A Lecture on Penrose Tilings of The Plane*

Sir Roger Penrose

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Abstract

These notes are from a lecture given by Sir. Roger Penrose at The Pennsylvania State University MASS Colloquium. The topic of the lecture was on tiling the plane and 5-fold symmetry. The lecture was given on October 24th, 2002.

1 Penrose Tiling

A crystalline pattern is a covering of the plane by copies of the same shape without overlapping boundries. For example a 3-fold tiling would be a tiling of the plane using triangles as shown in Figure **??** by [**?**] . The following Theorem gives us a necessary condition for a tiling to be a crystal symmetry.

Theorem 1. *The only crystal symmetries are those constructed using 1-fold, 2-fold, 3-fold, 4 fold, and 6-fold figures.*

Proof. For one of the symmetries above we can tile the plane periodically since a rotation by $\frac{2\pi}{n}$ gives us an equal or larger distance between the two vertices which indicates there will be no gaps between the pieces of the tiling. Consider the distance between two vertices of a tile that has 5-fold. If we rotate our 5-fold tile by an angle of $\frac{2\pi}{5}$ degrees we get that the vertices of our tile are closer together which means that we can't tile our space in a symmetric pattern. For an n-fold with $n > 6$ a rotation by an angle of $\frac{2\pi}{n}$ would cause the two vertices to cross which would give us a negative distance between the two vertices which prevents a tiling. □

The reason that 5-fold tiling doesn't work is because of the "gaps" left when one tries to construct a tiling of the plane. This can be seen in Figure **??** due to [**?**].

By considering a tiling that is less than perfect, that is a tiling that has an "efficiency" of less than 1 we may cover the plane with a tiling that doesn't have a true crystal structure. This idea can be thought of in a sense as a "delta-epsilon proof," except for tiling. It is possible to tile the plane without a true crystal structure and furthermore it is possible to tile it using 5-fold tiles.

*Lecture notes and typesetting by Adam T. Ringler and John Hamilton.

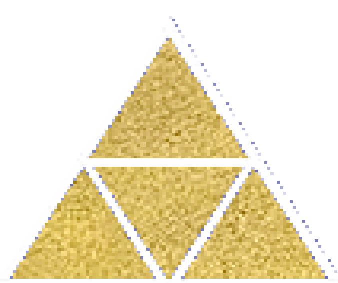

Figure 1: Here is a 3-fold tiling of the plane.

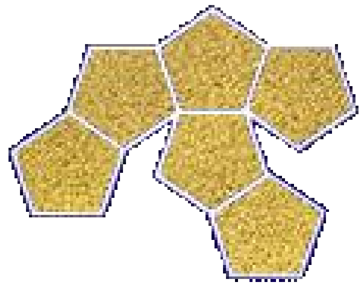

Figure 2: We can see that there are gaps created by the pentagon tiling.

The way R. Penrose constructed a 2-tile aperiodic tiling of the plane in 1974 is as follows. First take a pentagon. Then place six pentagons inside of the original pentagon so that there are three areas in the original pentagon that are not contained in any of the six pentagons. Also note that one of the pentagons is placed in the middle. Upon this construction one obtains decagons. One of the key steps is that at each point of the construction the inner pentagons must be enlarged. By continuing in this fashion one obtains a tiling of the entire plane. For a more comprehensive explanation we invite the reader check [**?**]. The interesting thing is that each decagon contains 3 pentagons, 2 rhombus, and a jester's cap. Also this tiling has the property that every decagon is surrounded by the ring of a pentagon. This can be seen in Figure **??** which is due to [**?**].

Another interesting find in this tiling is that it contains the golden ratio, along with the Fibonacci sequence. The ratio of narrow and broad lines is the golden ratio $\tau = \frac{1+\sqrt{5}}{2}$. Now labeling each band with its respective number that of τ or 1 we obtain the Fibonacci numbers. It should be noted that de Bruijn generalized a way of finding the Fibonacci sequence in different tiling schemes.

2 Other Aperiodic Tiling Schemes

Another tiling constructed by Penrose is the tiling that uses only two shapes. The shapes used in the construction are kites and darts. The infinite star pattern tiling can be seen in Figure **??**, due to [**?**].

Beyond the Penrose tiling there are other ways of tiling the plane. J. Kepler produced a few different ways of tiling the plane. Some of these tilings motivated Penrose into looking into further tilings of the plane. The interesting thing about Kepler's tilings is that some of them are now found in biological models, however during the time of Kepler the equipment to see these patterns in nature was not available. A few of these Kepler tilings as produced by [**?**] are shown in Figure **??**.

An interesting result about tiling the plane is due to Robert Berser and is stated as the Theorem below.

Theorem 2. *There does not exist an algorithm to tile the plane using a 5-fold tiling scheme.*

The proof to this theorem was done using a computer and 200,000 different tiling schemes. Although 200,000 different tiling schemes is a lot, there exist $2^{\aleph_0}$ different ways of tiling the plane. Because there are so many tilings it can be impossible to distinguish between two different tilings locally, because the tilings may be exactly the same locally, however globally they maybe different.

There are other tiling schemes that are aperiodic and we give you just a few of these. P. Steinhardt produced a tiling of the plane using fat and skinny rhombi. R. Ammann presented an 8-fold quasi-symmetry tiling in 1976. J. Socolar presented a 12-fold tiling. An Ammann tiling that is aperiodic can be seen in Figure **??**, from [**?**].

One of the most visually stimulating tiling schemes is the pinwheel tiling. The tiling is a combination of right triangles with sides of length $1, 2, \sqrt{5}$. The pinwhell construction can be seen in Figure **??** due to [**?**]. It is interesting to note that Figure **??** is only the 5th iteration of the construction.

Penrose presented a still open question which is "Does there exist a single tile shape that is aperiodic and tiles the plane?" Later Penrose gave two examples of a single shape using 1-key and 2-key pieces, however the question still remains open for a single shape without using a key.

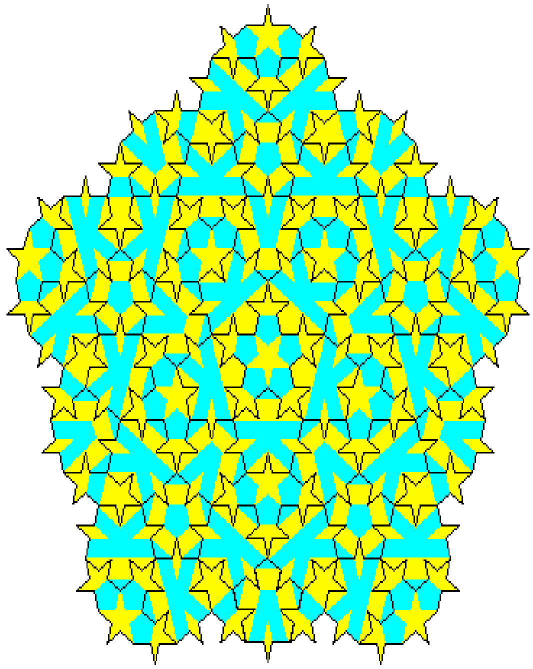

Figure 3: This is an aperiodic tiling of the plane which is similar to that of a tiling due to Kepler.

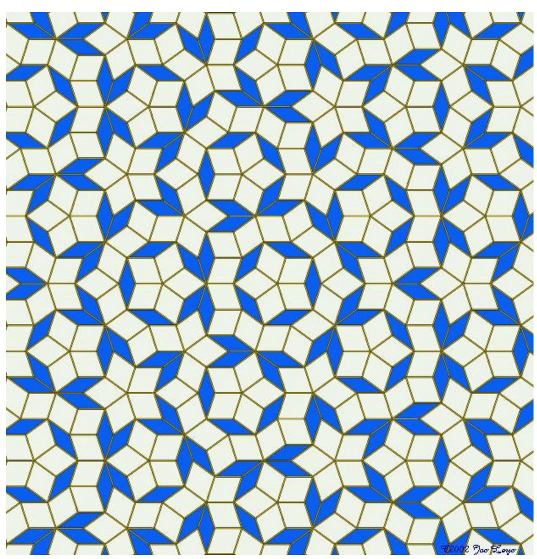

Figure 4: The Penrose infinite star pattern tiling is aperiodic and uses only two shapes.

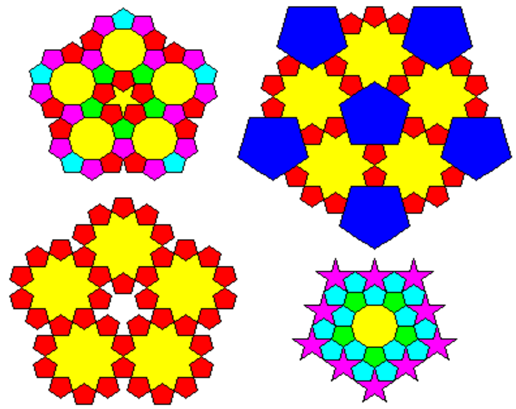

Figure 5: These are only a few of the many tiling schemes created by Kepler.

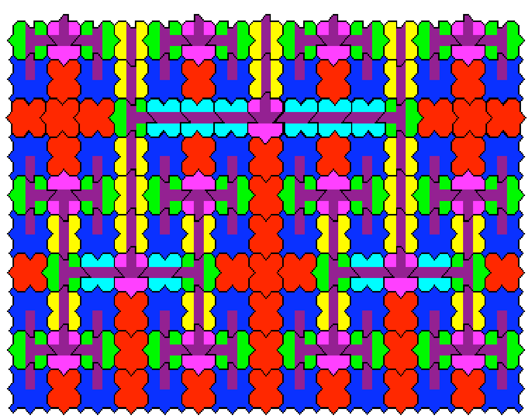

Figure 6:

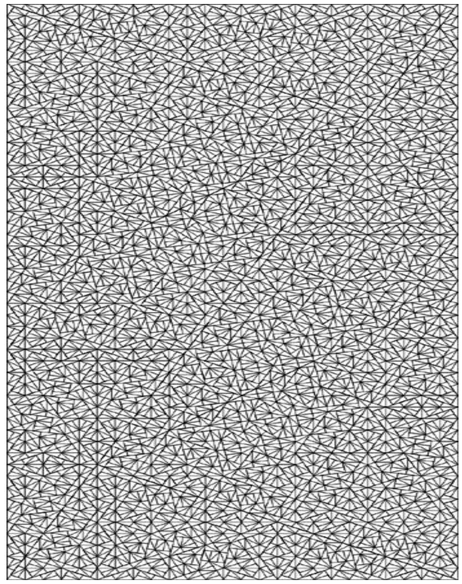

Figure 7: This is an aperiodic tiling using triangles known as the pinwheel construction.

3 Notes

To construct Penrose tiling schemes on a computer we refer the reader to [?]. For a short discussion about the geometry of Penrose tiling we refer the reader to [?]. For some examples of Socolar 12-fold tiling we refer the reader to [?]. We refer the reader to [?] for an extensive treatment of tiling schemes. For books written by R. Penrose on subjects other than tiling we refer the reader to [?], [?], [?], and [?].

References

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