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Asymptotics of the principal eigenvalue and expected hitting time for positive recurrent elliptic operators in a domain with a small puncture ☆

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Dedicated to the Memory of Bob Brooks (1952–2002)

Abstract

Let $X(t)$ be a positive recurrent diffusion process corresponding to an operator L on a domain $D \subseteq R^d$ with oblique reflection at ∂D if $D \neq R^d$. For each $x \in D$, we define a volume-preserving norm that depends on the diffusion matrix $a(x)$. We calculate the asymptotic behavior as $\varepsilon \to 0$ of the expected hitting time of the ε-ball centered at x and of the principal eigenvalue for L in the exterior domain formed by deleting the ball, with the oblique derivative boundary condition at ∂D and the Dirichlet boundary condition on the boundary of the ball. This operator is non-self-adjoint in general. The behavior is described in terms of the invariant probability density at x and $Det(a(x))$. In the case of normally reflected Brownian motion, the results become isoperimetric-type equalities.

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Let $D \subseteq R^d$, $d \geqslant 2$, be a domain. If $D \not\equiv R^d$, assume that D has a smooth boundary and let $v : \partial D \to S^d$ be smooth and satisfy $v(x) \cdot n(x) > 0$ for all $x \in \partial D$, where $n(x)$ denotes the inward unit normal to D at $x \in \partial D$. We will call v a *reflection vector*. Let $X(t)$ be the diffusion process in D with v-reflection at ∂D (if $D \not\equiv R^d$), and

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corresponding to the operator

$$L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla,$$

where $a = \{a_{i,j}\}_{i,j=1}^{d} \in C^{2,\alpha}_{\text{loc}}(R^d)$ is positive definite and $b = (b_1, \ldots, b_d) \in C^{1,\alpha}_{\text{loc}}(R^d)$, for some $\alpha \in (0, 1]$. The smoothness assumptions have been made in order to ensure that the adjoint operator has C^{α}-coefficients which then guarantees that invariant densities, which are solutions to the adjoint equation, are classical solutions. But this can be relaxed considerably since all we use is the existence of a continuous invariant density. In the case that the coefficients $a_{i,j}$, $\frac{\partial a_{i,j}}{\partial x_k}$, and b_i are bounded, the existence and uniqueness in law of such a diffusion process follows from [15] via the submartingale problem. In the case that the coefficients are not necessarily bounded, there exists a unique solution to the generalized submartingale problem up to a possibly finite explosion time (see [9] for the passage from the martingale problem to the generalized martingale problem in the case of diffusions on R^d; the passage from the submartingale problem to the generalized submartingale problem for reflected diffusions is treated similarly). Let P_x denote the probability measure corresponding to the diffusion starting from $x \in \bar{D}$.

We investigate the asymptotic behavior of the expected hitting time of a small ball starting from outside the ball and of the principal eigenvalue for L in the punctured domain obtained by deleting the small ball and placing the Dirichlet boundary condition on the resulting boundary. Note that the operator in question is in general non-self-adjoint because of the oblique derivative boundary condition as well as because of the drift term b. We will assume throughout the paper that the diffusion process is positive recurrent, which of course is always true if D is bounded. Indeed, the expected hitting time is finite if and only if the process is positive recurrent; that is, if and only if there exists an invariant probability density μ (see [9, Theorem 4.9.6] for a proof of the equivalence in the case $R^d = D$). For the investigation of the principal eigenvalue, we will need an additional assumption which also always holds if D is bounded.

The original motivation for this investigation was a recent paper [2] in which it was shown that if $\mathcal{T}(x, \varepsilon)$ is the first hitting time of the disc of radius ε for Brownian motion on the two-dimensional unit torus T, then $\lim_{\varepsilon \to 0} \sup_{x \in T^2} \frac{\mathcal{T}(x,\varepsilon)}{|\log \varepsilon|^2} = \frac{2}{\pi}$ a.s. A basic first step was to obtain estimates on the asymptotic behavior as $\varepsilon \to 0$ of the expected value of $\mathcal{T}(x, \varepsilon)$ starting from points $y \in T - \{x\}$.

For a positive definite $d \times d$ matrix Γ, define the norm

$$||v||_{\Gamma} = \left(v, \frac{\Gamma}{Det^{\frac{1}{d}}(\Gamma)} v \right)^{\frac{1}{2}} \quad \text{for } v \in R^d.$$

Note that this norm preserves the Euclidean volume but distorts directions. For $x \in D$ and $r > 0$, let $B_r^{\Gamma}(x) = \{y \in R^d : ||y - x||_{\Gamma} < r\}$ denote the open ball of radius r in the

Γ-norm and centered at x, and define $\tau_{B_r^\Gamma(x)} = \inf\{t \geq 0 : X(t) \in \bar{B}_r^\Gamma(x)\}$. In the case of the standard Euclidean norm, when Γ is a scalar multiple of I, we will use the notation $|v|$ and $B_r(x)$ in place of $||v||_I$ and $B_r^I(x)$. Let ω_d denote the volume of the unit ball in R^d.

Let $a^{\text{inv}}(x)$ denote the inverse matrix to $a(x)$. Here is the main result with regard to expected hitting times.

Theorem 1. *Let $X(t)$ be a positive recurrent diffusion in a domain $D \subseteq R^d$ with v-reflection at ∂D (if $D \not\equiv R^d$) and corresponding to an operator of the form L as above. Let μ denote the invariant probability density. Let $x \in D$ and $y \in \bar{D} - \{x\}$:*

(i) *If $d = 2$, then*

$$\lim_{\varepsilon \to 0} \frac{E_y \tau_{B_\varepsilon^{a^{\text{inv}}(x)}(x)}}{-\log \varepsilon} = \frac{1}{\pi Det^{\frac{1}{2}}(a(x))\mu(x)}; \tag{1.1i}$$

(ii) *If $d \geq 3$, then*

$$\lim_{\varepsilon \to 0} \frac{E_y \tau_{B_\varepsilon^{a^{\text{inv}}(x)}(x)}}{\varepsilon^{2-d}} = \frac{2}{d(d-2)\omega_d Det^{\frac{1}{d}}(a(x))\mu(x)}. \tag{1.1ii}$$

Remark. In the case that the diffusion process is reversible, the invariant density can be given explicitly. The diffusion is reversible if and only if the drift vector b is of the form $b = a\nabla Q$, for some function Q, and the reflection vector v is in the conormal direction; that is, $v(x) = c(x)a(x)n(x)$, where $c(x)$ is the normalizing scalar so that $v \in S^d$. In this case, positive recurrence is equivalent to the condition $\int_D \exp(2Q(y))\, dy < \infty$, and we have

$$\mu(x) = \frac{\exp(2Q(x))}{\int_D \exp(2Q(y))\, dy}.$$

Thus, in the reversible case, the right-hand side of (1.1) is given explicitly in terms of the coefficients of L.

If $X(t)$ is normally reflected Brownian motion, then $X(t)$ is positive recurrent if and only if D has finite volume, in which case the invariant probability density is $\frac{1}{Vol(D)}$. In this case, Theorem 1 becomes an asymptotic isoperimetric-type equality.

Corollary 1. *Let $X(t)$ be normally reflected Brownian motion in a domain $D \subset R^d$ of finite volume. Let $x \in D$ and $y \in \bar{D} - \{x\}$:*

(i) *If $d = 2$, then*

$$\lim_{\varepsilon \to 0} \frac{E_y \tau_{B_\varepsilon(x)}}{-\log \varepsilon} = \frac{Vol(D)}{\pi};$$

(ii) *If $d \geqslant 3$, then*

$$\lim_{\varepsilon \to 0} \frac{E_y \tau_{B_\varepsilon(x)}}{\varepsilon^{2-d}} = \frac{2Vol(D)}{d(d-2)\omega_d}.$$

The above results are asymptotic ones. In the case of Brownian motion with oblique reflection, we can give an exact calculation for the hitting time of a ball of fixed radius from distinguished starting points. Let $x \in D$ and let $0 < R < l$ be such that $\bar{B}_l(x) \subset D$. Then there exists a point $z_{l;R} \in \partial B_l(x)$ for which $E_{z_{l;R}} \tau_{B_R(x)}$ can be calculated explicitly. Before stating the result, we recall a few facts about positive recurrent obliquely reflected Brownian motion. As noted above, the process is reversible if and only if $v = n$ and, if it is reversible, it is positive recurrent if and only if $Vol(D) < \infty$, in which case the invariant density is $\frac{1}{Vol(D)}$. When the process is not reversible, this simple condition for positive recurrence fails. If D is bounded, then the process is always positive recurrent; however, if D is unbounded, then the question of positive recurrence is highly non-trivial. We can write $v(x) = c(x)n(x) - T(x)$, where $c \in (0, 1]$ and $T \not\equiv 0$ is a tangent vector field on ∂D. The density μ of an invariant measure must satisfy the adjoint equation: $\frac{1}{2}\Delta\mu = 0$ in D and $\nabla\mu \cdot n + \nabla \cdot (\frac{T}{c}\mu) = 0$ on ∂D. In particular, Lebesgue measure will be invariant only if $\nabla \cdot \frac{T}{c} \equiv 0$ on ∂D, which is in fact impossible if d is odd and ∂D is compact. It is not hard to give examples where $Vol(D) < \infty$ but the process is not positive recurrent, as well as examples where $Vol(D) = \infty$ but the process is positive recurrent.

Theorem 2. *Let $X(t)$ be v-reflected Brownian motion in a domain $D \subset R^d$. Assume that the process is positive recurrent and let μ denote the invariant probability density. Let $x \in D$ and let $l > 0$ be such that $\bar{B}_l(x) \subset D$. For each $R \in (0, l)$, there exists a $z_{l;R} \in \partial B_l(x)$ such that*

(i) *if $d = 2$, then*

$$E_{z_{l;R}} \tau_{B_R(x)} = \frac{1}{\mu(B_R(x))} R^2 \log \frac{l}{R} - \frac{1}{2}(l^2 - R^2); \tag{1.2i}$$

(ii) *if $d \geqslant 3$, then*

$$E_{z_{l;R}} \tau_{B_R(x)} = \frac{2R^d}{d(d-2)\mu(B_R(x))} (R^{2-d} - l^{2-d}) - \frac{1}{d}(l^2 - R^2). \tag{1.2ii}$$

Remark. Consider Theorem 2 when the reflection vector is normal, in which case we assume that $Vol(D) < \infty$ and we have $\mu(B_R(x)) = \frac{\omega_d R^d}{Vol(D)}$. Then the theorem indicates that for $x \in D$ and for $0 < R < l$ such that $\bar{B}_l(x) \subset D$, one can find a point $z_{l;R}$ such that $E_{z_{l;R}} \tau_{B_R(x)}$ is equal to the common value that one obtains for the expected value of $\tau_{B_R(x)}$ starting from any point on $\partial B_l(x)$ in the case that the domain is a ball of the same volume centered at x.

We note that Theorem 1 can also be thought of as giving a formula for the invariant density in terms of the asymptotic behavior of expected hitting times of small balls. Green's function and potential theory afficianados might want to represent this as follows: let $G_{B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)}(\cdot,\cdot)$ denote the Green's function for L in $D - \bar{B}_\varepsilon^{a^{\mathrm{inv}}(x)}(x)$ with the oblique derivative boundary condition in the direction $v = cn - T$ at ∂D (if $D \not\equiv R^d$) and the Dirichlet boundary condition at $\partial B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)$. (The probabilistic representation is given by $G_{B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)}(z, A) = E_z \int_0^{\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)}} 1_{\{A\}}(X(t))\,dt$, where $G_{B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)}(z, A) = \int_A G_{B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)}(z, y)\,dy$.) Then

$$E_z \tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)} = \int_{D - B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)} G_{B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)}(z, y)\,dy$$

for $z \in D - B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)$; thus, the unique solution $\mu > 0$ to the adjoint equation $\tilde{L}\mu = 0$ in D and $\nabla \mu \cdot n + \nabla \cdot (\frac{T}{c}\mu) = 0$ on ∂D, where $\tilde{L} = \frac{1}{2}\nabla \cdot a\nabla - b \cdot \nabla - \nabla \cdot b$, is given by

$$\mu(x) = \lim_{\varepsilon \to 0} \frac{-\log \varepsilon}{\pi Det^{\frac{1}{2}}(a(x)) \int_{D - B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)} G_{B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)}(z, y)\,dy}, \quad \text{if } d = 2 \tag{1.3i}$$

and

$$\mu(x) = \lim_{\varepsilon \to 0} \frac{2\varepsilon^{2-d}}{d(d-2)\omega_d Det^{\frac{1}{d}}(a(x)) \int_{D - B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)} G_{B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)}(z, y)\,dy}, \quad \text{if } d \geqslant 3 \tag{1.3ii}$$

for any $z \in D - \{x\}$.

In fact, (1.3) along with a slightly modified version of it will be used to prove our result concerning the asymptotic behavior of the principal eigenvalue, which we now consider. First consider the case that D is bounded. Of course, the principal eigenvalue for the operator $-L$ in D with the oblique derivative boundary condition in the direction v on ∂D is 0 and the corresponding eigenvector is constant. For $x \in D$, let $\lambda_\varepsilon(x)$ denote the classical principal eigenvalue for $-L$ in $D - \bar{B}^{a^{\mathrm{inv}}(x)}(x)$ with the oblique derivative boundary condition in the direction v on ∂D and the Dirichlet boundary condition on $\partial B^{a^{\mathrm{inv}}(x)}(x)$. Since the domain is bounded, the operator in question has a compact resolvent and it follows from the Krein–Rutman theorem that $\lambda_\varepsilon(x)>0$ (see [7] and also [9, Chapter 3] which treats the case of Dirichlet boundary rather than oblique reflection). If D is unbounded, then we need to define the principal eigenvalue $\lambda_\varepsilon(x)$ carefully and we need to make the assumption that it is in fact strictly positive. We define the *generalized* principal eigenvalue as follows: let $\{D_k\}$ be an increasing sequence of bounded domains with smooth boundaries satisfying $\cup_{k=1}^\infty \bar{D}_k = \bar{D}$ and consider the operator $-L$ on $D_k - \bar{B}_\varepsilon^{a^{\mathrm{inv}}(x)}(x)$ with the oblique derivative boundary condition in the direction v on the relative interior of $\partial D_k \cap \partial D$ in ∂D_k and with the Dirichlet boundary condition on the relative interior of $\partial D_k - \partial D$ in ∂D_k and on $\partial B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)$. No boundary condition is imposed on the relative boundary of ∂D_k in ∂D. A principal eigenvalue $\lambda_\varepsilon^{(k)}(x)$ exists for this problem [7] and is positive and monotone non-increasing. The generalized principal eigenvalue is defined as $\lambda_\varepsilon(x) = \lim_{k\to\infty} \lambda_\varepsilon^{(k)}(x)$.

We will need the following hypothesis.

Hypothesis 1. For some $\varepsilon_0>0$ and some $\bar{\lambda}>0$,

$$\sup_{y\in\bar{D}-B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}(x)} E_y \exp\left(\bar{\lambda}\tau_{B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}(x)}\right) < \infty.$$

Lemma 1. (i) *Hypothesis* 1 *always holds if D is bounded.*

(ii) *If*

$$E_y \exp\left(\bar{\lambda}\tau_{B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}(x)}\right) < \infty \quad \text{for } y\in\bar{D} - B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}(x)$$

for some $\varepsilon_0>0$ and some $\bar{\lambda}>0$, then $\lambda_\varepsilon(x)>0$ for all $\varepsilon>0$.

From Lemma 1 it follows in particular that $\lambda_\varepsilon(x)>0$ for $\varepsilon>0$ whenever Hypothesis 1 is in effect.

We can now state the theorem.

Theorem 3. *Let $x\in D$ and let $\lambda_\varepsilon(x)$ denote the (generalized) principal eigenvalue for L in $D - \bar{B}_\varepsilon^{a^{\mathrm{inv}}(x)}(x)$ with the oblique derivative boundary condition at ∂D (if $D\not\equiv R^d$) and the Dirichlet boundary condition at $\partial B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)$. If D is unbounded, assume that Hypothesis* 1 *holds.*

(i) *If $d=2$, then*

$$\lim_{\varepsilon\to 0} (-\log\varepsilon)\lambda_\varepsilon(x) = \pi Det^{\frac{1}{2}}(a(x))\mu(x); \tag{1.4i}$$

(ii) *If* $d \geqslant 3$, *then*

$$\lim_{\varepsilon\to 0} \varepsilon^{2-d}\lambda_\varepsilon(x) = \frac{d(d-2)\omega_d Det^{\frac{1}{d}}(a(x))\mu(x)}{2}. \tag{1.4ii}$$

Remark 1. Note that Theorems 1 and 3 show that $E_x\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)}$ and $\lambda_\varepsilon(x)$ have reciprocal asymptotic behavior. This is not surprising because if $\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)}$ had an exact exponential distribution, then its expectation and the asymptotic rate of decay of its tail probabilities would be reciprocals, and this latter quantity is essentially $\lambda_\varepsilon(x)$.

Remark 2. From the remark following Theorem 1, it follows that in the reversible case, the right-hand side of (1.4) is given explicitly in terms of the coefficients of L.

Remark 3. A corresponding formula in the case that the operator is Δ and the *Dirichlet* boundary condition is placed on ∂D was obtained in [6], which actually treated all the eigenvalues, not just the principal one. A similar result in the case of a closed manifold was obtained in [1]. Note that this problem is self-adjoint.

Remark 4. In another paper, the technique used in the proof of Theorem 1 is used along with other techniques to obtain the asymptotic behavior of the principal eigenvalue in regions with many small holes in the case of the Laplacian with the Neumann boundary condition [10]. For other papers concerning the shift of the principal eigenvalue in regions with many holes, see for example [4,11,12] as well as the exposition in [14, Chapter 22].

Consider the case of Brownian motion with normal reflection in a domain D of finite volume. By Lemma 1(i), if the domain is bounded then Hypothesis 1 is always satisfied. An interesting question is whether Hypothesis 1 is *ever* satisfied (or whether $\lambda_\varepsilon(x)$ is *ever* positive) when D is unbounded. As evidence that $\lambda_\varepsilon(x) = 0$ whenever D is unbounded, consider the case that D is horn shaped of the form $D = \{z = (s,w) \in R \times R^{d-1} : |w| < H(|s|)\}$, where H is a positive, continuous function satisfying $\int_0^\infty H^{d-1}(s)\,ds < \infty$. Then $D \subset R^d$ has finite volume. Look now at the punctured domain $D - \bar{B}_\varepsilon(0)$, for small $\varepsilon > 0$. The operator $-\frac{1}{2}\Delta$ in this domain with the Neumann boundary condition at ∂D and the Dirichlet boundary condition at $\partial B_\varepsilon(0)$ is self-adjoint; thus, an upper estimate on the principal eigenvalue can be obtained via the Rayliegh–Ritz quotient [13]. For $\delta > 0$, define the test function $f_\delta(z) = \sin \pi\delta(s-\varepsilon)$ for $s \in (\varepsilon, \frac{1}{\delta} + \varepsilon)$, $f_\delta(z) = 0$ for $s \in [0, \varepsilon]$ and $s \geqslant \frac{1}{\delta} + \varepsilon$, and f_δ extended to negative values of s as an even function. Then f_δ vanishes on $\partial B_\varepsilon(0)$ and $\frac{\frac{1}{2}\int_{D-B_\varepsilon(0)} |\nabla f_\delta(z)|^2\,dz}{\int_{D-B_\varepsilon(0)} f_\delta^2(z)\,dz} \leqslant \frac{\pi^2}{2}\delta^2$. Thus, by the Rayliegh–Ritz formula, $\lambda_\varepsilon(0) = 0$, and by Lemma 1 it also follows that Hypothesis 1 is not satisfied.

Analogous to Corollary 1, we have the following corollary, which in light of the above discussion, might be vacuous if D is unbounded.

Corollary 2. *Let $X(t)$ be normally reflected Brownian motion in a domain $D\subset R^d$ of finite volume. If D is unbounded, assume that Hypothesis* 1 *holds. Let $x\in D$.*

(i) *If $d=2$, then*

$$\lim_{\varepsilon\to 0} -\log \varepsilon\lambda_\varepsilon(x)=\frac{\pi}{Vol(D)};$$

(ii) *If $d\geqslant 3$, then*

$$\lim_{\varepsilon\to 0} \varepsilon^{2-d}\lambda_\varepsilon(x)=\frac{d(d-2)\omega_d}{2Vol(D)}.$$

We will prove Theorem 2 in Section 2, Theorem 1 in Section 3, and Theorem 3 and Lemma 1 in Section 4.

2. Proof of Theorem 2

Let x, R and l be as in the statement of the theorem. Define inductively stopping times $\sigma_1=\inf\{t\geqslant 0: X(t)\in\partial B_l(x)\}$, $\eta_n=\inf\{t\geqslant\sigma_n: X(t)\in\partial B_R(x)\}$, and $\sigma_{n+1}=\inf\{t\geqslant\eta_n: X(t)\in\partial B_l(x)\}$, $n=1,2,\ldots$. Note that under P_z with $z\in\partial B_l(x)$, we have $\eta_1=\inf\{t\geqslant 0: X(t)\in\partial B_R(x)\}$. Under P_z, with $z\in\partial B_R(x)\cup\partial B_l(x)$, the sequence $X(\sigma_1), X(\eta_1), X(\sigma_2), X(\eta_2),\ldots$ is a Markov process on a compact space and consequently possesses an invariant probability measure. Thus, there exist probability measures m_1 and m_2 on $\partial B_R(x)$ and $\partial B_l(x)$, respectively, such that $P_{m_1}(X(\sigma_1)\in\cdot)=m_2(\cdot)$ and $P_{m_2}(X(\eta_1)\in\cdot)=m_1(\cdot)$.

We now use Hasminski's construction of the invariant measure for a recurrent diffusion process. (See [3] where the construction is carried out in the case of an unrestricted diffusion on all of space; the same construction works for any Feller process.) With an abuse of notation, we let μ denote the invariant probability measure as well as its density. By Hasminskii's construction

$$\mu(A)=\frac{E_{m_1}\int_0^{\eta_1} 1_A(X(t))\,dt}{E_{m_1}\eta_1}\quad\text{for } A\subset\bar{D}. \tag{2.1}$$

We now express $E_{m_2}\tau_{B_R(x)}$ as the sum of two terms.

$$E_{m_2}\tau_{B_R(x)}=E_{m_2}\int_0^{\eta_1} 1_{D-B_l(x)}(X(t))\,dt+E_{m_2}\int_0^{\eta_1} 1_{B_l(x)-B_R(x)}(X(t))\,dt. \tag{2.2}$$

Using (2.1) and the invariance property of m_2, we write the first term on the right-hand side of (2.2) as

$$E_{m_2}\int_0^{\eta_1} 1_{D-B_l(x)}(X(t))\,dt = E_{m_1}\int_0^{\eta_1} 1_{D-B_l(x)}(X(t))\,dt$$
$$= \mu(D-B_l(x))E_{m_1}\eta_1 \tag{2.3}$$

and the second term on the right-hand side of (2.2) as

$$\begin{aligned}
&E_{m_2}\int_0^{\eta_1} 1_{B_l(x)-B_R(x)}(X(t))\,dt\\
&= E_{m_1}\int_0^{\eta_1} 1_{B_l(x)-B_R(x)}(X(t))\,dt - E_{m_1}\int_0^{\sigma_1} 1_{B_l(x)-B_R(x)}(X(t))\,dt\\
&= \mu(B_l(x)-B_R(x))E_{m_1}\eta_1 - E_{m_1}\int_0^{\sigma_1} 1_{B_l(x)-B_R(x)}(X(t))\,dt.
\end{aligned}\tag{2.4}$$

Setting $A = B_R(x)$ in (2.1), we have

$$\begin{aligned}
E_{m_1}\eta_1 &= \frac{E_{m_1}\int_0^{\eta_1} 1_{B_R(x)}(X(t))\,dt}{\mu(B_R(x))}\\
&= \frac{E_{m_1}\int_0^{\sigma_1} 1_{B_R(x)}(X(t))\,dt}{\mu(B_R(x))}.
\end{aligned}\tag{2.5}$$

From (2.2)–(2.5), we obtain

$$\begin{aligned}
E_{m_2}\tau_{B_R(x)} &= \frac{1-\mu(B_R)}{\mu(B_R)}E_{m_1}\int_0^{\sigma_1} 1_{B_R(x)}(X(t))\,dt\\
&\quad - E_{m_1}\int_0^{\sigma_1} 1_{B_l(x)-B_R(x)}(X(t))\,dt.
\end{aligned}\tag{2.6}$$

Note that the two expectations on the right-hand side of (2.6) are actually independent of the particular measure m_1 because of symmetry considerations.

Let $v_{R,l}(r)$ denote the solution to

$$\begin{aligned}
&\frac{1}{2}v''(r) + \frac{d-1}{2r}v'(r) = -1_{[0,R]}(r), \quad r\in(0,l),\\
&v'(0) = 0, \quad v(l) = 0.
\end{aligned}\tag{2.7}$$

Then, as is well known,

$$E_{m_1}\int_0^{\sigma_1} 1_{B_R(x)}(X(t))\,dt = v_{R,l}(R). \tag{2.8}$$

Solving the differential equation in (2.7) separately on $[0, R]$ and $[R, l]$, and matching the solutions and their first derivatives at $r = R$, we obtain

$$E_{m_1}\int_0^{\sigma_1} 1_{B_R(x)}(X(t))\,dt = v_{R,l}(R) = \begin{cases} R^2\log\frac{l}{R} & \text{if } d=2,\\ \frac{2R^d}{d(d-2)}(R^{2-d}-l^{2-d}) & \text{if } d\geqslant 3.\end{cases} \tag{2.9}$$

Similarly, let $u_{R,l}(r)$ denote the solution to

$$\begin{aligned}&\frac{1}{2}u'' + \frac{d-1}{2r}u' = -1, \quad r\in(0,l),\\ &u'(0)=0, \quad u(l)=0.\end{aligned} \tag{2.10}$$

Then $E_{m_1}\sigma_1 = u_{R,l}(R)$ and consequently,

$$E_{m_1}\int_0^{\sigma_1} 1_{B_l(x)-B_R(x)}(X(t))\,dt = u_{R,l}(R) - v_{R,l}(R). \tag{2.11}$$

Solving (2.10) for $u_{R,l}$ and using (2.11), we obtain

$$E_{m_1}\int_0^{\sigma_1} 1_{B_l(x)-B_R(x)}(X(t))\,dt = \tfrac{1}{d}(l^2-R^2) - v_{R,R}(R) \quad \text{for all } d\geqslant 2. \tag{2.12}$$

Now (1.2) follows from (2.6), (2.9), (2.12) and the fact that $E_z\tau_{B_R(x)}$ is continuous in z. $\square$

3. Proof of Theorem 1

Let $x\in D$. Define stopping times σ_n and η_n and identify probability measures m_1 and m_2 in the same way as at the beginning of the proof of Theorem 2, except this time instead of using the domains $B_R(x)$ and $B_l(x)$, use the domains $B_\varepsilon^{a^{\mathrm{inv}}(x)}(x)$ and $B_l(x)$, where l satisfies $\bar{B}_l(x)\subset D$ and ε is sufficiently small so that $\bar{B}_\varepsilon^{a^{\mathrm{inv}}(x)}(x)\subset B_l(x)$. As the notation is awkward, in the sequel we will write $B_\cdot^{a^{\mathrm{inv}}(x)}$ for $B_\cdot^{a^{\mathrm{inv}}(x)}(x)$. The calculations from (2.1) to (2.6) hold in the present context. Substituting $B_\varepsilon^{a^{\mathrm{inv}}(x)}$ for $B_R(x)$ in (2.6), we have

$$\begin{aligned}E_{m_2}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}} = &\frac{1-\mu\left(B_\varepsilon^{a^{\mathrm{inv}}(x)}\right)}{\mu\left(B_\varepsilon^{a^{\mathrm{inv}}(x)}\right)}E_{m_1}\int_0^{\sigma_1} 1_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}(X(t))\,dt\\ &- E_{m_1}\int_0^{\sigma_1} 1_{B_l(x)-B_\varepsilon^{a^{\mathrm{inv}}(x)}}(X(t))\,dt.\end{aligned} \tag{3.1}$$

The second term on the right-hand side of (3.1) remains bounded when $\varepsilon\to 0$; thus the asymptotic behavior of the left-hand side of (3.1) coincides with the asymptotic behavior of the first term on the right-hand side of (3.1) as $\varepsilon \to 0$. Let $G^{(l)}(x, y)$ denote the Green's function corresponding to the diffusion process in $B_l(x)$ which is killed upon hitting $\partial B_l(x)$. The expected value appearing in the first term on the right-hand side of (3.1) can be rewritten in terms of $G^{(l)}$ as follows:

$$E_{m_1} \int_0^{\sigma_1} 1_{B_\varepsilon^{a^{\text{inv}}(x)}}(X(t))\,dt = \int_{\partial B_\varepsilon^{a^{\text{inv}}(x)}} \int_{B_\varepsilon^{a^{\text{inv}}(x)}} G^{(l)}(z, y)\,dy\ m_1(dz). \tag{3.2}$$

The Green's function exhibits the following behavior at its pole:

$$G^{(l)}(z, y) = -\frac{1}{2\pi Det^{\frac{1}{2}}(a(z))} \log((y - z), a^{\text{inv}}(z)(y - z))$$
$$+ \text{lower order terms, as } y \to z \text{ if } d = 2,$$

$$G^{(l)}(z, y) = \frac{2}{d(d-2)\omega_d Det^{\frac{1}{2}}(a(z))}((y - z), a^{\text{inv}}(z)(y - z))^{\frac{2-d}{2}}$$
$$+ \text{lower order terms, as } y \to z \quad \text{if } d \geqslant 3. \tag{3.3}$$

(See [5, p. 17] and note that in this reference ω_d is the surface area of the unit ball rather than the volume.) Since $a(z)$ is continuous, it follows that as $\varepsilon \to 0$, the leading term in the asymptotics for the right-hand side of (3.2) is the same as what one would get with $G^{(l)}(z, y)$ replaced by the explicit term on the right-hand side of (3.3), but with $a(z)$ and $a^{\text{inv}}(z)$ replaced by $a(x)$ and $a^{\text{inv}}(x)$. Letting $(a^{\text{inv}})^{\frac{1}{2}}(x)$ denote the positive definite square root of $a^{\text{inv}}(x)$ and making the change of variables $\frac{(a^{\text{inv}})^{\frac{1}{2}}(x)}{Det^{\frac{1}{2d}}(a^{\text{inv}}(x))}(y - z) = u$, we calculate for $d \geqslant 3$,

$$\frac{1}{Det^{\frac{1}{2}}(a(x))} \int_{B_\varepsilon^{a^{\text{inv}}(x)}} ((y - z), a^{\text{inv}}(x)(y - z))^{\frac{2-d}{2}}\,dy$$
$$= \frac{1}{Det^{\frac{1}{d}}(a(x))} \int_{\left|u + \frac{(a^{\text{inv}})^{\frac{1}{2}}(x)}{Det^{\frac{1}{2d}}(a^{\text{inv}}(x))}(z - x)\right| < \varepsilon} |u|^{2-d}\,du. \tag{3.4}$$

Since $z \in \partial B_\varepsilon^{a^{\text{inv}}(x)}$, it follows that $\left|\frac{(a^{\text{inv}})^{\frac{1}{2}}(x)}{Det^{\frac{1}{2d}}(a^{\text{inv}}(x))}(z - x)\right| = \varepsilon$, and thus the integral on the right-hand side of (3.4) is independent of $z \in \partial B_\varepsilon^{a^{\text{inv}}(x)}$. This is imperative because we have no control over the probability measure $m_1(dz)$ on $\partial B_\varepsilon^{a^{\text{inv}}(x)}$. We conclude then that

$$\frac{2}{d(d-2)\omega_d Det^{\frac{1}{2}}(a(x))}\int_{\partial B_\varepsilon^{a^{\mathrm{inv}}(x)}}\int_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}((y-z),a^{\mathrm{inv}}(x)(y-z))^{\frac{2-d}{2}}\,dy\;m_1(dz)$$
$$=\frac{2}{d(d-2)\omega_d Det^{\frac{1}{2}}(a(x))}\int_{|u+w_\varepsilon|<\varepsilon}|u|^{2-d}\,du,\quad\text{for any } w_\varepsilon \text{ satisfying } |w_\varepsilon|=\varepsilon. \tag{3.5}$$

We have now established that the leading term in the asymptotics for $E_{m_1}\int_0^{\sigma_1}1_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}(X(t))\,dt=\int_{\partial B_\varepsilon^{a^{\mathrm{inv}}(x)}}\int_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}G^{(l)}(z,y)\,dy\;m_1(dz)$ coincides with the leading term in the asymptotics for the right-hand side of (3.5). Note that when $L=\frac{1}{2}\Delta$, in which case $a(x)=I$, an exact expression for $E_{m_1}\int_0^{\sigma_1}1_{B_\varepsilon(x)}(X(t))\,dt$ has already been given in (2.9) (with R replaced by ε). Comparing (2.9) with (3.5) then allows us to deduce that

$$\int_{|u+w_\varepsilon|<\varepsilon}|u|^{2-d}\,du=\omega_d\varepsilon^2+o(\varepsilon^2),\quad\text{as } \varepsilon\to 0. \tag{3.6}$$

From (3.5) and (3.6) we conclude that

$$\int_{\partial B_\varepsilon^{a^{\mathrm{inv}}(x)}}\int_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}G^{(l)}(z,y)\,dy\;m_1(dz)\sim\frac{2\varepsilon^2}{d(d-2)Det^{\frac{1}{d}}(a(x))},$$
$$\text{as } \varepsilon\to 0 \text{ if } d\geqslant 3. \tag{3.7}$$

The same argument in the case $d=2$ leads to

$$\int_{\partial B_\varepsilon^{a^{\mathrm{inv}}(x)}}\int_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}G^{(l)}(z,y)\,dy\;m_1(dz)\sim-\frac{\varepsilon^2\log\varepsilon}{Det^{\frac{1}{2}}(a(x))}\quad\text{as } \varepsilon\to 0 \text{ if } d=2. \tag{3.8}$$

From (3.1), (3.2), (3.7), (3.8) and the fact that $Vol(B_\varepsilon^{a^{\mathrm{inv}}(x)})=Vol(B_\varepsilon(x))=\omega_d\varepsilon^d$, we conclude that

$$\lim_{\varepsilon\to 0}\frac{E_{m_2}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}}{-\log\varepsilon}=\frac{1}{\pi Det^{\frac{1}{2}}(a(x))\mu(x)}\quad\text{if } d=2 \tag{3.9i}$$

and

$$\lim_{\varepsilon\to 0}\frac{E_{m_2}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}}{\varepsilon^{2-d}}=\frac{2}{d(d-2)\omega_d Det^{\frac{1}{d}}(a(x))\mu(x)}\quad\text{if } d\geqslant 3. \tag{3.9ii}$$

Recall that m_2 is a certain probability measure on $B_l(x)$, where l has been chosen so that $\bar{B}_l(x)\subset D$. Thus, in light of (3.9), to complete the proof of Theorem 1 it is enough to show that

$$\lim_{\varepsilon\to 0}\frac{E_{y_1}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}}{E_{y_2}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}}=1,\ \text{uniformly for } y_1,y_2 \text{ in compact subsets of } \bar{D}-\{x\}. \quad (3.10)$$

Let B_1 and B_2 be balls centered at x and satisfying $\bar{B}_1\subset B_2\subset D$. Redefine the stopping times σ_n and η_n and the probability measures m_1 and m_2, defined at the beginning of the proof in terms of the domains $B_\varepsilon^{a^{\mathrm{inv}}(x)}$ and $B_l(x)$, in terms of the domains B_1 and B_2. For $y\in\bar{D}-B_2$, we have for ε sufficiently small

$$E_y\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}=E_y\sigma_1+\int_{\partial B_2}E_z\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}P_y(X(\sigma_1)\in dz). \quad (3.11)$$

Since $E_y\sigma_1$ is bounded for y in a compact subset of $\bar{D}$, it follows from (3.11) that in order to prove (3.10) for an arbitrary compact subset of $\bar{D}-\{x\}$, it is enough to prove (3.10) for the compact subset ∂B_2.

We now set out to prove (3.10) in this particular case. By Harnack's inequality, it follows that there exists a $C>0$ such that

$$\frac{1}{C}P_{y_1}(X(\eta_1)\in\cdot)\leqslant P_{y_2}(X(\eta_1)\in\cdot)\leqslant CP_{y_1}(X(\eta_1)\in\cdot)\quad\text{for all } y_1,y_2\in\partial B_2. \quad (3.12)$$

(See, for example, [9, Theorem 7.4.5] which treats the case of diffusions that are killed rather than reflected at the boundary; however, the boundary is irrelevant since the interior Harnack inequality is used.) Thus, writing

$$E_{y_i}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}=E_{y_i}\eta_1+\int_{\partial B_1}E_w\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}P_{y_i}(X(\eta_1)\in dw)\quad\text{for } y_i\in\partial B_2,$$

it follows that

$$\frac{1}{C}\leqslant\liminf_{\varepsilon\to 0}\frac{E_{y_1}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}}{E_{y_2}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}}\leqslant\limsup_{\varepsilon\to 0}\frac{E_{y_1}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}}{E_{y_2}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}}\leqslant C\quad\text{for all } y_1,y_2\in\partial B_2. \quad (3.13)$$

Now (3.12) continues to hold with y_1 replaced by m_2. Since $P_{m_2}(X(\eta_1)\in\cdot)=m_1(\cdot)$, we conclude in particular that

$$P_y(X(\eta_1)\in\cdot)\geqslant\frac{1}{C}m_1(\cdot)\quad\text{for } y\in\partial B_2. \quad (3.14)$$

Using (3.14), we can make a Doeblin-type coupling argument to conclude that

$$P_y(X(\sigma_n)\in\cdot)=a_{n-1}m_2(\cdot)+(1-a_{n-1})\mu_{y,n}(\cdot)\quad\text{for all } y\in\partial B_2,\ n=2,3,\ldots, \quad (3.15)$$

where $\mu_{y,n}(\cdot)$ is a probability measure on ∂B_2, and a_n is defined recursively by $a_1=\frac{1}{C}$, $a_{n+1}=a_n+\frac{1}{C}(1-a_n)$ for $n\geqslant 1$. In particular, $\lim_{n\to\infty}a_n=1$. The coupling is achieved as follows. Start the process from $y\in\partial B_2$ at time $t=0=\sigma_1$ and wait until time $t=\eta_1$. By (3.14), the measure $P_y(X(\eta_1)\in\cdot)$ dominates $\frac{1}{C}m_1$. Since $P_{m_1}(X(\sigma_1)\in\cdot)=m_2(\cdot)$, it follows that the measure $P_y(X(\sigma_2)\in\cdot)$ dominates $\frac{1}{C}m_2$. Thus, by time $t=\sigma_2$, the process is running from equilibrium with probability $\frac{1}{C}$. With probability $1-\frac{1}{C}$ it is running from some arbitrary distribution, but applying the same reasoning again on another circuit shows that by time $t=\sigma_3$, the process is running from equilibrium with probability $\frac{1}{C}+\frac{1}{C}(1-\frac{1}{C})$. Continuing like this gives the coupling as above. (For more details, see for example [8, pp. 6–8].) Using (3.15), we have

$$\begin{aligned} E_y\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}} &= E_y\sigma_n\wedge\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}} + a_{n-1}P_y\left(\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}>\sigma_n\right)E_{m_2}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}} \\ &\quad + (1-a_{n-1})P_y\left(\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}>\sigma_n\right)E_{\mu_{y,n}}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}} \quad \text{for } y\in\partial B_2, \\ &\quad n=1,2,\dots . \end{aligned} \tag{3.16}$$

The first term on the right-hand side of (3.16) remains bounded when $\varepsilon\to 0$. Also, note that $P_y\left(\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}>\sigma_n\right)$ as function of $y\in\partial B_2$ is increasing pointwise to 1 as $\varepsilon\to 0$. Thus, by Dini's Theorem, the convergence is uniform. By (3.13), $E_{\mu_{y,n}}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}\leqslant CE_{m_2}\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}$. Using these facts along with (3.16) and the fact that $\lim_{n\to\infty}a_n=1$, (3.10) now follows for $y_1,y_2\in\partial B_2$. This completes the proof of (3.10). □

4. Proof of Theorem 3 and Lemma 1

We first prove Theorem 3 and then Lemma 1, although the proof of Theorem 3 uses Lemma 1.

Proof of Theorem 3. Recall the definition of $G_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}$ before (1.3). We prove the theorem by using a slight variation of (1.3) along with (1.3) and some ideas from criticality theory for elliptic operators. Let $\{D_n\}_{n=1}^\infty$ be a sequence of bounded domains satisfying $\bar{D}_n\subset D_{n+1}$ and $\bigcup_{n=1}^\infty D_n=D$. The variant of (1.3) that we need is this:

$$\mu(x)=\lim_{n\to\infty,\delta\to 0}\lim_{\varepsilon\to 0}\frac{-\log\varepsilon}{\pi Det^{\frac{1}{2}}(a(x))\int_{D_n-B_\delta(x)}G_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}(z,y)\,dy}\quad \text{if } d=2 \tag{4.1i}$$

and

$$\mu(x) = \lim_{n\to\infty,\delta\to 0} \lim_{\varepsilon\to 0} \frac{2\varepsilon^{2-d}}{d(d-2)\omega_d Det^{\frac{1}{d}}(a(x)) \int_{D_n - B_\delta(x)} G_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}(z,y)\,dy}$$

$$\text{if } d\geqslant 3, \tag{4.1ii}$$

for any $z\in D-\{x\}$. To prove (4.1), let $B_l(x), B_\varepsilon^{a^{\mathrm{inv}}(x)}, \sigma_n$ and η_n be as defined at the beginning of Section 3. Consider $\varepsilon>0$, $\delta\in(0,l)$ and n such that $\bar{B}_\varepsilon^{a^{\mathrm{inv}}(x)}\subset B_\delta(x)$ and $\bar{B}_l(x)\subset D_n$. Using (2.1), we make a calculation very similar to (2.2)–(2.6). We have

$$\begin{aligned} E_{m_2}\int_0^{\eta_1} 1_{D_n-B_\delta(x)}(X(t))\,dt &= E_{m_2}\int_0^{\eta_1} 1_{D_n-B_l(x)}(X(t))\,dt \\ &\quad + E_{m_2}\int_0^{\eta_1} 1_{B_l(x)-B_\delta(x)}(X(t))\,dt. \end{aligned} \tag{4.2}$$

Using (2.1) and the invariance property of m_2, we write the first term on the right-hand side of (4.2) as

$$\begin{aligned} E_{m_2}\int_0^{\eta_1} 1_{D_n-B_l(x)}(X(t))\,dt &= E_{m_1}\int_0^{\eta_1} 1_{D_n-B_l(x)}(X(t))\,dt \\ &= \mu(D_n - B_l(x))E_{m_1}\eta_1 \end{aligned} \tag{4.3}$$

and the second term on the right-hand side of (4.2) as

$$\begin{aligned} &E_{m_2}\int_0^{\eta_1} 1_{B_l(x)-B_\delta(x)}(X(t))\,dt \\ &\quad = E_{m_1}\int_0^{\eta_1} 1_{B_l(x)-B_\delta(x)}(X(t))\,dt - E_{m_1}\int_0^{\sigma_1} 1_{B_l(x)-B_\delta(x)}(X(t))\,dt \\ &\quad = \mu(B_l(x)-B_\delta(x))E_{m_1}\eta_1 - E_{m_1}\int_0^{\sigma_1} 1_{B_l(x)-B_\delta(x)}(X(t))\,dt. \end{aligned} \tag{4.4}$$

Setting $A = B_\varepsilon^{a^{\mathrm{inv}}(x)}$ in (2.1), we have

$$\begin{aligned} E_{m_1}\eta_1 &= \frac{E_{m_1}\int_0^{\eta_1} 1_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}(X(t))\,dt}{\mu\left(B_\varepsilon^{a^{\mathrm{inv}}(x)}\right)} \\ &= \frac{E_{m_1}\int_0^{\sigma_1} 1_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}(X(t))\,dt}{\mu\left(B_\varepsilon^{a^{\mathrm{inv}}(x)}\right)}. \end{aligned} \tag{4.5}$$

From (4.2)–(4.5), we obtain

$$E_{m_2}\int_0^{\eta_1} 1_{D_n-B_\delta(x)}(X(t))\,dt=\frac{\mu(D_n-B_\delta(x))}{\mu\left(B_\varepsilon^{a^{\mathrm{inv}}(x)}\right)}E_{m_1}\int_0^{\sigma_1} 1_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}(X(t))\,dt - E_{m_1}\int_0^{\sigma_1} 1_{B_l(x)-B_\delta(x)}(X(t))\,dt. \tag{4.6}$$

If we replace (3.1) with (4.6) and continue with the argument in Section 3, we obtain instead of (1.1)

$$\lim_{\varepsilon\to 0}\frac{E_y\int_0^{\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}} 1_{D_n-B_\delta(x)}(X(t))\,dt}{-\log\varepsilon}=\frac{\mu(D_n-B_\delta(x))}{\pi Det^{\frac{1}{2}}(a(x))\mu(x)}\quad \text{if } d=2, \tag{4.7i}$$

$$\lim_{\varepsilon\to 0}\frac{E_y\int_0^{\tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}} 1_{D_n-B_\delta(x)}(X(t))\,dt}{\varepsilon^{2-d}}=\frac{2\mu(D_n-B_\delta(x))}{d(d-2)\omega_d Det^{\frac{1}{d}}(a(x))\mu(x)}\quad \text{if } d\geqslant 3. \tag{4.7ii}$$

Now just as (1.3) is the analytical equivalent of (1.1), (4.1) without the term $\lim_{n\to\infty,\delta\to 0}$ is the analytical equivalent of (4.7). Thus, (4.1) follows by noting that $\lim_{n\to\infty,\delta\to 0}\mu(D_n-B_\delta(x))=1$.

The rest of the proof uses some ideas from criticality theory for elliptic operators—see [7,9, Chapter 4]. (We will give references from [9], which treats the case that the entire boundary is given the Dirichlet boundary condition implicitly.) We say "implicitly", because in fact no boundary condition is given in [9], but when the boundary is smooth, this is equivalent to the Dirichlet boundary condition. The results carry over to the case at hand as can be seen from [7]. Note, however, that our eigenvalue corresponds to the operator $-L$, while in [9] the operator in question is L.) The eigenvalue $\lambda_\varepsilon(x)$ is monotone non-decreasing in ε; let $\lambda_0(x)=\lim_{\varepsilon\to 0}\lambda_\varepsilon(x)$. Then it follows that $\lambda_0(x)$ is the generalized principal eigenvalue for $-L$ on $D-\{x\}$ with the oblique derivative boundary condition in the direction v [9, Theorem 4.4.1]. Since we have assumed that the diffusion corresponding to L on D with v-reflection at ∂D is recurrent, and since the point x is polar for the L diffusion, the diffusion corresponding to L on $D-\{x\}$ with v-reflection at ∂D and absorption at $\{x\}$ is also recurrent. Equivalently, in the language of criticality theory, L on $D-\{x\}$ with the oblique derivative boundary condition in the direction v is a critical operator [9, Theorem 4.3.3]. From this we conclude that $\lambda_0(x)=0$ [9, Theorem 4.3.2] and that the cone of positive harmonic functions for $L+\lambda_0(x)=L$ on $D-\{x\}$ which satisfy the oblique derivative boundary condition in the direction v is one dimensional [9, Theorem 4.3.4]; thus the only such harmonic functions are the constants.

We now show that for small $\varepsilon>0$, the operator $L+\lambda_\varepsilon(x)$ on $D-\bar{B}_\varepsilon^{a^{\mathrm{inv}}(x)}$ with the oblique derivative boundary condition in the direction v is also critical. By Lemma 1 and Hypothesis 1 we have $\lambda_\varepsilon(x)>0$ for $\varepsilon>0$. Fix $\varepsilon_1>0$. Since $\lambda_\varepsilon(x)$ is monotone non-decreasing in ε and since $\lim_{\varepsilon\to 0}\lambda_\varepsilon(x)=0$, it follows that $\lambda_\varepsilon(x)<\lambda_{\varepsilon_1}(x)$ for sufficiently small $\varepsilon>0$. The criticality now follows from [9, Theorem 4.7.2].

By the criticality, it follows again from [9, Theorem 4.3.4] that up to constant multiples, there exists a unique positive $(L+\lambda_\varepsilon(x))$-harmonic function ϕ_ε on $D-B_\varepsilon^{a^{\mathrm{inv}}(x)}$ which satisfies $v\nabla\phi_\varepsilon=0$ on ∂D. Because the boundary $\partial B_\varepsilon^{a^{\mathrm{inv}}(x)}$ is smooth, we have $\phi_\varepsilon=0$ on $\partial B_\varepsilon^{a^{\mathrm{inv}}(x)}$. Fixing some $z_0\in D$ with $z_0\neq x$ and considering $\varepsilon>0$ sufficiently small so that $z_0\notin \bar{B}_\varepsilon^{a^{\mathrm{inv}}(x)}$, we normalize ϕ_ε by requiring $\phi_\varepsilon(z_0)=1$. By standard Schauder estimates and Harnack's inequality, it follows that $\{\phi_\varepsilon\}$ is precompact in the $C^2_{\mathrm{loc}}(\bar{D})$-norm, that the convergence along a subsequence to a limiting function is uniform on compact subsets of $\bar{D}-\{x\}$, and that any limiting function ϕ is positive and satisfies $L\phi=0$ in $D-\{x\}$ and $v\nabla\phi=0$ on ∂D. By the above proved uniqueness of L harmonic functions on $D-\{x\}$ satisfying the oblique derivative boundary condition in the direction v, and by the normalization, we conclude that

$$\lim_{\varepsilon\to 0}\phi_\varepsilon(y)=1 \quad \text{for all } y\in\bar{D}-\{x\}$$
$$\text{and the convergence is uniform on compact subsets of } \bar{D}-\{x\}. \tag{4.8}$$

We now show that

$$\phi_\varepsilon(z)=\int_{D-B_\varepsilon^{a^{\mathrm{inv}}(x)}} G_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}(z,y)\lambda_\varepsilon(x)\phi_\varepsilon(y)\,dy. \tag{4.9}$$

To see this, we note that if there exists a positive solution to $Lu=-\lambda_\varepsilon(x)\phi_\varepsilon$ in $D-\bar{B}_\varepsilon^{a^{\mathrm{inv}}(x)}$ with the oblique derivative boundary condition in the direction v on ∂D, then the right-hand side of (4.9) is the smallest such solution (this is a slight generalization of [9, Theorem 4.3.8]). Since ϕ_ε is a positive solution, it follows that the right-hand side of (4.9), which we will denote by u_ε, is also a solution and $\phi_\varepsilon\geqslant u_\varepsilon$. Define $w_\varepsilon=\phi_\varepsilon-u_\varepsilon$. Then $w_\varepsilon\geqslant 0$, $Lw_\varepsilon=0$ and $v\nabla w_\varepsilon=0$ on ∂D. By the above proved uniqueness, it follows that $w_\varepsilon=c_\varepsilon$ for some non-negative constant. However, $\phi_\varepsilon=0$ on $\partial B_\varepsilon^{a^{\mathrm{inv}}(x)}$ which allows us to conclude that $c_\varepsilon=0$; hence $u_\varepsilon=\phi_\varepsilon$, proving (4.9).

Recalling that $\phi_\varepsilon(z_0)=1$, we can use (4.9) to represent the eigenvalue as

$$\lambda_\varepsilon(x)=\frac{1}{\int_{D-B_\varepsilon^{a^{\mathrm{inv}}(x)}} G_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}(z_0,y)\phi_\varepsilon(y)\,dy}. \tag{4.10}$$

We will show below that

$$\limsup_{\varepsilon\to 0}\ \sup_{y\in D-B_\varepsilon^{a^{\mathrm{inv}}(x)}}|\phi_\varepsilon(y)|<\infty. \tag{4.11}$$

The theorem is now an immediate consequence of (1.3), (4.1), (4.8), (4.10) and (4.11).

It remains to prove (4.11). It follows from (4.8) that it is enough to show that $|\phi_\varepsilon(y)|$ remains uniformly bounded as $\varepsilon \to 0$ for y close to x and for y outside a fixed neighborhood of x. Choose $r_0 > 0$ so that $B_{2r_0}(x) \subset D$. Let $\bar{\lambda} > 0$ denote the principal eigenvalue of $-L$ on $B_{2r_0}(x)$. (As in [9], we impose no boundary condition, but implicitly, the eigenvalue in question is the one corresponding to the Dirichlet boundary condition on $\partial B_{2r_0}(x)$.) Consider $\varepsilon > 0$ sufficiently small so that $\lambda_\varepsilon(x) < \frac{1}{2}\bar{\lambda}$. Since $L + \frac{1}{2}\bar{\lambda}$ on $B_{2r_0}(x)$ is subcritical, we can find a positive solution u to

$$\left(L + \frac{1}{2}\bar{\lambda}\right)u = 0 \quad \text{in } B_{2r_0}(x).$$

We now show that for some $c > 0$:

$$\phi_\varepsilon(y) \leqslant c \sup_{z \in B_{r_0}(x)} u(z) \quad \text{for } y \in B_{r_0}(x) - B_\varepsilon^{a^{\text{inv}}(x)}. \tag{4.12}$$

Define the h-transformed operator $(L + \lambda_\varepsilon(x))^u$ of $L + \lambda_\varepsilon(x)$ (via the function u) by $(L + \lambda_\varepsilon(x))^u f \equiv \frac{1}{u}(L + \lambda_\varepsilon(x))(fu)$ so that $(L + \lambda_\varepsilon(x))^u = L + a\frac{\nabla u}{u} \cdot \nabla + (\lambda_\varepsilon(x) - \frac{1}{2}\bar{\lambda})$. Note that since u is bounded away from 0 in $B_{r_0}(x)$, the coefficient $a\frac{\nabla u}{u}$ is bounded in $B_{r_0}(x)$. We have $(L + \lambda_\varepsilon(x))^u(\frac{\phi_\varepsilon}{u}) = 0$ in $B_{r_0}(x)$. Since the zeroth order term, $\lambda_\varepsilon(x) - \frac{1}{2}\bar{\lambda}$, of the operator $(L + \lambda_\varepsilon(x))^u$ is negative, it follows from the maximum principal that $\sup_{y \in B_{r_0}(x)} \frac{\phi_\varepsilon}{u}(y) = \sup_{y \in \partial B_{r_0}(x)} \frac{\phi_\varepsilon}{u}(y)$. Thus

$$\sup_{y \in B_{r_0}(x)} \phi_\varepsilon(y) \leqslant \sup_{y \in B_{r_0}(x)} u(y) \sup_{y \in \partial B_{r_0}(x)} \frac{\phi_\varepsilon(y)}{u}. \tag{4.13}$$

By (4.8), ϕ_ε is bounded on $\partial B_{r_0}(x)$, uniformly as $\varepsilon \to 0$. Thus, (4.12) follows from (4.13).

Now fix ε_0 and $\bar{\lambda}$ as in Hypothesis 1. By the Feynman–Kac formula,

$$v(y) \equiv E_y \exp(\bar{\lambda}\tau_{B_{\varepsilon_0}^{a^{\text{inv}}(x)}}), \; y \in \bar{D} - \bar{B}_{\varepsilon_0}^{a^{\text{inv}}(x)} \tag{4.14}$$

is a positive solution to $(L + \bar{\lambda})u = 0$ in $D - \bar{B}_{\varepsilon_0}^{a^{\text{inv}}(x)}$ satisfying $v\nabla v = 0$ on ∂D. Consider $\varepsilon < \varepsilon_0$ sufficiently small so that $\lambda_\varepsilon(x) < \bar{\lambda}$. We now show that for some $c > 0$,

$$\phi_\varepsilon \leqslant cv \quad \text{on } D - \bar{B}_{\varepsilon_0}^{a^{\text{inv}}(x)}. \tag{4.15}$$

Since the operator $L + \lambda_\varepsilon(x)$ on $D - \bar{B}_\varepsilon^{a^{\text{inv}}(x)}$ with the oblique derivative boundary condition in the direction v at ∂D is critical, the generalized eigenfunction ϕ_ε corresponding to $\lambda_\varepsilon(x)$ is called a "ground state" and is a positive solution of minimal growth at infinity for $L + \lambda_\varepsilon(x)$. (See [9 section 7.3 in general and Theorem 7.3.8 in particular].) This means in particular that the following maximum principal holds in $D - B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}$: if $w>0$ satisfies $(L+\lambda_\varepsilon(x))w \leqslant 0$ in $D - B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}$, $v\nabla w = 0$ on ∂D and $\phi_\varepsilon \leqslant w$ on $\partial B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}$, then $\phi_\varepsilon \leqslant w$ in $D - B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}$. We apply this with $w = \frac{\sup_{y\in\partial B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}} \phi_\varepsilon(y)}{\inf_{y\in\partial B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}} v(y)} v$. Note that $(L+\lambda_\varepsilon(x))w<0$ because $(L+\lambda_\varepsilon(x))v = (\lambda_\varepsilon(x) - \bar{\lambda})v < 0$. Thus, we conclude that

$$\phi_\varepsilon(z) \leqslant \frac{\sup_{y\in\partial B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}} \phi_\varepsilon(y)}{\inf_{y\in\partial B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}} v(y)} v(z) \quad \text{for } z \in D - B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}. \tag{4.16}$$

By (4.8), ϕ_ε is bounded on $\partial B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}$, uniformly as $\varepsilon \to 0$. Thus, (4.15) follows from (4.16). By Hypothesis 1 and (4.14), v is bounded on $D - \bar{B}_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}$. Using this fact along with (4.8), (4.12) and (4.15) gives us (4.11). □

Proof of Lemma 1. (i) Fix any $\varepsilon_0 > 0$ such that $\bar{B}_{\varepsilon_0}^{a^{\mathrm{inv}}(x)} \subset D$. Let $\rho = \sup_{y\in\bar{D}-B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}} P_y\left(\tau_{B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}} \geqslant 1\right)$. Since $\bar{D}$ is compact, $\rho<1$. An application of the strong Markov property then shows that $P_y(\tau_{B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}} \geqslant n) \leqslant \rho^n$ for all $y \in \bar{D} - B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}$. Thus,

$$\sup_{y\in\bar{D}-B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}} E_y \exp(\bar{\lambda}\tau_{B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}}) \leqslant \sum_{n=0}^{\infty} \exp(\bar{\lambda}(n+1))\rho^n < \infty$$

for $\bar{\lambda} < -\log\rho$.

(ii) Let ε_0 be as in the statement of the lemma and let $\varepsilon \in (0,\varepsilon_0)$. We will show below that

$$E_y \exp\left(\bar{\lambda}_\varepsilon \tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}\right) < \infty \quad \text{for } y \in \bar{D} - B_\varepsilon^{a^{\mathrm{inv}}(x)} \text{ and for some } \bar{\lambda}_\varepsilon > 0. \tag{4.17}$$

This is enough to prove the lemma. Indeed, let

$$v(y) = E_y \exp(\bar{\lambda}_\varepsilon \tau_{B_\varepsilon^{a^{\mathrm{inv}}(x)}}) \quad \text{for } y \in \bar{D} - \bar{B}_\varepsilon^{a^{\mathrm{inv}}(x)}.$$

Then by the Feynman–Kac formula, v is a positive solution to $(L+\bar{\lambda}_\varepsilon)v = 0$ in $\bar{D} - \bar{B}_\varepsilon^{a^{\mathrm{inv}}(x)}$ with $v\nabla v = 0$ on ∂D. Thus, $\lambda_\varepsilon(x) \geqslant \bar{\lambda}_\varepsilon > 0$. We now prove (4.17).

Choose $\varepsilon_1 > \varepsilon_0$ such that $B_{\varepsilon_1}^{a^{\mathrm{inv}}(x)} \subset D$. For $\delta \in (0, \varepsilon_1]$, let $\tau_{\partial B_\delta^{a^{\mathrm{inv}}(x)}} = \inf\left\{t \geqslant 0 : X(t) \in \partial B_\delta^{a^{\mathrm{inv}}(x)}\right\}$. Define $\rho_\varepsilon = \sup_{y\in\partial B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}} P_y\left(\tau_{\partial B_{\varepsilon_1}^{a^{\mathrm{inv}}(x)}} < \tau_{\partial B_\varepsilon^{a^{\mathrm{inv}}(x)}}\right)$. Of course, $\rho_\varepsilon < 1$. Let $\bar{\lambda}$ be as in the statement of the lemma and choose $\bar{\lambda}_\varepsilon \in (0,\bar{\lambda})$

sufficiently small so that $\sup_{y\in\partial B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}} E_y\left(\exp\left(\bar{\lambda}_\varepsilon\tau_{\partial B_{\varepsilon_1}^{a^{\mathrm{inv}}(x)}}\right)|\tau_{\partial B_{\varepsilon_1}^{a^{\mathrm{inv}}(x)}}<\tau_{\partial B_{\varepsilon}^{a^{\mathrm{inv}}(x)}}\right)\leqslant\rho_\varepsilon^{-\frac{1}{2}}$,
$\sup_{y\in\partial B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}} E_y(\exp\left(\bar{\lambda}_\varepsilon\tau_{\partial B_{\varepsilon}^{a^{\mathrm{inv}}(x)}}\right)|\tau_{\partial B_{\varepsilon}^{a^{\mathrm{inv}}(x)}}<\tau_{\partial B_{\varepsilon_1}^{a^{\mathrm{inv}}(x)}})<\rho_\varepsilon^{-\frac{1}{2}}$, and

$$E_y\exp\left(\bar{\lambda}_\varepsilon\tau_{\partial B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}}\right)<\infty\quad\text{for } y\in B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}. \tag{4.18}$$

An application of the strong Markov property then shows that

$$E_y\exp\left(\bar{\lambda}_\varepsilon\tau_{B_{\varepsilon}^{a^{\mathrm{inv}}(x)}}\right)\leqslant\begin{cases} E_y\exp\left(\bar{\lambda}_\varepsilon\tau_{B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}}\right)\sum_{n=0}^{\infty}\rho_\varepsilon^n\left(\rho_\varepsilon^{-\frac{1}{2}}\right)^{n+1} & \text{for } y\in\bar{D}-B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)},\\ E_y\exp\left(\bar{\lambda}_\varepsilon\tau_{\partial B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}}\right)\sum_{n=0}^{\infty}\rho_\varepsilon^n\left(\rho_\varepsilon^{-\frac{1}{2}}\right)^{n+1} & \text{for } y\in B_{\varepsilon_0}^{a^{\mathrm{inv}}(x)}.\end{cases} \tag{4.19}$$

The upper expression on the right-hand side of (4.19) is finite by the assumption in the lemma while the lower one is finite by (4.18). □

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