**14.1.** Let $\mathscr{P}(\mathbb{T})$ denote the closure of $\mathbb{C}[z]$ in $C(\mathbb{T})$, where $z$ is the coordinate on $\mathbb{C}$. Recall that the *disk algebra* $\mathscr{A}(\bar{\mathbb{D}})$ consists of those $f \in C(\bar{\mathbb{D}})$ that are holomorphic on the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Show that each $f \in \mathscr{P}(\mathbb{T})$ uniquely extends to $\tilde{f} \in \mathscr{A}(\bar{\mathbb{D}})$, and that $\sigma_{\mathscr{P}(\mathbb{T})}(f) = \tilde{f}(\bar{\mathbb{D}})$.

**14.2.** A unital commutative algebra $A$ is *local* if $A$ has a unique maximal ideal. Construct a local Banach algebra without zero divisors.

*Hint.* Consider the subalgebra of $\mathbb{C}[[z]]$ that consists of formal series $a = \sum c_n z^n$ satisfying $\|a\| = \sum |c_n| w_n < \infty$. Here $(w_n)$ is a sequence of positive numbers satisfying some special conditions.

**14.3.** Prove that for each unital algebra $A$ and each $a \in A$ we have $\sigma_{A_+}(a) = \sigma(a) \cup \{0\}$.

**14.4.** **(a)** Let $A$ be a Banach algebra, $a, b \in A$, $ab = ba$. Prove that $r(a+b) \leqslant r(a) + r(b)$ and $r(ab) \leqslant r(a)r(b)$ (where $r$ is the spectral radius).
**(b)** Does (a) hold if we drop the assumption that $ab = ba$?

**14.5.** Let $c_{00} \subset c_0$ denote the ideal of finite sequences (i.e., of those sequences $a = (a_n)$ such that $a_n = 0$ for all but finitely many $n \in \mathbb{N}$). Prove that $c_{00}$ is not contained in a maximal ideal of $c_0$.

**14.6.** Let $A = \{f \in C[0,1] : f(0) = 0\}$, and let $I = \{f \in A : f$ vanishes on a neighborhood of $0\}$. Prove that $I$ is not contained in a maximal ideal of $A$.

**14.7.** Let $X$ be a compact Hausdorff topological space. For each closed subset $Y \subset X$ let $I_Y = \{f \in C(X) : f|_Y = 0\}$. Prove that the assignment $Y \mapsto I_Y$ is a 1-1 correspondence between the collection of all closed subsets of $X$ and the collection of all closed ideals of $C(X)$.

**14.8.** A commutative algebra $A$ is *semisimple* if the intersection of all maximal modular ideals of $A$ (the *Jacobson radical* of $A$) is $\{0\}$. Show that every homomorphism from a Banach algebra to a commutative semisimple Banach algebra is continuous.

**14.9.** Describe the maximal spectrum and the Gelfand transform for the algebras **(a)** $C^n[0,1]$; **(b)** $\mathscr{A}(\bar{\mathbb{D}})$; **(c)** $\mathscr{P}(\mathbb{T})$.

**14.10.** Let $A(\mathbb{T}) = \{f \in C(\mathbb{T}) : \sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty\}$, where $\hat{f}(n)$ is the $n$th Fourier coefficient of $f$ w.r.t. the trigonometric system $(e_n)$ on $\mathbb{T}$ (i.e., $e_n(z) = z^n$ for all $z \in \mathbb{T}$ and $n \in \mathbb{Z}$). Prove that $A(\mathbb{T})$ is a spectrally invariant subalgebra of $C(\mathbb{T})$.

**14.11.** Let $X$ be a topological space, let $\beta X = \operatorname{Max} C_b(X)$, and let $\varepsilon \colon X \to \beta X$ take each $x \in X$ to the evaluation map $\varepsilon_x \colon C_b(X) \to \mathbb{C}$ given by $\varepsilon_x(f) = f(x)$.
**(a)** Prove that $(\beta X, \varepsilon)$ is the Stone-Čech compactification of $X$ (i.e., for each compact Hausdorff topological space and each continuous map $f \colon X \to Y$ there exists a unique continuous map $\tilde{f} \colon \beta X \to Y$ such that $\tilde{f} \circ \varepsilon = f$).
**(b)** Prove that $\varepsilon(X)$ is dense in $\beta X$.
**(c)** Prove that $\varepsilon$ is a homeomorphism onto $\varepsilon(X)$ if and only if $X$ is completely regular.

**14.12.** Let $A$ be a commutative algebra, and $I$ be a maximal ideal of $A$. Prove that $I$ is either modular or a codimension 1 ideal containing $A^2 = \operatorname{span}\{ab : a, b \in A\}$.

**14.13.** Let $A$ be a commutative algebra, and let $\operatorname{Max}_+(A) = \operatorname{Max}(A) \cup \{A\}$. Prove that the map $\operatorname{Max}(A_+) \to \operatorname{Max}_+(A)$, $I \mapsto I \cap A$, is a bijection.

**14.14.** **(a)** Does there exist a norm and an involution on $C^1[a,b]$ making it into a $C^*$-algebra?
**(b)** Does there exist a norm and an involution on $\mathscr{A}(\bar{\mathbb{D}})$ making it into a $C^*$-algebra?

**14.15.** Let $X$ be a locally compact Hausdorff topological space, and let $X_+$ denote the one-point compactification of $X$. For each $f \in C_0(X)$, define $f_+ \colon X_+ \to \mathbb{C}$ by $f_+(x) = f(x)$ for $x \in X$ and $f_+(\infty) = 0$. Prove that $f_+$ is continuous, and that the map $C_0(X)_+ \to C(X_+)$, $f + \lambda 1_+ \mapsto f_+ + \lambda$, is an isometric $*$-isomorphism. (Here we assume that $C_0(X)_+$ is equipped with the canonical $C^*$-norm extending the supremum norm on $C_0(X)$.)

**14.16.** Let $A$ and $B$ be $C^*$-algebras. Show that if $B$ is commutative, then each homomorphism from $A$ to $B$ is a $*$-homomorphism. Does the above result hold without the commutativity assumption?

**14.17.** Let $A = C^1[0,1]$. **(a)** Is $A$ hermitian? **(b)** Does the identity $\|a\| = r(a)$ hold in $A$?

**14.18.** Let $A = \mathscr{A}(\bar{\mathbb{D}})$. **(a)** Is $A$ hermitian? **(b)** Does the identity $\|a\| = r(a)$ hold in $A$?

**14.19.** **(a)** Let $H$ be a $*$-module over a Banach $*$-algebra $A$. Assume that $\mathrm{End}_A(H) = \mathbb{C}\mathbf{1}_H$. Show that $H$ is irreducible.
**(b)** Does (a) hold if $H$ is a Banach $A$-module (but is not necessarily a $*$-module)?