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Polynomially bounded solutions to the Loewner differential equation in several complex variables

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ABSTRACT

We determine the form of polynomially bounded solutions to the Loewner differential equation that is satisfied by univalent subordination chains of the form $f(z,t)=e^{tA}z+\cdots$, where $A\in L(\mathbb{C}^n,\mathbb{C}^n)$ has the property $m(A)>0$. Here $m(A)=\min\{\Re\langle A(z),z\rangle\colon \|z\|=1\}$. We also give sufficient conditions for $g(z,t)=L(f(z,t))$ to be polynomially bounded, where $f(z,t)$ is an A-normalized polynomially bounded Loewner chain solution to the Loewner differential equation.

1. Introduction

Becker ([2,3]; cf. [4]) investigated the general form of solutions to the Loewner differential equation in one complex variable,

$$\frac{\partial f}{\partial t}(z,t)=zf'(z,t)p(z,t)\quad \text{a.e. } t\geqslant 0,\ \forall z\in U, \tag{1.1}$$

where $p(\cdot,t)\in\mathcal{P}$ (the well-known Carathéodory class of holomorphic functions q on U with $q(0)=1$ and $\Re q(z)>0$ for $z\in U$) for any fixed $t\in[0,\infty)$, and $p(z,\cdot)$ is measurable on $[0,\infty)$ for $z\in U$. In one complex variable there exists a unique univalent solution $f(z,t)=e^tz+\cdots$ of (1.1) (called the canonical solution). He also proved that any other solution $g(z,t)$ of (1.1) that is holomorphic on U and locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z\in U$, has the form $g(z,t)=\Phi(f(z,t))$, where $f(z,t)$ is the canonical solution and Φ is an entire function (compare also [27]). In particular, if $g(\cdot,t)$ is univalent on U and $g(0,t)=g'(0,t)-e^t=0$ for $t\geqslant 0$, then $g(z,t)\equiv f(z,t)$.

In recent years, the general form of solutions to the Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t)=Df(z,t)h(z,t)\quad \text{a.e. } t\geqslant 0,\ \forall z\in B^n, \tag{1.2}$$

which have the normalization $h(z,t)=Az+\cdots$, where $A\in L(\mathbb{C}^n,\mathbb{C}^n)$ and $m(A)>0$ has been studied. The case in which $A=I_n$ was considered by Graham, Kohr, and Pfaltzgraff [17] and the case in which $k_+(A)<2m(A)$ was considered by Duren, Graham, Hamada, and Kohr [9]. One of the results in [9] is as follows. Any bounded solution $g(z,t)$ to the Loewner differential equation (1.2) has the form $g(z,t)=L(f(z,t))$, where $L\in L(\mathbb{C}^n,\mathbb{C}^n)$ and $f(z,t)$ is the unique A-normalized bounded solution to the Loewner differential equation (1.2). The proof of the above result requires a generalization to higher dimensions of the well-known Carathéodory kernel convergence result for univalent functions [9,23].

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On the other hand, Voda [36] finds an A-normalized polynomially bounded solution to the Loewner differential equation (1.2) for any $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $m(A) > 0$. In this paper, we determine the form of arbitrary polynomially bounded univalent solutions $g(z,t)$ to the Loewner differential equation (1.2) for any $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $m(A) > 0$. The proof is elementary and we do not need the Carathéodory kernel convergence result for univalent mappings. We also give sufficient conditions for $g(z,t) = L(f(z,t))$ to be polynomially bounded, where $f(z,t)$ is an A-normalized polynomially bounded Loewner chain solution to the Loewner differential equation (1.2).

Any solution $f(z,t)$ to the Loewner differential equation (1.2) that is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B^n$ is a subordination chain (see Proposition 3.1). In one complex variable, if $a : [0,\infty) \to \mathbb{C}$ is a function of class C^1 such that $a(t) \neq 0$ for $t \geqslant 0$ and $|a(\cdot)|$ is strictly increasing on $[0,\infty)$, and if $f(z,t) = a(t)z + \cdots$ is a non-normalized univalent subordination chain on the unit disc U, then $f_*(z,t^*) = f(e^{-i\theta(t)}z,t)/a(0)$ is a normalized univalent subordination chain, where $t^* = \log(|a(t)/a(0)|)$ and $\theta(t) = \arg(a(t)/a(0))$. That is, there exists a normalized univalent subordination chain with essentially the same geometric properties as the original one. However, the situation is different in dimension $n \geqslant 2$. There exist non-normalized subordination chains $f(z,t) = e^{tA}z + \cdots$ which cannot be normalized by an analogous change of variable. On the other hand, there exist biholomorphic mappings f which have useful embeddings in non-normalized subordination chains. For example, if $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ is such that $m(A) > 0$ and if f is a spirallike mapping with respect to A, then $f(z,t) = e^{tA}f(z)$ is a univalent subordination chain. In connection with this observation, Graham, Hamada, Kohr, and Kohr [13] introduced the class $S_A^0(B^n)$ of mappings which have A-parametric representation, i.e. the subclass of $S(B^n)$ which consists of those mappings f that can be embedded in univalent subordination chains $f(z,t) = e^{tA}z + \cdots$ such that $\{e^{-tA}f(\cdot,t)\}_{t\geqslant 0}$ is a normal family on B^n. They proved that certain results which hold for the class S (embedding results, compactness) can be generalized to the above classes. It is therefore of interest and important to consider subordination chains which do not have the standard normalization in the study of univalent mappings on B^n for $n \geqslant 2$.

For several results on subordination chains in several complex variables, the readers may consult [1,6,10–16,19–21,24–26, 28–32] and the references therein.

2. Preliminaries

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open unit ball $\{z \in \mathbb{C}^n\colon\ \|z\| < 1\}$ is denoted by B^n. In the case of one complex variable, B^1 is denoted by U.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ be the space of linear and continuous operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm and let I_n be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$. Let $H(B^n)$ be the set of holomorphic mappings from B^n into $\mathbb{C}^n$. If $f \in H(B^n)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$.

If $f \in H(B^n)$, we say that f is locally biholomorphic on B^n if $J_f(z) \neq 0$, $z \in B^n$, where $J_f(z) = \det Df(z)$ and $Df(z)$ is the complex Jacobian matrix of f at z.

If $f \in H(B^n)$, then f can be expanded in a power series of homogeneous polynomials

$$f(z) = \sum_{k=0}^{\infty} A_k\big(z^k\big), \quad z \in B^n,$$

where $A_k(z^k) = \frac{1}{k!}D^k f(0)(z^k)$. Here, for $h \in \mathbb{C}^n$, $D^0 f(0)(h^0) = f(0)$ and for $k \geqslant 1$, $D^k f(0)(h^k) = D^k f(0)(\underbrace{h, \ldots, h}_{k\text{-times}})$.

Several notions from operator theory play a role in studying special classes of holomorphic mappings of B^n or in proving estimates or existing theorems for the general Loewner differential equation. These notions involve properties of the numerical radius or the spectrum of a linear operator.

If $A \in L(\mathbb{C}^n, \mathbb{C}^n)$, let

$$m(A) = \min\big\{\Re\langle A(z), z\rangle\colon\ \|z\| = 1\big\} \quad \text{and} \quad k(A) = \max\big\{\Re\langle A(z), z\rangle\colon\ \|z\| = 1\big\}.$$

Also let

$$\big|V(A)\big| = \max\big\{\big|\langle A(z), z\rangle\big|\colon\ \|z\| = 1\big\}$$

be the numerical radius of the operator A. Then $\|A\| \leqslant 2|V(A)|$ by [18, Theorem 1.3.1] (see also [5] and [22]). The upper exponential index of A is defined by

$$k_+(A) = \max\big\{\Re\lambda\colon\ \lambda \in \sigma(A)\big\},$$

where $\sigma(A)$ is the spectrum of A. Then it is known that for each $\omega > k_+(A)$, there exists a positive number $\delta = \delta(\omega)$ such that

$$\big\|e^{tA}\big\| \leqslant \delta e^{\omega t}, \quad t \geqslant 0, \tag{2.1}$$

by [7]; see also [8] and [33, p. 311].

Remark 2.1. If $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ is such that $m(A) > 0$, then it is obvious that

$$m(A) \leqslant k_+(A) \leqslant k(A) \leqslant \|A\|.$$

The following classes of mappings in $H(B^n)$ occur throughout the theory of univalent mappings in several complex variables [10,15,16,25,26,34,35]:

$$\mathcal{N} = \{h \in H(B^n): \ h(0) = 0, \ \Re\langle h(z), z\rangle > 0, \ z \in B^n \setminus \{0\}\}$$

and

$$\mathcal{M} = \{h \in \mathcal{N}: \ Dh(0) = I_n\}.$$

In one complex variable, we have $f \in \mathcal{M}$ if and only if $f(z)/z \in \mathcal{P}$, where

$$\mathcal{P} = \{p \in H(U): \ p(0) = 1, \ \Re p(z) > 0, \ z \in U\}$$

is the Carathéodory class.

The next lemma ([13, Lemma 1.2]; cf. [10, Theorem 1.2]) plays an important role in the study of the Loewner differential equation and in the theory of univalent subordination chains in $\mathbb{C}^n$. The proof is similar to that of [10, Theorem 1.2], which considers the case $A = I_n$.

Lemma 2.2. *Let $h: B^n \to \mathbb{C}^n$ be a mapping such that $h \in \mathcal{N}$, $Dh(0) = A$ and $m(A) > 0$. Then $\|h(z)\| \leqslant \frac{4r}{(1-r)^2}|V(A)|$ for $\|z\| \leqslant r < 1$.*

We next consider the notions of subordination and subordination chains on B^n. If $f, g \in H(B^n)$, we say that f is subordinate to g (written $f \prec g$) if $f = g \circ v$ for some Schwarz mapping v. In other words, $v \in H(B^n)$ and $\|v(z)\| \leqslant \|z\|$, $z \in B^n$.

Definition 2.3. A mapping $f: B^n \times [0, \infty) \to \mathbb{C}^n$ is called a subordination chain if $f(\cdot, t)$ is holomorphic on B^n, $f(0, t) = 0$ for $t \geqslant 0$, and $f(\cdot, s) \prec f(\cdot, t)$, $0 \leqslant s \leqslant t < \infty$. In addition, if $f(\cdot, t)$ is biholomorphic on B^n for $t \geqslant 0$, we say that $f(z, t)$ is a Loewner chain. Also, if $f(z, t)$ is a subordination (Loewner) chain such that $Df(0, t) = e^{tA}$ for $t \geqslant 0$, we say that $f(z, t)$ is an A-normalized subordination (Loewner) chain.

We need the following existence result for the initial value problem (2.2) ([13, Theorem 2.1]; cf. [15, Theorem 8.1.3]). This result is a generalization of [25, Theorem 2.1], [10, Theorem 1.4] and [16, Lemma 1.3]. In the case of one complex variable, see [2, Lemma 1].

Lemma 2.4. *Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $m(A) > 0$ and let $h = h(z, t): B^n \times [0, \infty) \to \mathbb{C}^n$ be a mapping which satisfies the following conditions*:

(i) *$h(\cdot, t) \in \mathcal{N}$, $Dh(0, t) = A$ for all $t \geqslant 0$.*
(ii) *$h(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in B^n$.*

Then for each $s \geqslant 0$ and $z \in B^n$, the initial value problem

$$\frac{\partial v}{\partial t} = -h(v, t) \quad \text{a.e. } t \geqslant s, \ v(z, s, s) = z, \tag{2.2}$$

has a unique solution $v = v(z, s, t)$ such that $v(\cdot, s, t)$ is a univalent Schwarz mapping, $v(z, s, \cdot)$ is Lipschitz continuous on $[s, \infty)$ locally uniformly with respect to $z \in B^n$, and $Dv(0, s, t) = \exp(-A(t - s))$ for $t \geqslant s \geqslant 0$.

Definition 2.5. A mapping $h = h(z, t): B^n \times [0, \infty) \to \mathbb{C}^n$ which satisfies the assumptions (i) and (ii) of Lemma 2.4 will be called a generating vector field.

We next mention the following growth result that is satisfied by the solution $v(z, s, t)$ of the initial value problem (2.2) (see [13, Theorem 2.1]).

Lemma 2.6. *Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be such that $m(A) > 0$ and let $h = h(z, t): B^n \times [0, \infty) \to \mathbb{C}^n$ be a generating vector field such that $Dh(0, t) = A$, $t \geqslant 0$. Also let $v = v(z, s, t)$ be the unique Lipschitz continuous solution on $[s, \infty)$ of the initial value problem* (2.2)*. Then*

$$\frac{\|v(z, s, t)\|}{(1 - \|v(z, s, t)\|)^2} \leqslant e^{-m(A)(t-s)} \frac{\|z\|}{(1 - \|z\|)^2}, \quad z \in B^n, \ t \geqslant s \geqslant 0, \tag{2.3}$$

and

$$e^{-k(A)(t-s)}\frac{\|z\|}{(1+\|z\|)^2}\leqslant\frac{\|v(z,s,t)\|}{(1+\|v(z,s,t)\|)^2},\quad z\in B^n,\ t\geqslant s\geqslant 0.$$

Definition 2.7. Let $g(z,t):B^n\times[0,\infty)\to\mathbb{C}^n$ be a mapping such that $g(\cdot,t)\in H(B^n)$, $g(0,t)=0$, $t\geqslant 0$, and $g(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z\in B^n$. Assume that $g(z,t)$ satisfies the Loewner differential equation (1.2). $g(z,t)$ will be called a standard solution of the Loewner differential equation (1.2).

We next give the definition of polynomially boundedness (Voda [36]).

Definition 2.8. (i) A standard solution $f:B^n\times[0,\infty)\to\mathbb{C}^n$ to (1.2) is said to be polynomially bounded (bounded) if $\{e^{-tA}f(\cdot,t)\}_{t\geqslant 0}$ is locally polynomially bounded (locally bounded), i.e. for any compact set $K\subset B^n$, there exist a constant C_K and a polynomial (constant polynomial) P such that

$$\left\|e^{-tA}f(z,t)\right\|\leqslant C_K P(t),\quad z\in K,\ t\geqslant 0.$$

(ii) A function $F_k:[0,\infty)\to\mathcal{P}^k(\mathbb{C}^n)$, where $\mathcal{P}^k(\mathbb{C}^n)$ denote the Banach space of homogeneous polynomial mappings of degree k from $\mathbb{C}^n$ to $\mathbb{C}^n$, is said to be polynomially bounded (bounded) if there exists a polynomial (constant polynomial) P such that $\|F_k(t)\|\leqslant P(t)$, $t\geqslant 0$.

If $k_+(A)<2m(A)$, then we obtain the existence and uniqueness of A-normalized bounded solution to the Loewner differential equation (1.2) [9, Corollary 4.4] (cf. [13, Theorem 2.3]).

Lemma 2.9. *Let h be as in Lemma 2.4 and let $v=v(z,s,t)$ be the unique Lipschitz continuous solution on $[s,\infty)$ of the initial value problem (2.2). If $k_+(A)<2m(A)$, then the limit*

$$\lim_{t\to\infty}e^{tA}v(z,s,t)=f(z,s)$$

exists locally uniformly on B^n for each $s\geqslant 0$. Moreover, $f(z,t)$ is the unique A-normalized bounded Loewner chain solution to the Loewner differential equation (1.2).

3. Main results

We begin this section with the following proposition.

Proposition 3.1. *Let $g(z,t)$ be a standard solution to the Loewner differential equation (1.2). Then $g(z,s)=g(v(z,s,t),t)$ holds for $z\in B^n$ and $t\geqslant s\geqslant 0$, where $v=v(z,s,t)$ is the unique solution of the initial value problem (2.2). Moreover, $Dg(0,t)=Dg(0,0)e^{tA}$ holds.*

Proof. Let $g(z,s,t)=g(v(z,s,t),t)$ for $z\in B^n$ and $t\geqslant s\geqslant 0$. We prove that $g(z,s,t)=g(z,s,s)$, i.e. $g(v(z,s,t),t)=g(z,s)$ for $z\in B^n$ and $t\geqslant s\geqslant 0$. Fix $r\in(0,1)$ and $T>0$. Since $g(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z\in B^n$, we deduce that $g(z,t)$ is continuous on $B^n\times[0,\infty)$. Then there exists a constant $L_1(r,T)>0$ such that

$$\left\|g(z,t)\right\|\leqslant L_1(r,T),\quad \|z\|\leqslant r,\ t\in[0,T].$$

In view of the Cauchy integral formula for holomorphic mappings, we deduce that there exists a constant $L_1^*(r,T)>0$ such that

$$\left\|Dg(z,t)\right\|\leqslant L_1^*(r,T),\quad \|z\|\leqslant r,\ t\in[0,T]. \tag{3.1}$$

On the other hand, letting $b=|V(A)|$ and taking into account the relations (1.2), (3.1) and Lemma 2.2, we deduce that there exists a constant $L_2(r,T,b)>0$ such that

$$\left\|\frac{\partial g}{\partial t}(z,t)\right\|\leqslant L_2(r,T,b)\quad \text{a.e. } t\in[0,T],\ \|z\|\leqslant r.$$

Hence, in view of the local absolute continuity of $g(z,\cdot)$ locally uniformly with respect to $z\in B^n$, we obtain that

$$\left\|g(z,t_1)-g(z,t_2)\right\|=\left\|\int_{t_1}^{t_2}\frac{\partial g}{\partial t}(z,t)\,dt\right\|\leqslant L_2(r,T,b)(t_2-t_1)$$

for $\|z\| \leqslant r$ and $0 \leqslant t_1 \leqslant t_2 \leqslant T$. Therefore, $g(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z \in B^n$. Since $v(z,s,\cdot)$ is Lipschitz continuous on $[s,\infty)$ locally uniformly with respect to $z \in B^n$ in view of Lemma 2.4, it is not difficult to deduce that the same is true for $g(z,s,\cdot)$. Then $g(z,s,t)$ is differentiable for almost all $t \in [s,\infty)$ and in view of (1.2) and (2.2), we deduce that $\frac{\partial g}{\partial t}(z,s,t) = 0$ for a.e. $t \geqslant s$ and for all $z \in B^n$. Hence $g(z,s) = g(v(z,s,t),t)$ for $t \geqslant s$ and $z \in B^n$.

Moreover, we have

$$Dg(0,0) = Dg(0,t)Dv(0,0,t) = Dg(0,t)e^{-tA}$$

by Lemma 2.4. Thus, $Dg(0,t) = Dg(0,0)e^{tA}$. This completes the proof. □

In the following theorem, we determine the form of locally biholomorphic polynomially bounded solutions to the Loewner differential equation. For A-normalized polynomially bounded solution to the Loewner differential equation, the following theorem is due to [36, Proposition 2.1].

Theorem 3.2. *Let $g(z,t)$ be a polynomially bounded solution to the Loewner differential equation* (1.2) *such that $L = Dg(0,0)$ is a non-singular matrix. Then*

$$g(z,s) = L\big(f(z,s)\big), \quad z \in B^n,\ s \geqslant 0, \tag{3.2}$$

where $f(z,s)$ is an A-normalized subordination chain solution to the Loewner differential equation (1.2)*. $f(z,s)$ can be written as follows*:

$$f(z,s) = \lim_{t\to\infty} e^{tA}\left(v(z,s,t) + \sum_{k=2}^{n_0} F_k\big(v(z,s,t)^k,t\big)\right) \tag{3.3}$$

locally uniformly in z, where $v = v(z,s,t)$ is the unique solution of the initial value problem (2.2)*, $n_0 = [k_+(A)/m(A)]$ and*

$$f(z,t) = e^{tA}\left(z + \sum_{k=2}^{\infty} F_k\big(z^k,t\big)\right).$$

Moreover, if $Dg(0,0)$ commutes with A, then $f(z,t)$ is an A-normalized polynomially bounded Loewner chain solution to the Loewner differential equation (1.2)*.*

Proof. Let $f(z,t) = [Dg(0,0)]^{-1}g(z,t)$. Then

$$f(z,s) = f\big(v(z,s,t),t\big) = e^{tA}\left(v(z,s,t) + \sum_{k=2}^{n_0} F_k\big(v(z,s,t)^k,t\big)\right) + e^{tA}R\big(v(z,s,t),t\big),$$

where $R(z,t) = \sum_{k=n_0+1}^{\infty} F_k(z^k,t)$. Since g is polynomially bounded, using the formula for the remainder of the Taylor series and Cauchy's formula, we obtain

$$\big\|e^{-tA}Dg(0,0)e^{tA}R(z,t)\big\| \leqslant C_r P(t)\|z\|^{n_0+1}, \quad \|z\| \leqslant r,$$

where C_r is a constant which depends only on r and $P(t)$ is a polynomial in t. From (2.1), (2.3) and this inequality, for each $\omega > k_+(A)$, there exists $\delta > 0$ such that

$$\begin{aligned}\big\|e^{tA}R\big(v(z,s,t),t\big)\big\| &= \big\|\big[Dg(0,0)\big]^{-1}e^{tA}e^{-tA}Dg(0,0)e^{tA}R\big(v(z,s,t),t\big)\big\| \\ &\leqslant \big\|\big[Dg(0,0)\big]^{-1}\big\|\delta e^{\omega t}C_r P(t)\big\|v(z,s,t)\big\|^{n_0+1} \\ &\leqslant C_{\omega,r,s}e^{(\omega-(n_0+1)m(A))t}P(t), \quad \|z\| \leqslant r,\end{aligned}$$

where $C_{\omega,r,s}$ is a constant which depends only on ω, r and s. Since we can take ω sufficiently close to $k_+(A)$, we conclude that $e^{tA}R(v(z,s,t),t) \to 0$ locally uniformly in z. Thus, we obtain (3.2) and (3.3).

Next, assume that $Dg(0,0)$ commutes with A. Since g is polynomially bounded, $F_k(\cdot,t) = [Dg(0,0)]^{-1}e^{-tA}Dg(0,0) \times e^{tA}F_k(\cdot,t)$, $2 \leqslant k \leqslant n_0$, are polynomially bounded locally uniformly on B^n. Thus, by [36, Theorem 2.8], $[Dg(0,0)]^{-1}g(z,t)$ is an A-normalized polynomially bounded Loewner chain solution to the Loewner differential equation (1.2). □

If $n_0 = 1$, i.e. $k_+(A) < 2m(A)$, then we can determine the form of an arbitrary polynomially bounded solution to the Loewner differential equation. For bounded solution to the Loewner differential equation, the following theorem is due to [9, Corollary 5.2].

Theorem 3.3. *Assume that $n_0 = 1$. Let $g(z,t)$ be a polynomially bounded solution to the Loewner differential equation* (1.2)*. Then*

$$g(z,s) = L\big(f(z,s)\big), \quad z \in B^n,\ s \geqslant 0, \tag{3.4}$$

where $L = Dg(0,0)$ and

$$f(z,s) = \lim_{t\to\infty} e^{tA} v(z,s,t)$$

is the unique A-normalized bounded Loewner chain solution to the Loewner differential equation (1.2)*.*

Proof. Let

$$g(z,t) = Dg(0,0)e^{tA}z + e^{tA}\sum_{k=2}^{\infty} G_k\big(z^k,t\big)$$

be a polynomially bounded solution to the Loewner differential equation (1.2). Then

$$g(z,s) = g\big(v(z,s,t),t\big) = Dg(0,0)e^{tA}v(z,s,t) + e^{tA}R\big(v(z,s,t),t\big),$$

where $R(z,t) = \sum_{k=2}^{\infty} G_k(z^k,t)$. Since g is polynomially bounded, using the formula for the remainder of the Taylor series and Cauchy's formula, we obtain

$$\big\|R(z,t)\big\| \leqslant C_r P(t)\|z\|^2, \quad \|z\| \leqslant r,$$

where C_r is a constant which depends only on r and $P(t)$ is a polynomial in t. From (2.1), (2.3) and this inequality, for each $\omega > k_+(A)$, there exists $\delta > 0$ such that

$$\big\|e^{tA}R\big(v(z,s,t),t\big)\big\| \leqslant \delta e^{\omega t} C_r P(t)\big\|v(z,s,t)\big\|^2 \leqslant C_{\omega,r,s}e^{(\omega-2m(A))t}P(t), \quad \|z\| \leqslant r,$$

where $C_{\omega,r,s}$ is a constant which depends only on ω, r and s. Since we can take ω sufficiently close to $k_+(A)$, we conclude that $e^{tA}R(v(z,s,t),t) \to 0$ locally uniformly in z. Thus, we obtain (3.4). By Lemma 2.9, $f(z,s) = \lim_{t\to\infty} e^{tA}v(z,s,t)$ is the unique A-normalized bounded Loewner chain solution to the Loewner differential equation (1.2). This completes the proof. □

As a corollary of the above theorem, we obtain the following uniqueness of polynomially bounded solutions to the Loewner differential equation (1.2). For bounded solution to the Loewner differential equation, the following corollary is due to [9, Corollary 4.4].

Corollary 3.4. *Assume that $n_0 = 1$. Let $f(z,t)$ be a polynomially bounded solution to the Loewner differential equation* (1.2) *such that $Df(0,0) = I_n$. Then*

$$f(z,s) = \lim_{t\to\infty} e^{tA} v(z,s,t)$$

and it is the unique A-normalized bounded Loewner chain solution to the Loewner differential equation (1.2)*.*

Next, assume that $g(z,t) = L(f(z,t))$, where $f(z,t)$ is an A-normalized polynomially bounded Loewner chain solution to the Loewner differential equation (1.2) and $L \in L(\mathbb{C}^n,\mathbb{C}^n)$. The following examples show that even if $f(z,t)$ is bounded, $g(z,t)$ may be polynomially bounded or not polynomially bounded.

Example 3.5. Let

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix},$$

where $\alpha > 2$ and let $f(z) = z$, where $z \in \mathbb{C}^2$. Then $m(A) \geqslant \alpha - 1 > 0$ and f is a spirallike mapping with respect to A. Since $k_+(A) = \alpha < 2(\alpha-1) \leqslant 2m(A)$, we obtain $n_0 = 1$. Then

$$f(z,t) = e^{tA}z = e^{t\alpha}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} z$$

is the unique bounded solution to the Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)Az \quad \text{a.e. } t \geqslant 0,\ \forall z \in B^2. \tag{3.5}$$

However, if

$$L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

then

$$L\big(f(z,t)\big) = L\big(e^{tA}z\big) = e^{t\alpha}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} z = e^{t\alpha}\begin{bmatrix} 0 & -1 \\ 1 & t \end{bmatrix} z$$

is polynomially bounded and not bounded solution to the Loewner differential equation (3.5).

Example 3.6. Let

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix},$$

where $0 < \alpha < \beta < 2\alpha$. Then $m(A) > 0$ and $n_0 = 1$. Let $f(z) = z$, where $z \in \mathbb{C}^2$. Then f is a spirallike mapping with respect to A and $f(z,t) = e^{tA}z = (e^{t\alpha}z_1, e^{t\beta}z_2)$ is the unique bounded solution to the Loewner differential equation (3.5). However, if

$$L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

then $L(f(z,t)) = L(e^{tA}z) = (-e^{t\beta}z_2, e^{t\alpha}z_1)$ is not polynomially bounded.

In the following theorem, we will give sufficient conditions for the mapping $g(z,t) = L(f(z,t))$ to be polynomially bounded.

Theorem 3.7. *Let $g(z,t) = L(f(z,t))$, where $f(z,t)$ is an A-normalized polynomially bounded Loewner chain solution to the Loewner differential equation* (1.2) *and $L \in L(\mathbb{C}^n, \mathbb{C}^n)$. If one of the following conditions is satisfied, then $g(z,t)$ is polynomially bounded.*

(i) *L commutes with A;*
(ii) *$k_+(A) = k_-(A)$, where $k_-(A) = \min\{\Re\lambda\colon \lambda \in \sigma(A)\}$.*

Proof. (i) Assume that L commutes with A. Then

$$\left\|e^{-tA}g(z,t)\right\| = \left\|L\big(e^{-tA}f(z,t)\big)\right\| \leqslant \|L\|\left\|e^{-tA}f(z,t)\right\|, \quad z \in B^n,\ t \geqslant 0.$$

This implies that $g(z,t)$ is polynomially bounded.

(ii) Let $\alpha = k_+(A) = k_-(A)$. Then $e^{tA} = e^{t\alpha}M(t)$, where $M(t)$ is a matrix such that $\|M(t)\| \leqslant Q(|t|)$ for any $t \in \mathbb{R}$ with some polynomial Q. Since

$$\left\|e^{-tA}g(z,t)\right\| \leqslant \left\|M(-t)\right\|\|L\|\left\|M(t)\right\|\left\|e^{-tA}f(z,t)\right\| \leqslant Q(t)^2\|L\|\left\|e^{-tA}f(z,t)\right\|,$$

for $z \in B^n$ and $t \geqslant 0$, $g(z,t)$ is polynomially bounded. □

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