frac{d}{ds}\log\bigl((1-|z|^2)|f'(z)|\bigr) = 2\,\mathrm{Re}\left(A_f(z)\frac{dz}{ds}\right).$$

Since $|dz/ds| = 1$ it follows that

$$A_f(z)\frac{dz}{ds} = i|A_f(z)|,$$

and since $A_f(z) \neq 0$ we can continue this local parametrization to a parametrization of whole component C_n. We see that C_n is an open Jordan arc.

(b) Now we show that $C(t_0)$ is connected. Let $w_j \in C(t_0)$ for $j = 1, 2$ and let $w \in [w_1, w_2]$. We consider the solutions $\psi(\tau, w)$ of the differential equation (34) that satisfy the initial condition $\psi(0, w) = w$. The parameter τ is a w-dependent shift of the trajectory parameter t. By Proposition 5.1 there is a unique $\tau(w)$ such that $\psi(\tau(w), w) \in C(t_0)$. Hence $\tau(w)$ is continuous. In particular we have $\tau(w_j) = 0$ and $\psi(0, w_j) = w_j$.

The map $\Phi\colon [w_1, w_2] \to C(t_0)$ defined by $\Phi(w) = \psi(\tau(w), w)$ is continuous so that $\Phi([w_1, w_2])$ is a connected subset of $C(t_0)$ containing the given points w_1 and w_2. It follows that $C(t_0)$ is connected and is thus an open Jordan arc.

(c) Since the level sets $C(t)$ are connected, the sets

$$B(t) = \bigl\{z \in \mathbb{D}\colon (1-|z|^2)|f'(z)| \geq t\bigr\} = \bigcup_{\tau \geq t} C(\tau) \tag{38}$$

are also connected. Furthermore we have $B(t) \neq \emptyset$ for $0 < t < \infty$ because of (36) and Proposition 5.1. It follows that the closed set

$$B = \bigcap_t \overline{B(t)} \subset \partial\mathbb{D} = \mathbb{T} \tag{39}$$

is connected and non-empty, so that B is either a point or an arc of $\mathbb{T}$.

Now suppose that B is a proper arc of $\mathbb{T}$. No analytic function can tend uniformly to ∞ on an arc of $\mathbb{T}$ by the Privalov uniqueness theorem [18, p. 140]. Hence there are $z_n \in \mathbb{D}$ with $z_n \to z_0 \in B$ such that $|f'(z_n)|$ remains bounded. But this contradicts (38) for large t. Hence B is a single point which we call ζ_∞, and (11) follows from (38) and (39). ∎

It follows from (36) and (11) that all trajectories $\Gamma\colon z(t)$, $0 < t < \infty$ satisfy

$$z(t) \to \zeta_\infty \qquad \text{as } t \to \infty. \tag{40}$$

Proof of Theorem 2.2. Let $\delta > 0$. By (11) we can find t_2 depending only on f and δ such that $|z_2 - \zeta_\infty| < \delta/2$ for $z_2 \in C(t_2)$. Now let $|\zeta - \zeta_\infty| \geq \delta$ and $0 < r < 1$. We consider the trajectory Γ through $z_1 = r\zeta$. There exists $z_2 \in \Gamma \cap C(t_2)$ such that

$$t_1 = (1 - r^2)|f'(r\zeta)| < t_2 = (1 - |z_2|^2)|f'(z_2)| \tag{41}$$

and therefore, by Proposition 5.1,

$$\frac{t_2}{t_1} \geq \exp(2\mu\lambda_{\mathbb{D}}(r\zeta, z_2))\,. \tag{42}$$

We have $|\arg\zeta - \arg z_2| > \delta/2$ and therefore [15, p. 342]

$$\lambda_{\mathbb{D}}(r\zeta, z_2) \geq \frac{1}{2}\log\frac{1+r}{1-r} + \frac{1}{2}\log\frac{1+|z_2|}{1-|z_2|} - \log\left(\sin\frac{4}{\delta}\right).$$

Hence we obtain from (41) and (42) that

$$(1 - r^2)|f'(r\zeta)| \leq t_2\left(\frac{1-r}{1+r}\right)^{\mu}\left(\sin\frac{4}{\delta}\right)^{2\mu},$$

which implies (13). By a standard argument [18, p. 47], we deduce from (13) that f has a continuous extension to $\overline{\mathbb{D}} \setminus \{\zeta_\infty\}$, which is even locally $\mu(f)$-Hölder continuous. ■

Proof of Theorem 2.3. For fixed t, let ζ be a point of $\overline{C(t)} \setminus C(t)$. Then $\zeta \in \mathbb{T}$. Suppose that $\zeta \neq \zeta_\infty$. Then it follows from (13) that

$$(1 - |z|^2)|f'(z)| \to 0 \qquad \text{as } z \to \zeta,\ z \in C(t)$$

which is false by (11). Hence we have $\overline{C(t)} \setminus C(t) = \{\zeta_\infty\}$. Since $C(t)$ is a Jordan arc it follows that $\overline{C(t)}$ is a Jordan curve. ■

Proof of Theorem 2.4. (a) Suppose that $\overline{C(t)}$ is not tangential to $\mathbb{T}$ at ζ_∞ for some t. We choose a trajectory $\Gamma\colon w(\tau)$, $0 < \tau < \infty$. There are two cases.

Case 1. There exists $z_n \in C(t)$ with $z_n \to \zeta_\infty$ as $n \to \infty$ that lie in a fixed Stolz angle at ζ_∞. Since f is univalent there exist [18, Cor. 4.18] a constant c_1 and hyperbolic rays $R_n^{\pm}$ that begin at z_n and end at points $\zeta_n^{\pm} \in \mathbb{T}$ on different sides of ζ_∞, such that $\zeta_n^{\pm} \to \zeta_\infty$ as $n \to \infty$ and, for $z \in R_n^{\pm}$,

$$(1 - |z|^2)|f'(z)| < c_1 t. \tag{43}$$

But $R_n^+ \cup R_n^-$ intersects the trajectory Γ at some point $w(\tau_n)$. It follows from (10) and (43) that τ_n remains bounded so that we may assume $\tau_n \to \tau^* < \infty$. Then $w(\tau^*) \in \Gamma \subset \mathbb{D}$ by Proposition 5.1, which contradicts the fact that $w(\tau_n) \in R_n^+ \cup R_n^-$ tends to ζ_∞.

Case 2. The two ends of $C(t)$ are tangential to $\mathbb{T}$ but from the same side Γ of $\mathbb{T} \setminus \{\zeta_\infty, -\zeta_\infty\}$. We choose z_n on the end of $C(t)$ furthest away from T. There exist [18, Cor. 4.18] hyperbolic rays R_n from z_n to some point $\zeta_n \in T$ such that

(43) holds for $z \in R$. Now Γ has to intersect R_n and we obtain a contradiction as in the first case.

(b) Let $r \in (0,1)$ be sufficiently close to 1. Since the Jordan curves $\overline{C(t)}$ are tangential to $\mathbb{T}$ at ζ_∞, there exist $t(r)$ such that $r\,\zeta_\infty \in C(t(r))$. If $r \to 1$ then $t(r) \to \infty$ by Proposition 5.1.

One of the Koebe distortion theorems [10, Th. 2.7] shows that

$$|f(r\zeta_\infty) - f(0)| \geq \frac{1-r}{1+r}|rf'(r\zeta_\infty)| = \frac{rt(r)}{(1+r)^2}.$$

It follows that $|f(r\zeta_\infty)| \to \infty$ as $r \to 1$. Since f is univalent and therefore normal, we conclude [18, Cor. 4.5] that f has the angular limit ∞ at ζ_∞. ■

References

1. L. V. Ahlfors and G. Weill, A uniqueness theorem for Beltrami equations, *Proc. Amer. Math. Soc.* **13** (1962), 975–978.
2. F. G. Avkhadiev, Ch. Pommerenke and K.-J. Wirths, Sharp inequalities for the coefficients of concave schlicht functions, *Comment. Math. Helv.* **81** (2006), 801–807.
3. F. G. Avkhadiev and K.-J. Wirths, Convex holes produce lower bounds for coefficients, *Complex Variables* **47** (2002), 553–563.
4. ———, On a conjecture of Livingston, *Mathematica* **46** (2004), 19–23.
5. J. Becker and Ch. Pommerenke, Schlichtheitskriterien und Jordangebiete, *J. Reine Angew. Math.* **354** (1984), 74–94
6. E. Berkson and H. Porta, Semigroups of analytic functions and composition operators, *Michigan Math. J.* **25** (1978), 101–115.
7. M. Chuaqui and B. Osgood, Sharp distortion theorems associated with the Schwarzian derivative, *J. London Math. Soc. (2)* **48** (1993), 289–298.
8. M. Chuaqui and Ch. Pommerenke, Characteristic properties of Nehari functions, *Pacific J. Math.* **188** (1999), 83–94.
9. L. Cruz and Ch. Pommerenke, On concave univalent functions, to appear in: *Complex Variables and Elliptic Equations* (2007).
10. P. L. Duren, *Univalent Functions*, Springer, New York, 1983.
11. F. W. Gehring and Ch. Pommerenke, On the Nehari univalence criterion and quasicircles, *Comment. Math. Helv.* **59** (1984), 226–242.
12. W. K. Hayman, *Meromorphic Functions*, Oxford University Press, 1975.
13. ———, *Multivalent Functions*, Cambridge University Press, 1994.
14. Z. Lewandowski, Sur l'identité de certaines classes de fonctions univalentes II, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **14** (1960), 19–46.
15. D. Mejía and Ch. Pommerenke, A Möbius-invariant family of conformal maps, *Comput. Methods Funct. Theory* **2** (2002), 337–351.
16. Z. Nehari, The Schwarzian derivative and schlicht functions, *Bull. Amer. Math. Soc.* **55** (1949), 545–551.

17. Ch. Pommerenke, Linear-invariante Familien analytischer Funktionen I, *Math. Ann.* **155** (1964), 108–154.
18. ———, *Boundary Behaviour of Conformal Maps*, Springer, Berlin, 1992.

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