11.1. Let G be a locally compact group. As was shown in the lectures, $L^1(G)$ is a Banach algebra under convolution.
(a) Show that $L^1(G)$ is a Banach $*$-algebra w.r.t. the involution $f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})$ ($f \in L^1(G)$, $x \in G$).
(b) Show that $L^1(G)$ (equipped with the standard L^1-norm and with the involution defined in (a)) is not a C^*-algebra unless $G = \{e\}$.
(c) Show that $L^1(G)$ is commutative if and only if G is commutative.
(d) Show that $L^1(G)$ is unital if and only if G is discrete.

11.2. Let G be a locally compact group, and let $p, q \in (1, +\infty)$ satisfy $1/p + 1/q = 1$. Show that, for each $f \in L^p(G)$ and $g \in L^q(G)$, the convolution $f * Sg$ (where $(Sg)(x) = g(x^{-1})$) is defined everywhere on G, belongs to $C_0(G)$, and that $\|f * Sg\|_\infty \leqslant \|f\|_p \|g\|_q$.

11.3. Let G be a locally compact group.
(a) Show that $M(G)$ is a unital Banach $*$-algebra w.r.t. the involution $\nu^*(B) = \overline{\nu(B^{-1})}$ ($\nu \in M(G)$, $B \subset G$ is a Borel set). In particular, show that convolution is associative on $M(G)$ (this was not proved at the lectures).
(b) Show that $M(G)$ is commutative if and only if G is commutative.
(c) Let μ be a left Haar measure on G. Show that the map $i\colon L^1(G) \to M(G)$, $f \mapsto f \cdot \mu$, is an isometric $*$-algebra homomorphism.
(d) Identify $L^1(G)$ with its canonical image in $M(G)$ (see (c) above). Show that $L^1(G)$ is a closed 2-sided ideal of $M(G)$, and that for each $f \in L^1(G)$, $\nu \in M(G)$, and for almost all $x \in G$ we have

$$(\nu * f)(x) = \int_G f(yx^{-1})\, d\nu(y), \qquad (f * \nu)(x) = \int_G f(xy^{-1})\Delta(y^{-1})\, d\nu(y). \tag{1}$$

11.4. Let G be a locally compact group, and let λ (resp. ρ) denote the left (resp. right) regular representation of G on $L^1(G)$. Show that $\lambda(x)f = \delta_x * f$ and $\rho(x)f = f * \delta_{x^{-1}}$ ($f \in L^1(G)$, $x \in G$).

11.5. Let X be a locally compact Hausdorff topological space.
(a) Construct a bounded approximate identity in $C_0(X)$.
(b) Show that $C_0(X)$ has a sequential bounded approximate identity if and only if X is σ-compact.

11.6. Let H be a Hilbert space.
(a) Construct a bounded approximate identity in $\mathscr{K}(H)$.
(b) Show that $\mathscr{K}(H)$ has a sequential bounded approximate identity if and only if H is separable.

11.7. Let G be a locally compact group, and let (u_i) be a Dirac net in $L^1(G)$. Show that (u_i) converges to $\delta_e \in M(G)$ w.r.t. the weak* topology on $M(G)$.

11.8. Let A be a Banach algebra with a bounded approximate identity (e_α), and let E be a left Banach A-module. Recall that the *essential part* of E is $E_{\mathrm{ess}} = \overline{\operatorname{span}\{av : a \in A,\ v \in E\}}$.
(a) Show that E_{ess} is the largest essential submodule of E.
(b) Show that $E_{\mathrm{ess}} = \{v \in E : v = \lim e_\alpha v\}$.
(c) Does (a) hold without the assumption that A has a b.a.i.?

11.9. Let G be a locally compact group, $\nu \in M(G)$, and $f \in L^p(G)$ (where $1 \leqslant p \leqslant \infty$).
(a) Show that the convolution $\nu * f$ given by (1) is defined a.e. on G, that $\nu * f \in L^p(G)$, and that the action $(\nu, f) \mapsto \nu * f$ makes $L^p(G)$ into a left unital Banach $M(G)$-module.
(b) Show that $L^p(G)$ is essential over $L^1(G)$ if $p < \infty$.
(c) Find the essential part of $L^\infty(G)$ over $L^1(G)$.

11.10. Define $f\colon [0,1] \to c_0$ by

$$f(t) = (\chi_{(0,1]}(t), 2\chi_{(0,1/2]}(t), \ldots, n\chi_{(0,1/n]}(t), \ldots) \qquad (t \in [0,1]).$$

Show that f is Dunford integrable (w.r.t. the Lebesgue measure on $[0,1]$), but is not Pettis integrable.

11.11. Let (X, μ) be a measure space, and let E, F be Banach spaces.
(a) Suppose that $f\colon X \to E$ is Dunford integrable and that $\|f\|\colon x \mapsto \|f(x)\|$ is integrable. Show that $\| \int f\, d\mu\| \leqslant \int \|f\|\, d|\mu|$.
(b) Suppose that $f\colon X \to E$ is Pettis integrable. Show that for each bounded linear map $T\colon E \to F$ the function $T \circ f$ is Pettis integrable, and that $T(\int f\, d\mu) = \int (T \circ f)\, d\mu$.

11.12. Let G be a locally compact group, and let π be a uniformly bounded continuous representation of G on a reflexive Banach space E. Recall that the *canonical extension* of π to $M(G)$ is given by $\tilde{\pi}(\nu)v = \int_G \pi(x)v\, d\nu(x)$ ($\nu \in M(G)$, $v \in E$).
(a) Show that $\tilde{\pi}$ is indeed a representation of $M(G)$ on E.
(b) Choose a left Haar measure μ on G. Show that for each $g \in L^1(G)$ (where $L^1(G)$ is canonically embedded into $M(G)$) we have $\tilde{\pi}(g)v = \int_G g(x)\pi(x)v\, d\mu(x)$.
(c) Suppose that E is a Hilbert space. Show that $\tilde{\pi}$ is a $*$-representation if and only if π is unitary.

11.13. Let G be a locally compact group. Show that, for each $\nu \in M(G)$ and $f \in L^p(G)$ (where $1 < p < \infty$), we have $\nu * f = \tilde{\lambda}(\nu)f$, where λ is the left regular representation of G on $L^p(G)$ and $\tilde{\lambda}$ is the canonical extension of λ to $M(G)$.

11.14. Let A be a Banach algebra, and let $B \subset A$ be a closed 2-sided ideal with a bounded approximate identity. Show that
(a) if E is a B-essential Banach A-module and $E_0 \subset E$ is a closed B-submodule, then E_0 is an A-submodule;
(b) if E and F are B-essential Banach A-modules, then $\mathrm{Hom}_A(E, F) = \mathrm{Hom}_B(E, F)$;
(c) if H is a Hilbert space equipped with an action of A which makes H into a B-essential Banach A-module, then H is a $*$-module over B iff H is a $*$-module over A.

11.15. Define a representation π of $\mathbb{R}$ on $L^2(\mathbb{R})$ by $(\pi(t)f)(x) = e^{-2\pi i t x} f(x)$. Find an explicit formula for the associated representation $\tilde{\pi}$ of $L^1(\mathbb{R})$. Show that π is unitarily isomorphic to the left regular representation.