Math. Z. 156, 157–164 (1977)

Mathematische Zeitschrift

On the Green's Fundamental Domain

Christian Pommerenke

Fachbereich Mathematik, Technische Universität Berlin, Straße des 17. Juni 135, D-1000 Berlin 12, Bundesrepublik Deutschland

1. The Green's Fundamental Domain

Let Γ be a Fuchsian group, that is a discontinuous group of Möbius transformations of $D=\{z\in\mathbb{C}: |z|<1\}$ onto itself. We assume throughout that 0 is not an elliptic fixed point and that Γ is of convergence type. Then the Blaschke product [7, 8]

$$g(z)=\prod_{\gamma\in\Gamma}[e^{-i\arg\gamma(0)}\gamma(z)]=g'(0)\,z+\cdots \quad (z\in D) \tag{1.1}$$

exists and is called the *Green's function* of Γ with respect to 0. It has the zeros $\gamma(0)(\gamma\in\Gamma)$ and satisfies

$$g(\gamma(z))\equiv v(\gamma)\,g(z) \quad \text{with} \quad |v(\gamma)|=1 \quad (\gamma\in\Gamma). \tag{1.2}$$

The positive harmonic function $-\log|g(z)|$ is invariant under Γ and thus becomes a function on the Riemann surface D/Γ, namely the usual Green's function with a logarithmic pole at the point corresponding to 0.

A *Green's line* L (with respect to 0) is defined as a maximal half-open arc in D beginning at 0 along which

$$\arg g(z)=\text{const}, \qquad g'(z)\neq 0. \tag{1.3}$$

The Green's line L either ends at a point $\zeta\in D$ with $g'(\zeta)=0$, or it ends at a definite point $\zeta\in\partial D$ as we see from Koebe's theorem on bounded analytic functions [11, p. 267]. If L ends at $\zeta\in\partial D$ then, by Lindelöf's theorem [11, p. 268], the angular limit $g(\zeta)$ exists and $g(z)\to g(\zeta)$ as $z\to\zeta$, $z\in L$. It follows that at most one Green's line ends at a given point of ∂D.

The union G of all Green's lines is a simply connected domain with $0\in G\subset D$. Since 0 is a simple zero it is easy to show that $g(z)$ is univalent in G. The image domain $g(G)\subset D$ is starlike because each Green's line is, by (1.3), mapped onto the radial segment $[0, g(\zeta))$.

Let D^* be obtained from D by deleting the countably many elliptic fixed points. Then D^*/Γ is a Riemann surface with the Green's function $-\log|g|$ and the projections of the Green's lines defined above become the Green's lines introduced by Brelot and Choquet [3]. They proved [3, p. 239] that

$$\operatorname{area}(D \setminus g(G)) = 0; \tag{1.4}$$

see also Sario-Nakai [13, p. 202] and Arsove-Johnson [1].

Theorem 1. *The union G of all Green's lines is a fundamental domain of Γ with area $\partial G = 0$.*

This domain will be called the *Green's fundamental domain* of Γ (with respect to 0). I wish to thank Professor A. Marden for our discussions about it.

Proof. We show first that the domains $\gamma(G)(\gamma \in \Gamma)$ are disjoint. Let $\gamma_1(z_1) = \gamma_2(z_2)$ for some $\gamma_1, \gamma_2 \in \Gamma$ and $z_1, z_2 \in G$. Then $z_j (j=1, 2)$ lies on some Green's line L_j. The arcs L_1 and $L'_1 = \gamma_1^{-1} \circ \gamma_2(L_2)$ intersect at z_1. Now $g'(z) \neq 0$ and $\arg g(z) = \text{const}$ on L_1 and on L'_1, by (1.3) and (1.2). Hence we conclude that $L_1 = L'_1$ and therefore that $\gamma_1 = \gamma_2$ and $z_1 = z_2$.

To complete the proof we show now that

$$\operatorname{area}(D \setminus \bigcup_{\gamma \in \Gamma} \gamma(G)) = 0. \tag{1.5}$$

It follows from (1.1) that $\log|g(\zeta)| = \sum \log|\gamma(\zeta)|$ and from (1.2) that

$$|g'(\gamma(\zeta))\,\gamma'(\zeta)| = |g'(\zeta)| \quad \text{for } \gamma \in \Gamma.$$

Hence (with $d_z\Omega \equiv dx\,dy$)

$$\sum_{\gamma \in \Gamma} \iint_{\gamma(G)} |g'(z)|^2 \log\frac{1}{|z|}\, d_z\Omega$$
$$= \iint_G |g'(\zeta)|^2 \log\frac{1}{|g(\zeta)|}\, d_\zeta\Omega = \iint_{g(G)} \log\frac{1}{|w|}\, d_w\Omega = \frac{\pi}{2} \tag{1.6}$$

because $g(\zeta)$ is univalent in G and because $g(G) = D \setminus (\text{set of zero area})$ by (1.4). Writing $g(z) = b_1 z + b_2 z + \cdots$ we obtain from Parseval's formula that, on the other hand,

$$\iint_D |g'(z)|^2 \log\frac{1}{|z|}\, d_z\Omega = 2\pi \int_0^1 \sum_{n=1}^{\infty} n^2 |b_n|^2 r^{2n-1} \log\frac{1}{r}\, dr = \frac{\pi}{2} \sum_{n=1}^{\infty} |b_n|^2 \leqq \frac{\pi}{2}$$

because $|g(z)| < 1$. The assertion (1.5) therefore follows from (1.6).

We show now by an example that the Green's fundamental domain G need not be a Jordan domain. This is in contrast to the normal fundamental domain which is always bounded by a Jordan curve. It easily follows from [9, Th. 1] that this Jordan curve is rectifiable.

Example. Let Γ be the Fuchsian group obtained by uniformizing the plane domain

$$H = D \setminus \left\{ \frac{1}{2} + \frac{i}{n} : n = 2, 3, \ldots \right\}.$$

Since cap $(D \setminus H)=0$ it is easy to see that $g(G)$ is obtained from H by deleting the radial segments S_n from $\frac{1}{2}+\frac{i}{n}$ from the unit circle. Hence $\{g(z): |z| \leqq r\}$ intersects all segments S_n if r is sufficiently close to 1. It follows that $g^{-1}(S_n) \cap \partial G$ intersects $\{|z| \leqq r\}$ for each n so that G cannot be a Jordan domain.

2. The Capacity of the Boundary

The Green's line L is called *regular* if its endpoint ζ lies on ∂D and if $|g(\zeta)|=1$. Let B denote the set of endpoints of regular Green's lines. The above remarks on Green's lines show that the angular limits $g(\zeta)(\zeta \in B)$ give a strictly increasing mapping of B into ∂D. Since $g(G)$ is a starlike domain, $g(B)$ is a Borel set on ∂D. Hence its pre-image B is also a Borel set on ∂D. It follows from (1.4) that

$$\operatorname{mes} g(B)=2\pi. \tag{2.1}$$

Theorem 2. *The set B of endpoints of regular Green's lines contains no Γ-equivalent points and satisfies*

$$\operatorname{cap} B \geqq \sqrt{g'(0)}>0. \tag{2.2}$$

It is not possible in general to replace (2.2) by the stronger relation $\operatorname{mes} B>0$. Indeed, for the group Γ constructed in [12, Example 1] there does not exist any set of positive measure that contains no Γ-equivalent points. We mention that there are non-constant bounded automorphic functions for this group.

The proof of (2.2) is based on two capacity distortion estimates for univalent functions.

Lemma 1 [11, p. 348]. *Let $f(s)=as+\cdots$ be univalent in D and let $|f(s)|<1$. If the angular limit exists on the Borel set $P \subset \partial D$ and if $f(P)$ is a Borel set on ∂D then*

$$\operatorname{cap} f(P) \geqq (\operatorname{cap} P)/\sqrt{|a|}. \tag{2.3}$$

The next result is somewhat stronger than what we need.

Lemma 2. *Let the univalent function $f(s)=as+\cdots$ map D onto a starlike subdomain of D and let*

$$\tau(\zeta)=\lim_{r \to 1-0} \arg f(r\zeta) \qquad (\zeta \in \partial D). \tag{2.4}$$

If B is a Borel set on $[0, 2\pi]$ with $\operatorname{mes} B=2\pi\lambda\,(0<\lambda \leqq 1)$ then its pre-image $P=\{\zeta \in \partial D\colon \tau(\zeta) \in B\}$ satisfies

$$\operatorname{cap} P \geqq 2^{-(1-\lambda)/\lambda}\,|a|^{1/(2\lambda)}. \tag{2.5}$$

Proof. The limit (2.4) exists and defines an increasing continuous function. We may assume that B is closed; otherwise we exhaust B by closed subsets. Then P is also closed. We can write (see, for instance, [10])

$$f(s)=as\exp\left(-\frac{1}{\pi}\int_{\partial D}\log(1-\bar{\zeta}s)\,d\tau(\zeta)\right)\qquad(s\in D).$$

Since $|1-\bar{\zeta}q|<2$ and $|f(s)|<1$ we conclude that, for $s\in D$,

$$\frac{1}{\pi}\int_P \log|1-\bar{\zeta}s|\,d\tau(\zeta)+2(1-\lambda)\log 2\geqq\log\left|\frac{as}{f(s)}\right|>\log|as|. \tag{2.6}$$

Because $d\tau(\zeta)\geqq 0$ and

$$\int_P d\tau(\zeta)=\operatorname{mes}B=2\pi\lambda,$$

it follows from the definition of the capacity and from (2.6) that

$$\begin{aligned}\log\operatorname{cap}P&\geqq(2\pi\lambda)^{-2}\int_P\int_P\log|\zeta-s|\,d\tau(\zeta)\,d\tau(s)\\&\geqq(2\lambda)^{-1}(\log|a|-2(1-\lambda)\log 2).\end{aligned}$$

Proof of Theorem 2. (a) Suppose that B contains two points ζ, ζ' with $\zeta'=\gamma(\zeta)$ and $\gamma(z)\not\equiv z$. They are endpoints of the regular Green's lines L, L' respectively, and the arcs $L'\subset G$ and $\gamma(L)\subset\gamma(G)$ both end at ζ'. If C is a Jordan arc in D that connects 0 and $\gamma(0)$ without otherwise meeting L' and $\gamma(L)$ then $L'\cup\gamma(L)\cup C\cup\{\zeta'\}$ is a Jordan curve whose inner domain H lies in D.

It follows from Lindelöf's theorem on bounded analytic functions that $g(z)\to g(\zeta)$ as $z\to\zeta$, $z\in H$. With $\rho=\sup_{z\in C}|g(z)|<1$, we conclude that $\partial g(H)\cap\{|w|>\rho\}$ lies on the line segment $[0,g(\zeta)]$, and this is impossible because the bounded domain $g(H)$ has points in $\{|w|>\rho\}$.

(b) Let the univalent function $f(s)=as+\cdots$ map D onto the simply connected domain G. Since $g(z)=bz+\cdots$ maps G one-to-one onto a starlike domain in D, the function

$$h(s)=g(f(s))=abs+\cdots\qquad(|s|<1) \tag{2.7}$$

is starlike and satisfies $|h(s)|<1$. We set

$$A=\{\sigma\in\partial D:\text{angular limit}\,f(\sigma)\in B\}. \tag{2.8}$$

Using Lindelöf's theorem we see that $h(A)=g(f(A))=g(B)$. Hence we conclude from (2.1) and from Lemma 2 (with $\lambda=1$) that

$$\operatorname{cap}A\geqq|h'(0)|^{\frac{1}{2}}=|ab|^{\frac{1}{2}} \tag{2.9}$$

and the assertion (2.2) follows because $\operatorname{cap}B\geqq|a|^{-\frac{1}{2}}\operatorname{cap}A$ by Lemma 1.

Remark. We have actually proved in (2.9) that $\operatorname{cap}A>0$, which is stronger than $\operatorname{cap}B>0$ because of Lemma 1. Using the Evans-Selberg function and a theorem of

Beurling [2] we can prove, as in the book of Sario-Nakai [13, p. 350], the following statement in the opposite direction:

Let Γ be a Fuchsian group of divergence type with no elliptic fixed points. If the univalent function $f(s)$ maps D onto the normal fundamental domain of Γ then $\operatorname{cap}\{s\in\partial D: |f(s)|=1\}=0$.

We do not know whether there is a converse to Theorem 2: If there exists a Borel set on ∂D of positive capacity that contains no Γ-equivalent points, does it follow that Γ is of convergence type?

3. The Green's Measure and the Angular Derivative

We consider again the one-to-one mapping $g(\zeta)$ from the Borel set $B\subset\partial D$ into ∂D and we define a measure on ∂D by

$$\mu(E)=\frac{1}{2\pi}\operatorname{mes} g(E\cap B) \qquad (E \text{ Borel set on } \partial D). \tag{3.1}$$

It follows from (2.1) that $\mu(\partial D)=1$. Hence μ is a probability measure which we call the *Green's measure* (with respect to 0) associated with Γ. Its analogue on Riemann surfaces has been extensively studied; see for instance [13, p. 205].

For $\zeta\in\partial D$, let $g'(\zeta)$ denote the *angular derivative*, that is the (finite) angular limit of $g'(z)$ at ζ. We say that $g'(\zeta)$ exists if this angular limit exists and if $g'(\zeta)\neq\infty$; this implies the existence of the angular limit $g(\zeta)$.

Theorem 3. *Let*

$$A=\{\zeta\in\partial D: |g(\zeta)|=1,\ g'(\zeta) \text{ exists}\}. \tag{3.2}$$

Then the symmetric difference satisfies

$$\operatorname{mes}(A\,\triangle \bigcup_{\gamma\in\Gamma}\gamma(B))=0. \tag{3.3}$$

The Green's measure can be decomposed as $\mu=\mu_0+\mu_1$ *where*

$$\mu_1(E)=\mu(E\cap A)=\frac{1}{2\pi}\int_{E\cap A\cap B}|g'(z)|\,|dz| \tag{3.4}$$

is absolutely continuous and where

$$\mu_0(E)=\mu(E\setminus A) \tag{3.5}$$

is singular. Furthermore $\mu_1(\partial D)=(2\pi)^{-1}\operatorname{mes} A$.

In a recent paper [5], Heins has shown that, for all analytic functions mapping D into itself, the existence of the angular derivative is closely connected with the injectivity on ∂D. His Theorem 7.1 is related to (3.3). We need his generalization of Löwner's lemma [5, formula (5.5)]:

Lemma 3. *Let $f(z)$ be analytic in D and $|f(z)|<1$. Let the angular derivative exist on the measurable set $E\subset\partial D$. If the angular limit function $f(\zeta)$ maps E injectively into ∂D then*

$$\operatorname{mes} f(E)=\int_E |f'(\zeta)|\,|d\zeta|.$$

Proof of Theorem 3. (a) We show first that

$$A_1=A\setminus\bigcup_{\gamma\in\Gamma}\gamma(B).$$

has measure zero. Let $\zeta\in A_1$. It follows from $\zeta\in A$ that $\zeta g'(\zeta)/g(\zeta)$ is real and positive. Hence $g(z)$ is univalent in some Stolz angle at ζ and there exists a curve $C(\zeta)$ ending at ζ orthogonal to ∂D that is mapped by $g(z)$ onto a radial segment.

Since $\zeta\notin\gamma(B)$ for $\gamma\in\Gamma$, this curve $C(\zeta)$ does not intersect the γ-image of a regular Green's line. Since $|g(\zeta)|=1$ it follows that $C(\zeta)$ is disjoint from $\bigcup\gamma(G)$. Hence we easily conclude from Theorem 1 that $\operatorname{mes} A_1=0$.

(b) In his paper on Blaschke products Frostman [4] has shown that

$$|g'(\zeta)|=\sum_{\gamma\in\Gamma}|\gamma'(\zeta)|<\infty \qquad \text{for } \zeta\in A \tag{3.6}$$

whereas this sum is infinite for $\zeta\in\partial D\setminus A$.

The sets $\gamma(B)(\gamma\in\Gamma)$ are disjoint by Theorem 2. Hence

$$\int_{B\setminus A}\sum_{\gamma\in\Gamma}|\gamma'(\zeta)|\,|d\zeta|=\sum_\gamma\int_{\gamma(B\setminus A)}|dz|\leqq\sum_\gamma\int_{\gamma(B)}|dz|<\infty$$

so that $\operatorname{mes}(B\setminus A)=0$ by (3.2) and Frostman's theorem. Therefore $A_2=\bigcup\gamma(B)\setminus A$ has also zero measure, and the assertion (3.3) follows.

(c) We can write

$$\mu(E)=\mu(E\cap A)+\mu(E\setminus A) \qquad (E \text{ Borel set in } \partial D).$$

Then (3.4) holds because, by (3.1) and Lemma 3,

$$\mu(E\cap A)=\frac{1}{2\pi}\operatorname{mes} g(E\cap A\cap B)=\frac{1}{2\pi}\int_{E\cap A\cap B}|g'(z)|\,|dz|.$$

Hence the measure μ_1 is absolutely continuous as an integral. It follows from (3.6) and (3.3) that

$$2\pi\mu_1(\partial D)=\int_{A\cap B}|g'(\zeta)|\,|d\zeta|=\sum_{\gamma\in\Gamma}\int_{A\cap B}|\gamma'(\zeta)|\,|d\zeta|$$

$$=\sum_\gamma\operatorname{mes}\gamma(A\cap B)=\operatorname{mes} A$$

because the sets $\gamma(B)$ are disjoint. Finally

$$\mu_0(\partial D)=\mu(\partial D\setminus A)=\frac{1}{2\pi}\operatorname{mes} g(B\setminus A)$$

$$=\mu(B\setminus A)=\mu_0(B\setminus A).$$

Since $\operatorname{mes}(B\setminus A)=0$ by (3.3), we conclude that μ_0 is singular.

The simply connected domain $H \subset D$ is called *tangential to D at ω* if H contains some small Stolz angle of vertex ω and opening $\pi-\delta$ for every $\delta>0$. This holds, in particular, if the univalent function mapping D onto H has a (finite) angular derivative at the corresponding point.

Theorem 4. *If T is the set of $\omega \in \partial D$ where $g(G)$ is tangential to D and if A is defined by* (3.2), *then*

$$\operatorname{mes} T = \operatorname{mes} A.$$

We need the following special case of a general result of McMillan [6, Th. 2] on conformal mapping.

Lemma 4. *Let the univalent function $f(s)$ map D into D. Let P be a measurable set on ∂D such that, for $\sigma \in P$, the angular limit satisfies $|f(\sigma)|=1$ and $f(D)$ is tangential to D at $f(\sigma)$. If*

$$Q=\{\sigma \in P : f'(\sigma) \text{ exists}\} \tag{3.7}$$

then $\operatorname{mes} f(Q) = \operatorname{mes} f(P)$.

Proof of Theorem 4. We consider again the univalent function $f(s)=as+\cdots$ that maps D onto G. With $P=\{\sigma \in \partial D:\ f(\sigma) \in A \cap B\}$ and with Q defined by (3.7), we obtain from Lemma 4 that

$$\operatorname{mes}(A \cap B) = \operatorname{mes} f(P) = \operatorname{mes} f(Q). \tag{3.8}$$

The starlike function $h(s)=g(f(s))$ maps D onto $g(G)$. Let $P^*=\{\sigma \in \partial D:\ h(\sigma) \in T\}$. We show now that

$$Q=\{\sigma \in P^*:\ h'(\sigma) \text{ exists}\}. \tag{3.9}$$

Let $\sigma \in Q$. Then $f'(\sigma)$ exists and $\zeta = f(\sigma) \in A \cap B$. Hence $g'(\zeta)$ exists, and it easily follows that the angular derivative $h'(\sigma)=g'(\zeta) f'(\sigma)$ exists. Since G is tangential to D at ζ this implies that $h(\sigma) \in T$, so that $\sigma \in P^*$.

Conversely, we assume that $\sigma \in P^*$ and that $h'(\sigma)$ exists. We deduce [11, p. 308] that $f'(\sigma)$ and $g'(\zeta)$ exist where $\zeta = f(\sigma)$. Hence $\zeta \in A \cap B$ and thus $\sigma \in Q$. This proves (3.9).

We obtain from (3.9), Lemma 4 and Lemma 3 that

$$\operatorname{mes} T = \operatorname{mes} h(P^*) = \operatorname{mes} h(Q) = \int_Q |h'(\sigma)|\,|d\sigma|.$$

Since $\zeta = f(\sigma)$ is one-to-one and absolutely continuous on Q [11, p. 328] we see from (3.8) that

$$\operatorname{mes} T = \int_{f(Q)} |g'(\zeta)|\,|d\zeta| = \int_{A \cap B} |g'(\zeta)|\,|d\zeta|.$$

Because the sets $\gamma(A \cap B) = A \cap \gamma(B)\,(\gamma \in \Gamma)$ are disjoint it follows by (3.6) that

$$\operatorname{mes} T = \sum_{\gamma} \int_{A \cap B} |\gamma'(\zeta)|\,|d\zeta| = \operatorname{mes}\Big(A \cap \bigcup_{\gamma \in \Gamma} \gamma(B)\Big)$$

which is $= \operatorname{mes} A$ by Theorem 3.

These theorems can be used to give further characterizations of some concepts introduced in [12]. The Fuchsian group Γ is called *of accessible type* if $\operatorname{mes} A > 0$ where A is defined by (3.2). This holds if and only if $\operatorname{mes}(\partial F \cap \partial D) > 0$ where F is the normal fundamental domain. The group is called *of fully accessible type* if $\operatorname{mes} A = 2\pi$.

Corollary. *With the above notations,*

$$\Gamma \text{ of fully accessible type} \Leftrightarrow \mu \text{ absolutely continuous}$$

$$\Leftrightarrow \operatorname{mes} T = 2\pi;$$

$$\Gamma \text{ not of accessible type} \Leftrightarrow \mu \text{ singular}$$

$$\Leftrightarrow \operatorname{mes} T = 0 \Leftrightarrow \operatorname{mes} B = 0.$$

This corollary follows immediately from Theorems 3 and 4; the last equivalence is a consequence of (3.3).

References

1. Arsove, M.G., Johnson Jr., G.: A conformal mapping technique for infinitely connected regions. Mem. Amer. math. Soc. **91** (1970)
2. Beurling, A.: Études sur un problème de majoration. Thèse, Uppsala, 1935
3. Brelot, M., Choquet, G.: Espaces et lignes de Green. Ann. Inst. Fourier **3**, 199–263 (1951)
4. Frostman, O.: Sur les produits de Blaschke. Fysiogr. Sällsk. Lund Förhl. **12**, 169–182 (1942)
5. Heins, M.: On the finite angular derivatives of an analytic function mapping the open unit disk into itself. Preprint, Univ. of Maryland, 1976; J. London math. Soc., to appear
6. McMillan, J.E.: Boundary behavior of a conformal mapping. Acta math. **123**, 43–67 (1969)
7. Myrberg, P.J.: Über die Existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Fläche. Acta math. **61**, 39–79 (1933)
8. Poincaré, H.: Sur l'uniformisation des fonctions analytiques. Acta math. **31**, 1–63 (1908) = Oeuvres, Tome IV pp. 70–139. Paris: Gauthier-Villars 1950
9. Pommerenke, Ch.: On some metric properties of polynomials with real zeros II. Michigan math. J. **8**, 49–54 (1961)
10. Pommerenke, Ch.: On starlike and convex functions. J. London math. Soc. **37**, 209–224 (1962)
11. Pommerenke, Ch.: Univalent functions. Göttingen: Vandenhoeck & Ruprecht, 1975
12. Pommerenke, Ch.: On the Green's function of Fuchsian groups. Ann. Acad. Sci. Fennicae, Ser. A I Math. **2**, 409–427 (1976)
13. Sario, L., Nakai, M.: Classification theory of Riemann surfaces. Berlin-Heidelberg-New York: Springer 1970

Received January 19, 1977