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L^p BOUNDEDNESS OF COMMUTATOR OPERATOR ASSOCIATED WITH SCHRÖDINGER OPERATORS ON HEISENBERG GROUP*

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Abstract Let $L=-\triangle_{H^n}+V$ be a Schrödinger operator on Heisenberg group H^n, where $\triangle_{H^n}$ is the sublaplacian and the nonnegative potential V belongs to the reverse Hölder class $B_{Q/2}$, where Q is the homogeneous dimension of H^n. Let $T_1=(-\triangle_{H^n}+V)^{-1}V$, $T_2=(-\triangle_{H^n}+V)^{-1/2}V^{1/2}$, and $T_3=(-\triangle_{H^n}+V)^{-1/2}\nabla_{H^n}$, then we verify that $[b,T_i],\ i=1,2,3$ are bounded on some $L^p(H^n)$, where $b\in \mathrm{BMO}(H^n)$. Note that the kernel of $T_i,\ i=1,2,3$ has no smoothness.

Key words Commutator; BMO; Heisenberg group; boundedness; Riesz transforms associated to Schrödinger operators

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1 Introduction

Let $L=-\triangle_{H^n}+V$ be a Schrödinger differential operator on Heisenberg group H^n. Throughout this article, we will assume that $V(x)$ is a nonzero, nonnegative potential, and belongs to B_q for some $q>Q/2$, where $Q=2n+2$ is the homogeneous dimension of H^n. Let $T_1=(-\triangle_{H^n}+V)^{-1}V$, $T_2=(-\triangle_{H^n}+V)^{-1/2}V^{1/2}$, and $T_3=(-\triangle_{H^n}+V)^{-1/2}\nabla_{H^n}$, then L^p boundedness of the operator T_j was studied by Z. Shen in [6]. In [4], the authors considered the L^p boundedness of $[b,T_i],\ i=1,2,3$, on R^n. In this article, we will discuss the L^p boundedness of the commutator operators $[b,T_i]=bT_i-T_ib$, where $b\in \mathrm{BMO}(H^n)$.

This article is organized as follows. In the rest of this section, we state some knowledge, notations, and terminologies to be used throughout this article. In Section 2, we give some estimates of the kernels of the operators $T_i,\ i=1,2,3$. In Section 3, we will prove our main results.

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The $(2n+1)$-dimensional Heisenberg group H^n is a nilpotent Lie group with underlying manifold $R^{2n} \times R$. The group structure is given by

$$(x,t)(y,s) = \left(x + y, t + s + 2\sum_{j=1}^{n}(x_{n+j}y_j - x_j y_{n+j})\right). \tag{1.1}$$

The Lie algebra of left-invariant vector fields on H^n is spanned by

$$X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial t},\ X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial t} \quad j = 1, 2 \cdots n, \tag{1.2}$$

$$X_{2n+1} = \frac{\partial}{\partial t}. \tag{1.3}$$

All nontrivial commutation relations are given by $[X_j, X_{n+j}] = -4X_{2n+1}, j = 1, 2, \cdots n$. The sub-Laplacian $\triangle_{H^n}$ and the gradient ∇_{H^n} are defined, respectively, by

$$\triangle_{H^n} = \sum_{j=1}^{2n} X_j^2 \ \text{ and } \ \nabla_{H^n} = (X_1, X_2, \cdots X_{2n}). \tag{1.4}$$

The dilations on H^n have the form

$$\delta_r(x,t) = (rx, r^2 t) \quad t > 0. \tag{1.5}$$

The Haar measure on H^n coincides with the Lebesgue measure on $R^{2n} \times R$. The measure of any measurable set E is denoted by $|E|$, we define a homogeneous norm on H^n by

$$|g| = (|x|^4 + |t|^2)^{1/4}, \quad g = (x,t) \in H^n. \tag{1.6}$$

This norm satisfies the triangular inequality and leads to a left-invariant distance $d(g,h) = |g^{-1}h|$. the ball of radius r and centered at g is denoted by

$$B(g,r) = \{h \in H^n, |g^{-1}h| < r\} \tag{1.7}$$

whose volume is given by

$$|B(g,r)| = c_n r^Q,\ c_n = |B(0,1)| = \frac{2\pi^{n+1/2}\Gamma(n/2)}{(n+1)\Gamma(n)\Gamma(\frac{n+1}{2})}, \tag{1.8}$$

where $Q = 2n + 2$ is the homogeneous dimension of H^n.

Now, we turn to the Schrödinger operator L on Heisenberg group.

Definition 1 A nonnegative locally L^q integrable function V on H^n is said to belong to B_q $(1 < q < \infty)$, if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(g)^q dg\right)^{1/q} \le C\left(\frac{1}{|B|}\int_B V(g)\mathrm{d}g\right) \tag{1.9}$$

holds for every ball B in H^n.

Remark 1 Obviously, $B_{q_1} \subset B_{q_2}$ if $q_1 > q_2$. But it is important that the B_q class has a property of self-improvement, that is, if $V \in B_q$, then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$. In this article, we always assume that $0 \neq V \in B_{Q/2}$ and then $V \in B_{q_0}$ for some $q_0 > Q/2$. Of course, we may assume that $q_0 < Q$. It is also well known that if $V \in B_q, q > 1$, then, $V(g)\mathrm{d}g$ is a doubling measure, namely,

$$\int_{B(g_0,2r)} V(g)\mathrm{d}g \le C_0 \int_{B(g_0,r)} V(g)\mathrm{d}g. \tag{1.10}$$

The following theorem is a collection of properties of the potential V satisfying the reverse Hölder inequality on Heisenberg group H^n. We first introduce an auxiliary function.

Definition 2 For $g \in H^n$, the function $m(g,V)$ is defined by

$$\frac{1}{m(g,V)} = \sup\left\{r > 0, \frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)\mathrm{d}h \le 1\right\}. \tag{1.11}$$

Theorem A There exist $C > 0,\ c > 0$, and $k_0 > 0$ such that, for $g, h \in H^n$,

(a) $0 < m(g,V) < \infty,$ for every $g \in R^n$; (1.12)

(b) If $r = \dfrac{1}{m(g,V)},$ then $\dfrac{1}{r^{Q-2}} \displaystyle\int_{B(g,r)} V(h)\mathrm{d}h = 1$; (1.13)

(c) $m(g,V) \sim m(h,V),$ if $|gh^{-1}| \le \dfrac{c}{m(g,V)}$; (1.14)

(d) $m(h,V) \le C\left\{1 + |gh^{-1}|m(g,V)\right\}^{k_0} m(g,V)$; (1.15)

(e) $m(h,V) \ge Cm(g,V)\left\{1 + |gh^{-1}|m(g,V)\right\}^{-k_0/(1+k_0)}$; (1.16)

(f) $c\left\{1 + |gh^{-1}|m(h,V)\right\}^{\frac{1}{k_0+1}} \le 1 + |gh^{-1}|m(g,V) \le C\{1 + |gh^{-1}|m(h,V)\}^{k_0+1}$. (1.17)

Now, we turn to the estimate of the kernel. Let $\Gamma(g,h,t)$ denote the fundamental solution of the operator $-\Delta_{H^n} + \mathrm{i}\tau$, where $\tau \in R$. In [5], C. Lin, H. Liu and Y. Liu proved that, for any $k > 0$, there exists a constant $C_l > 0$, such that

$$|\Gamma(g,h,\tau)| \le \frac{C_l}{(1 + |gh^{-1}||\tau|^{1/2})^l} \frac{1}{|gh^{-1}|^{Q-2}}, \tag{1.18}$$

$$|\nabla_{H^n,g}\Gamma(g,h,\tau)| \le \frac{C_l}{(1 + |gh^{-1}||\tau|^{1/2})^l} \frac{1}{|gh^{-1}|^{Q-1}}, \tag{1.19}$$

where $\nabla_{H^n,g}$ denotes the gradient operator for g (See [5] for details). The above estimate still holds for $\nabla_{H^n,h}$ instead of $\nabla_{H^n,g}$ and can be easily reduced from the corresponding estimates of the heat kernel. We remark that the explicit expression of $\Gamma(g,h) = \Gamma(g,h,0)$ was obtained by Folland:

$$\Gamma(g,h) = \frac{2^{n-2}\gamma(n/2)^2}{\pi^{n+1}} \frac{1}{|gh^{-1}|^{Q-2}}. \tag{1.20}$$

Let $\Gamma^L(g,h,\tau)$ denote the fundamental solution for the operator $L + \mathrm{i}\tau$, where $\tau \in R$. For any $k > 0$, there exists a constant $C_l > 0$, such that

$$|\Gamma^L(g,h,\tau)| \le \frac{C_l}{(1 + |gh^{-1}||\tau|^{1/2})^l(1 + |gh^{-1}|(m(g,V) + m(h,V)))^k} \frac{1}{|gh^{-1}|^{Q-2}}. \tag{1.21}$$

2 The Estimate of the Kernel of T_i, $i = 1, 2, 3$

Lemma 1 Suppose $V \in B_{q_0}$, $q_0 > 1$. Assume that $-\Delta_{H^n} u + (V + \mathrm{i}\tau)u = 0$ in $B(g_0, 2R)$, then,

(1) For $x \in B(g_0, R)$,

$$|\nabla_{H^n} u(g)| \le C \sup_{B(g_0,2R)} |u| \int_{B(g_0,2R)} \frac{V(h)}{|gh^{-1}|} \mathrm{d}h + \frac{C}{R^{Q+1}} \int_{B(g_0,2R)} |u(h)| \mathrm{d}h; \tag{2.1}$$

(2) If $Q/2 < q_0 < Q$, let $\frac{1}{t} = \frac{1}{q_0} - \frac{1}{n}$, $k_0 > \log_2 C_0 + 1$, where C_0 is the constant in (1.10),

$$\left(\int_{B(g_0,R)} |\nabla_{H^n} u|^t \mathrm{d}g \right)^{1/t} \le C R^{Q/q_0 - 2} \{1 + Rm(g_0, V)\}^{k_0} \sup_{B(g_0,2R)} |u|. \tag{2.2}$$

Lemma 2 Suppose $V \in B_q$. For some $q > \frac{Q}{2}$, let $N > \log_2 C_0 + 1$, where C_0 is the constant in Remark 1, (1.10). Then, for any $g_0 \in H^n, R > 0$,

$$\frac{1}{\{1 + m(g_0, V)R\}^N} \int_{B(g_0,R)} V(h) \mathrm{d}h \le C R^{Q-2}. \tag{2.3}$$

Proof There exists an integer $j_0 \in Z$, such that $2^{j_0} R \le \frac{1}{m(g_0,V)} < 2^{j_0+1} R$.

Case 1: $j_0 < 0$. By the doubling property of $V(g)$, Lemma 1 and (b) of Theorem A, we have

$$\begin{aligned} \frac{1}{\{1 + m(g_0, V)R\}^N} \int_{B(g_0,R)} V(h) \mathrm{d}h &\le \frac{1}{\{1 + 2^{-(j_0+1)} R^{-1} R\}^N} \int_{B(g_0,R)} V(h) \mathrm{d}h \\ &\le \frac{1}{(2^{-j_0})^N} \int_{B(g_0,R)} V(h) \mathrm{d}h \\ &\le \frac{C}{(2^{-j_0})^N} C_0^{-j_0} (2^{j_0} R)^{Q-2} \\ &\le R^{Q-2}, \ (N > \log_2 C_0). \end{aligned} \tag{2.4}$$

Case 2: $j_0 \ge 0$. By (b) and (g) of Theorem A, we obtain

$$\begin{aligned} \frac{1}{\{1 + m(g_0, V)R\}^N} \int_{B(g_0,R)} V(h) \mathrm{d}h &\le \int_{B(g_0,R)} V(h) \mathrm{d}h \\ &\le R^{Q-2} \frac{1}{R^{Q-2}} \int_{B(g_0,R)} V(h) \mathrm{d}h \\ &\le R^{Q-2}. \end{aligned} \tag{2.5}$$

Now, we estimate the kernels $K_i(g,h)$ of T_i, $i = 1, 2, 3$.

Lemma 3 Suppose $V \in B_q$ for some $q > Q/2$, then there exists $\delta > 0$ such that, for any integer $k > 0$ and $0 < |\omega| < |gh^{-1}|/16$,

$$|K_1(g, h)| \le \frac{C_k}{\{1 + m(g, V)|gh^{-1}|\}^k} \frac{1}{|gh^{-1}|^{Q-2}} V(h); \tag{2.6}$$

$$|K_1(g\omega, h) - K_1(g, h)| \le \frac{C_k}{\{1 + m(g, V)|gh^{-1}|\}^k} \frac{|\omega|^\delta}{|gh^{-1}|^{Q-2+\delta}} V(h)^{1/2}. \tag{2.7}$$

Proof Because $T_1 = (-\Delta_{H^n} + V)^{-1}V$, then, we can see that $K_1(g,h) = \Gamma^L(g,h,0)V(h)$. By the estimate of fundamental solution, we have

$$|K_1(g,h)| \le \frac{C_k}{\{1+m(g,V)|gh^{-1}|\}^k}\frac{1}{|gh^{-1}|^{Q-2}}V(h). \tag{2.8}$$

For (2.7), taking $g,h \in H^n$ and $Q/2 < q_0 < \min(Q,q)$, we know $V \in B_{q_0}$. Let $R = \frac{|gh^{-1}|}{8}$, $\frac{1}{t} = \frac{1}{q_0} - \frac{1}{Q}$, then $\delta = 1 - Q/t > 0$. For any $0 < h < \frac{R}{2}$, it follows from the imbedding theorem of Morrey and Lemma 1,

$$\begin{aligned}
&|K_1(g\omega,h) - K_1(g,h)| \\
&\le |\Gamma^L(g\omega,h,0) - \Gamma^L(g,h,0)|V(h) \\
&\le C|\omega|^{1-Q/t}\bigg(\int_{B(g,R)}|\nabla_g\Gamma^L(u,h,0)|^t\mathrm{d}u\bigg)^{1/t}V(h) \\
&\le C|\omega|^{1-Q/t}R^{Q/q_0-2}\{1+Rm(g,v)\}^{k_0}\sup_{u\in B(g,2R)}|\Gamma^L(u,h,0)|V(h) \\
&\le C\bigg(\frac{|\omega|}{R}\bigg)^{1-Q/t}\{1+Rm(g,v)\}^{k_0}\sup_{u\in B(g,2R)}|\Gamma^L(u,h,0)|V(h) \\
&\le C\frac{|\omega|^\delta}{R^\delta}\{1+Rm(g,v)\}^{k_0}\sup_{u\in B(g,2R)}\frac{C_{k_1}}{\{1+m(h,V)|uh^{-1}|\}^{k_1}}\frac{1}{|uh^{-1}|^{Q-2}}V(h) \\
&\le C_k\frac{|\omega|^\delta}{|gh^{-1}|^\delta}\frac{1}{\{1+m(g,V)|gh^{-1}|\}^k}\frac{1}{|gh^{-1}|^{Q-2}}V(h).
\end{aligned} \tag{2.9}$$

Lemma 4 Suppose $V \in B_q$ for some $q > Q/2$. Then, there exists $\delta > 0$ and for any integer $k > 0$, $0 < |\omega| < |gh^{-1}|/16$,

$$|K_2(g,h)| \le \frac{C_k}{\{1+m(g,V)|gh^{-1}|\}^k}\frac{1}{|gh^{-1}|^{Q-1}}V(h)^{1/2}; \tag{2.10}$$

$$|K_2(g\omega,h) - K_2(g,h)| \le \frac{C_k}{\{1+m(h,V)|gh^{-1}|\}^k}\frac{|\omega|^\delta}{|gh^{-1}|^{Q-1+\delta}}V(h)^{1/2}. \tag{2.11}$$

Proof By functional calculus, we write

$$(-\Delta_{H^n} + V)^{-1/2} = -\frac{1}{2\pi}\int_R(-\mathrm{i}\tau)^{-1/2}(-\Delta_{H^n} + V + \mathrm{i}\tau)^{-1}\mathrm{d}\tau, \tag{2.12}$$

then, $K_2(g,h) = -\frac{1}{2\pi}\int_R(-\mathrm{i}\tau)^{-1/2}\Gamma(g,h,\tau)\mathrm{d}\tau V(h)^{1/2}$.

We claim that, for $k > 2$,

$$\int_R|\tau|^{-1/2}\{1+|\tau|^{1/2}|gh^{-1}|\}^{-k}\mathrm{d}\tau \le \frac{C_k}{|gh^{-1}|}. \tag{2.13}$$

In fact, we have

$$\begin{aligned}
&\int_R|\tau|^{-1/2}\{1+|\tau|^{1/2}|gh^{-1}|\}^{-k}\mathrm{d}\tau \\
&\le \bigg(\int_{|\tau|\le|gh^{-1}|^{-2}} + \int_{|\tau|>|gh^{-1}|^{-2}}\bigg)|\tau|^{-1/2}\{1+|\tau|^{1/2}|gh^{-1}|\}^{-k}\mathrm{d}\tau \\
&\le \int_{|\tau|\le|gh^{-1}|^{-2}}|\tau|^{-1/2}\mathrm{d}\tau + \int_{|\tau|>|gh^{-1}|^{-2}}|\tau|^{(-k-1)/2}|gh^{-1}|\mathrm{d}\tau \\
&\le \frac{C_k}{|gh^{-1}|}.
\end{aligned} \tag{2.14}$$

We first prove (2.10). We obtain

$$\begin{aligned}
&|K_2(g,h)| \\
&\le \frac{1}{2\pi}\int_R |\tau|^{-1/2}|\Gamma^L(g,h,\tau)|\mathrm{d}\tau V(h)^{1/2} \\
&\le \frac{1}{2\pi}\int_R |\tau|^{-1/2}\frac{C_k}{\{1+|gh^{-1}||\tau|^{1/2}\}^k}\frac{1}{|gh^{-1}|^{Q-2}}\mathrm{d}\tau V(h)^{1/2}\frac{1}{(1+|gh^{-1}|m(g,V))^k} \\
&\le \frac{1}{2\pi}\frac{1}{|gh^{-1}|^{Q-2}}V(h)^{1/2}\int_R |\tau|^{-1/2}\frac{C_k}{\{1+|gh^{-1}||\tau|^{1/2}\}^k}\mathrm{d}\tau\frac{1}{(1+|gh^{-1}|m(g,V))^k} \\
&\le \frac{C_k}{\{1+m(g,V)|gh^{-1}|\}^k}\frac{1}{|gh^{-1}|^{Q-1}}V(h)^{1/2}.
\end{aligned} \tag{2.15}$$

For (2.11), fix $g,h\in H^n$ and $Q/2<q_0<\min(Q,q)$, then, we know $V\in B_{q_0}$. Let $R=|gh^{-1}|/8$ and $1/t=1/q_0-1/Q$, then, $\delta=1-Q/t>0$. For any $0<|\omega|<\frac{R}{2}$, we have

$$|K_2(g\omega,h)-K_2(g,h)|\le\frac{1}{2\pi}\int_R|\tau|^{1/2}|\Gamma^L(g\omega,h,\tau)-\Gamma^L(g,h,\tau)|\mathrm{d}\tau V(y)^{1/2}. \tag{2.16}$$

By Morrey's imbedding theorem,

$$\begin{aligned}
&|K_2(g\omega,h)-K_2(g,h)| \\
&\le \frac{C}{2\pi}\int_R |\tau|^{1/2}|\omega|^{1-Q/t}\bigg(\int_{B(g,R)}|\nabla_{H^n,g}\Gamma^L(u,h,\tau)|^t\mathrm{d}t\bigg)^{1/t}\mathrm{d}\tau V(h)^{1/2} \\
&\le \frac{C}{2\pi}\int_R |\tau|^{1/2}|\omega|^{1-Q/t}R^{Q/q_0-2}\{1+Rm(g,V)\}^{k_0}\sup_{u\in B(g,2R)}|\Gamma^L(u,h,\tau)|\mathrm{d}\tau V(h)^{1/2} \\
&\le C\bigg(\frac{|\omega|}{R}\bigg)^{\delta}\{1+Rm(g,V)\}^{k_0}\int_R|\tau|^{1/2}\sup_{u\in B(g,2R)}|\Gamma^L(u,h,\tau)|\mathrm{d}\tau V(h)^{1/2}.
\end{aligned} \tag{2.17}$$

Because $|gu^{-1}|\le 2R=|gh^{-1}|/4$, then, $|uh^{-1}|\ge c(|gh^{-1}|-|gu^{-1}|)\ge c|gh^{-1}|$. So, we obtain

$$\begin{aligned}
&\sup_{u\in B(g,2R)}|\Gamma^L(u,h,\tau)| \\
&\le \sup_{u\in B(g,2R)}\frac{C_k}{\{1+|uh^{-1}||\tau|^{1/2}\}^k}\frac{1}{\{1+m(h,V)|uh^{-1}|\}^k}\frac{1}{|uh^{-1}|^{Q-2}} \\
&\le \frac{C_k}{\{1+|gh^{-1}||\tau|^{1/2}\}^k}\frac{1}{\{1+m(h,V)|gh^{-1}|\}^k}\frac{1}{|gh^{-1}|^{Q-2}}.
\end{aligned} \tag{2.18}$$

Finally, we have

$$\begin{aligned}
&|K_2(g\omega,h)-K_2(g,h)| \\
&\le C\bigg(\frac{|\omega|}{|gh^{-1}|}\bigg)^{\delta}\frac{1}{\{1+m(h,V)|gh^{-1}|\}^k}\frac{1}{|gh^{-1}|^{Q-2}}\int_R|\tau|^{1/2}\frac{C_k}{\{1+|gh^{-1}||\tau|^{1/2}\}^k}\mathrm{d}\tau V(h)^{1/2} \\
&\le \frac{C_k}{\{1+m(h,V)|gh^{-1}|\}^k}\frac{|\omega|^{\delta}}{|gh^{-1}|^{Q-1+\delta}}V(h)^{1/2}.
\end{aligned} \tag{2.19}$$

Lemma 5 Suppose $V\in B_q$ for some $Q/2<q<Q$. Then, there exists $\delta>0$ such that, for any integer $k>0$, $0<|\omega|<|gh^{-1}|/16$,

$$\begin{aligned}
&|K_3(g,h)| \\
&\le \frac{C_k}{\{1+m(g,V)|gh^{-1}|\}^k}\frac{1}{|gh^{-1}|^{Q-1}}\bigg(\int_{B(h,|gh^{-1}|)}\frac{V(u)}{|hu^{-1}|^{Q-1}}\mathrm{d}u+\frac{1}{|gh^{-1}|}\bigg);
\end{aligned} \tag{2.20}$$

$$\begin{aligned}&|K_3(g\omega,h)-K_3(g,h)|\\ \le\ &\frac{C_k}{\{1+m(g,V)|gh^{-1}|\}^k}\frac{\delta}{|gh^{-1}|^{Q-1+\delta}}\bigg(\int_{B(h,|gh^{-1}|)}\frac{V(u)}{|hu^{-1}|^{Q-1}}\mathrm{d}u+\frac{1}{|gh^{-1}|}\bigg).\end{aligned}\tag{2.21}$$

Proof By partial integral, we know that

$$K_3(g,h)=\frac{1}{2\pi}\int_{\mathbb{R}}(-\mathrm{i}\tau)^{-1/2}\nabla_{H^n,2}\Gamma^L(g,h,\tau)\mathrm{d}\tau.$$

Fix $g,\ h\in H^n$, let $R=|gh^{-1}|/8$, $1/t=1/q-1/Q$ and $\delta=Q/q-2>0$. For any $0<|\omega|<R/2$, we have

$$|K_3(g\omega,h)-K_3(g,h)|\le\frac{1}{2\pi}\int_{\mathbb{R}}|\tau|^{-1/2}|\nabla_{H^n,2}\Gamma^L(g\omega,h,\tau)-\nabla_{H^n,2}\Gamma^L(g,h,\tau)|\mathrm{d}\tau.\tag{2.22}$$

By Morrey's imbedding theorem, we have

$$\begin{aligned}&|\nabla_{H^n,2}\Gamma^L(g\omega,h,\tau)-\nabla_{H^n,2}\Gamma^L(g,h,\tau)|\\ \le\ &C|h|^{1-Q/t}\bigg(\int_{B(g,R)}|\nabla_{H^n,1}\nabla_{H^n,2}\Gamma^L(u,h,\tau)|^t\mathrm{d}u\bigg)^{1/t}\\ \le\ &C|h|^{1-Q/t}R^{Q/q-2}\{1+Rm(g,V)\}^{k_0}\sup_{u\in B(g,2R)}|\nabla_{H^n,2}\Gamma^L(u,h,\tau)|.\end{aligned}\tag{2.23}$$

As $\Gamma^L(u,h,\tau)=\Gamma^L(u,h,u,-\tau)$, then, $\nabla_{H^n,2}\Gamma^L(u,h,\tau)=\nabla_{H^n,1}\Gamma^L(h,u,\tau)$. So, we obtain

$$\begin{aligned}&\sup_{u\in B(g,2R)}|\nabla_{H^n,2}\Gamma^L(u,h,\tau)|\\ \le\ &\sup_{u\in B(g,2R)}|\nabla_{H^n,1}\Gamma^L(h,u,-\tau)|\\ \le\ &\sup_{u\in B(g,2R)}\bigg\{\sup_{\eta\in B(h,|hu^{-1}|/4)}|\Gamma^L(\eta,u,-\tau)|\int_{B(h,|hu^{-1}|/2)}\frac{V(v)}{|hv^{-1}|^{Q-1}}\mathrm{d}v\\ &+\frac{c}{|hu^{-1}|^{Q+1}}\int_{B(h,|hu^{-1}|/2)}\Gamma^L(v,u,-\tau)\mathrm{d}v\bigg\}.\end{aligned}\tag{2.24}$$

Using the fact that $|\eta u^{-1}|\sim|hu^{-1}|$, $|vu^{-1}|\sim|hu^{-1}|$, $|gh^{-1}|\sim|hu^{-1}|$, and choosing k_1 large enough, we have

$$\begin{aligned}&\sup_{u\in B(g,2R)}|\nabla_{H^n,2}\Gamma^L(u,h,\tau)|\\ \le\ &\sup_{u\in B(g,2R)}\frac{C_k}{\{1+|\tau|^{1/2}|hu^{-1}|\}^{k_1}\{1+m(u,V)|hu^{-1}|\}^{k_1}}\frac{1}{|hu^{-1}|^{Q-2}}\\ &\times\int_{B(h,|gh^{-1}|)}\frac{V(v)}{|hv^{-1}|^{Q-1}}\mathrm{d}v+\frac{C_k}{\{1+|\tau|^{1/2}|hu^{-1}|\}^{k_1}\{1+m(u,V)|hu^{-1}|\}^{k_1}}\frac{1}{|hu^{-1}|^{Q-1}}\\ \le\ &\frac{C_k}{\{1+|\tau|^{1/2}|gh^{-1}|\}^{k_1}\{1+m(g,V)|gh^{-1}|\}^{k_1}}\\ &\times\bigg(\frac{1}{|gh^{-1}|^{Q-2}}\int_{B(h,|gh^{-1}|)}\frac{V(v)}{|hv^{-1}|^{Q-1}}\mathrm{d}v+\frac{1}{|gh^{-1}|^{Q-1}}\bigg).\end{aligned}\tag{2.25}$$

So, we obtain

$$|\nabla_{H^n,2}\Gamma^L(g\omega,h,\tau)-\nabla_{H^n,2}\Gamma^L(g,h,\tau)|$$

$$\leq C_k \frac{|\omega|^\delta}{|gh^{-1}|^\delta} \frac{1}{\{1+m(g,V)|gh^{-1}|\}^k} \frac{1}{\{1+|gh^{-1}||\tau|^{1/2}\}^k} \times\left(\frac{1}{|gh^{-1}|^{Q-2}} \int_{B(h,|gh^{-1}|)} \frac{V(v)}{|hv^{-1}|^{Q-1}} \mathrm{d}v + \frac{1}{|gh^{-1}|^{Q-1}}\right). \tag{2.26}$$

Finally, we have

$$|K_3(g\omega,h)-K_3(g,h)| \leq C_k \frac{|h|^\delta}{|gh^{-1}|^\delta} \frac{1}{\{1+m(g,V)|gh^{-1}|\}^k} \times\left(\frac{1}{|gh^{-1}|^{Q-1}} \int_{B(h,|gh^{-1}|)} \frac{V(v)}{|hv^{-1}|^{Q-1}} \mathrm{d}v + \frac{1}{|gh^{-1}|^{Q}}\right). \tag{2.27}$$

3 The Main Theorems

In this section, we prove the main theorems. We first introduce a new Hömander condition on Heisenberg group H^n.

Definition 3 $K(g,h)$ is said to satisfy condition $H(m)$ for some $m \geq 1$, if there exists a constant $C>0$ such that for $\forall l>0,\ g,\ g_0 \in H^n$ with $gg_0^{-1} \leq l$, then,

$$\sum_{k=5}^{\infty} k(2^k l)^{Q/m'} \left(\int_{2^k l \leq |hg_0^{-1}| < 2^{k+1} l} |K(g,h)-K(g_0,h)|^m \mathrm{d}h\right)^{1/m} \leq C, \tag{3.1}$$

where $\frac{1}{m'} = 1-\frac{1}{m}$.

Proposition 1 Let $m>1$, and suppose T is bounded on $L^p(H^n)$ for $p \in (m',\infty)$ and K satisfies $H(m)$, then, $\forall b \in \mathrm{BMO}(H^n)$, $[b,T]$ is bounded on $L^p(H^n)$ for every $p \in (m',\infty)$, and

$$\|[b,T]f\|_p \leq C_p \|b\|_{\mathrm{BMO}} \|f\|_p. \tag{3.2}$$

Proof Proposition 1 is an easy corollary of the following lemma. Recall that the sharp function of Fefferman-Stein is defined by

$$M^\sharp f(g) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(h)-f_B| \mathrm{d}h, \tag{3.3}$$

where $f_B = \frac{1}{|B|}\int_B f(h)\mathrm{d}h$ and the supremum is taken on all balls B with $g \in B$.

Lemma 6 Let T satisfy the same condition in Proposition 1. Then, for $\forall s>m'$, there exists a constant $C_s>0$, such that for $\forall f \in L^1_{\mathrm{loc}}$, $b \in \mathrm{BMO}(H^n)$,

$$M^\sharp([b,T]f)(g) \leq C_s \|b\|_{\mathrm{BMO}} \{M_s(Tf)(g) + M_s(f)(g)\}, \tag{3.4}$$

where $M_s(f) = (M(|f|^s))^{1/s}$ and M is the Hardy-Littlewood maximal function.

Proof Fix $s>m'$, $f \in L^1_{\mathrm{loc}}$, $g \in H^n$ and fix a ball $I = B(g_0,l)$ with $g \in I$. We only need to control $J = \frac{1}{|I|}\int_I |[b,T]f(h) - ([b,T]f)_I| \mathrm{d}h$ by the right-hand side of (3.4). Let $f = f_1+f_2$, where $f_1 = f\chi_{32I}, f_2 = f-f_1$. Then,

$$[b,T]f = (b-b_I)Tf - T(b-b_I)f_1 - T(b-b_I)f_2 \triangleq A_1 f + A_2 f + A_3 f. \tag{3.5}$$

Then, we have

$$J \leq \frac{1}{|I|}\int_I |A_1 f(h) - (A_1 f)_I| \mathrm{d}h + \frac{1}{|I|}\int_I |A_2 f(h) - (A_2 f)_I| \mathrm{d}h + \frac{1}{|I|}\int_I |A_3 f(h) - (A_3 f)_I| \mathrm{d}h \triangleq J_1 + J_2 + J_3. \tag{3.6}$$

We estimate $J_1,\ J_2$, and J_3, separately. For J_1, we have

$$\begin{aligned} J_1 &\le \frac{2}{|I|}\int_I |A_1 f(h)|\mathrm{d}h = \frac{2}{|I|}\int_I |(b-b_I)Tf(h)|\mathrm{d}h \\ &\le 2\bigg(\frac{1}{|I|}\int_I |b-b_I|^{s'}\mathrm{d}h\bigg)^{1/s'}\bigg(\frac{1}{|I|}\int_I |Tf(h)|^{s}\mathrm{d}h\bigg)^{1/s} \\ &\le 2\|b\|_{\mathrm{BMO}(H^n)}M_s(Tf)(g). \end{aligned} \tag{3.7}$$

For J_2, by the L^p boundedness of T, we have

$$\begin{aligned} J_2 &\le \frac{2}{|I|}\int_I |A_2 f(h)|\mathrm{d}h \le 2\bigg(\frac{1}{|I|}\int_I |A_2 f(h)|^{s_1}\mathrm{d}h\bigg)^{1/s_1} \\ &\le 2\bigg(\frac{1}{|I|}\int_{32I} |A_2 f(h)|^{s_1}\mathrm{d}h\bigg)^{1/s_1} \\ &\le C_Q\bigg(\frac{1}{|32I|}\int_{32I} |b-b_I|^{s_2}\mathrm{d}h\bigg)^{1/s_2}\bigg(\frac{1}{|32I|}\int_{32I} |f(h)|^{s}\mathrm{d}h\bigg)^{1/s} \\ &\le C_Q\|b\|_{\mathrm{BMO}}M_s(f)(g). \end{aligned} \tag{3.8}$$

At last, we start to estimate J_3. Set $C_I=\int_{|ug_0^{-1}|>32l} K(g_0,u)(b(u)-b_I)f(u)\mathrm{d}u$, then, we have

$$\begin{aligned} J_3 &\le \frac{2}{|I|}\int_I |A_3 f(h)-C_I|\mathrm{d}h \\ &\le \frac{2}{|I|}\int_I \int_{|ug_0^{-1}|>32l} |K(h,u)-K(g_0,u)||b(u)-b_I||f(u)|\mathrm{d}u\mathrm{d}h \\ &\le \frac{2}{|I|}\int_I \sum_{k=5}^{\infty}\int_{2^k l\le|ug_0^{-1}|<2^{k+1}l} |K(h,u)-K(g_0,u)||b(u)-b_I||f(u)|\mathrm{d}u\mathrm{d}h \\ &\le \frac{2}{|I|}\int_I \sum_{k=5}^{\infty}\bigg(\int_{2^k l\le|g_0^{-1}u|<2^{k+1}l} |K(h,u)-K(g_0,u)|^m\mathrm{d}u\bigg)^{1/m} \\ &\quad\times\bigg(\int_{2^k l\le|g_0^{-1}u|<2^{k+1}l} |(b(u)-b_I)f(u)|^{m'}\mathrm{d}u\bigg)^{1/m'} \\ &\le \frac{2}{|I|}\int_I \sum_{k=5}^{\infty} k(2^k l)^{Q/m'}\frac{1}{(2^k l)^{Q/m'}k}\bigg(\int_{2^k l\le|g_0^{-1}u|<2^{k+1}l} |(b(u)-b_I)f(u)|^{m'}\mathrm{d}u\bigg)^{1/m'} \\ &\le C\sup_{k\ge5}\frac{k+2}{k}\|b\|_{\mathrm{BMO}}M_s f(g)\le C\|b\|_{\mathrm{BMO}}M_s f(g). \end{aligned} \tag{3.9}$$

Theorem 1 (i) Suppose $V\in B_q,\ q\ge Q/2$, and let $b\in\mathrm{BMO}$. Then, for $q'\le p<\infty$,

$$\|[b,T_1]f\|_p\le C_p\|b\|_{\mathrm{BMO}}\|f\|_p. \tag{3.10}$$

(ii) Suppose $V\in B_q,\ q\ge Q/2$, and let $b\in\mathrm{BMO}$, then, for $(2q)'\le p<\infty$,

$$\|[b,T_2]f\|_p\le C_p\|b\|_{\mathrm{BMO}}\|f\|_p. \tag{3.11}$$

(iii) Suppose $V\in B_q$ and $\frac{Q}{2}\le q<Q$, and let $b\in\mathrm{BMO}$. Then, for $p_0'\le p<\infty$,

$$\|[b,T_3]f\|_p\le C_p\|b\|_{\mathrm{BMO}}\|f\|_p,\quad \text{where}\ \ 1/p_0=1/q-1/Q. \tag{3.12}$$

Proof By use of Proposition 1, we only need to prove the kernels of $T_i,\ i=1,2,3$ satisfy the condition $H(m)$ for some m in Definition 3.

(i) We only need to prove that the kernel K_1 satisfies $H(q)$. By Lemma 3 and $V\in B_q$, we have

$$\begin{aligned}&\Big(\int_{2^kl\le|hg_0^{-1}|<2^{k+1}l}|K_1(g,h)-K_1(g_0,h)|^q\mathrm{d}y\Big)^{1/q}\\&\le C_N\frac{l^\delta}{(2^kl)^{Q-2+\delta}}\frac{1}{\{1+m(g_0,V)2^kl\}^N}\Big(\int_{B(g_0,2^{k+3}l)}V^q(h)\mathrm{d}h\Big)^{1/q}\\&\le C_N\frac{l^\delta}{(2^kl)^{Q-2+\delta}}\frac{1}{\{1+m(g_0,V)2^kl\}^N}(2^kl)^{-Q/q'}\int_{B(g_0,2^kl)}V(h)\mathrm{d}h\\&\le C_N\frac{l^\delta}{(2^kl)^{Q-2+\delta}}(2^kl)^{Q/q-2}\le C\frac{l^\delta}{(2^kl)^{(Q/q')+\delta}}.\end{aligned}\tag{3.13}$$

Then, we have

$$\sum_{k=5}^{\infty}k(2^kl)^{Q'/q}\Big(\int_{2^kl\le|g_0^{-1}h|<2^{k+1}l}|K_1(g,h)-K_1(g_0,h)|^q\,\mathrm{d}h\Big)^{1/q}\le C\sum_{k=5}^{\infty}\frac{k}{2^{k\delta}}\le C.\tag{3.14}$$

For the proof of (ii), we prove that K_2 satisfies $H(m)$ for $m=2q$. Form Lemma 4,

$$\begin{aligned}&\Big(\int_{2^kl\le|g_0^{-1}|<2^{k+1}l}|K_2(g,h)-K_2(g_0,h)|^{2q}\mathrm{d}h\Big)^{1/2q}\\&\le C_N\frac{l^\delta}{(2^kl)^{Q-1+\delta}}\frac{1}{\{1+m(g_0,V)2^kl\}^N}\Big(\int_{B(g_0,2^{k+3}l)}V^q(h)\mathrm{d}h\Big)^{1/2q}\\&\le C_N\frac{l^\delta}{(2^kl)^{Q-1+\delta}}\frac{1}{\{1+m(g_0,V)2^kl\}^N}(2^kl)^{-Q/(2q')}\Big(\int_{B(g_0,2^kl)}V(h)\mathrm{d}h\Big)^{1/2}\\&\le C_N\frac{l^\delta}{(2^kl)^{Q-1+\delta}}(2^kl)^{-Q/(2q')+(Q-2)/2}\le C\frac{l^\delta}{(2^kl)^{\delta}}(2^kl)^{-Q/(2q')}.\end{aligned}\tag{3.15}$$

Hence,

$$\sum_{k=5}^{\infty}k(2^kl)^{Q/(2q)'}\Big(\int_{2^kl\le|g_0^{-1}h|<2^{k+1}l}|K_2(g,h)-K_2(g_0,h)|^{2q}\mathrm{d}h\Big)^{1/2q}\le C\sum_{k=5}^{\infty}\frac{k}{2^{k\delta}}\le C.\tag{3.16}$$

For the proof of (iii), we prove that K_3 satisfies $H(p_0)$. From Lemma 5,

$$\begin{aligned}&\Big(\int_{2^kl\le|g_0^{-1}|<2^{k+1}l}|K_3(g,h)-K_3(g_0,h)|^{p_0}\mathrm{d}h\Big)^{1/p_0}\\&\le C_N\frac{l^\delta}{(2^kl)^{Q-1+\delta}}\frac{1}{\{1+m(g_0,V)2^kl\}^N}\Big\|\int_{B(g_0,2^{k+3}l)}\frac{V(u)}{|uh^{-1}|^{Q-1}}\mathrm{d}u\Big\|_{L^{p_0}(dh)}+\frac{l^\delta}{(2^kl)^{Q/p_0'}}\\&\le C_N\frac{l^\delta}{(2^kl)^{Q-1+\delta}}\frac{1}{\{1+m(g_0,V)2^kl\}^N}\Big(\int_{B(g_0,2^{k+3}l)}V^q(h)\mathrm{d}h\Big)^{1/q}+\frac{l^\delta}{(2^kl)^{(Q/p_0')+\delta}}\\&\le C_N\frac{l^\delta}{(2^kl)^{Q-1+\delta}}\frac{1}{\{1+m(g_0,V)2^kl\}^N}(2^kl)^{-Q/q'}\Big(\int_{B(g_0,2^kl)}V(h)\mathrm{d}h\Big)+\frac{l^\delta}{(2^kl)^{Q/p_0'+\delta}}\\&\le C\frac{l^\delta}{(2^kl)^{Q/p_0'+\delta}}.\end{aligned}\tag{3.17}$$

Therefore, we have

$$\sum_{k=5}^{\infty} k(2^k l)^{n/p_0'} \bigg(\int_{2^k l \le |g_0^{-1} h| < 2^{k+1} l} |K_3(g,h) - K_3(g_0,h)|^{p_0} \mathrm{d}y \bigg)^{1/p_0}$$
$$\le C \sum_{k=5}^{\infty} \frac{k}{2^{k\delta}} \le C. \tag{3.18}$$

By duality, we can get the following theorem.

Theorem 2 For the operators $T_1^* = V(-\triangle_{H^n} + V)^{-1}$, $T_2^* = V^{1/2}(-\triangle_{H^n} + V)^{-1/2}$, and $T_3^* = -\nabla_{H^n}(-\triangle_{H^n} + V)^{-1/2}$, we have

$$\|[b, T_1^*]f\|_p \le C_p \|b\|_{\mathrm{BMO}} \|f\|_p, \ 1 < p \le q; \tag{3.19}$$
$$\|[b, T_2^*]f\|_p \le C_p \|b\|_{\mathrm{BMO}} \|f\|_p, \ 1 < p \le 2q; \tag{3.20}$$
$$\|[b, T_3^*]f\|_p \le C_p \|b\|_{\mathrm{BMO}} \|f\|_p, \ 1 < p \le p_0, \ \text{where } 1/p_0 = 1/q - 1/Q. \tag{3.21}$$

From Theorem 1 (i), we can get the result concerning second order Riesz transform. Let $T_4 = (-\triangle_{H^n} + V)^{-1} \nabla_{H^n}^2$ and $T_4^* = \nabla_{H^n}^2 (-\triangle_{H^n} + V)^{-1}$, we have

Theorem 3 Suppose $V \in B_q, q \ge Q/2$, then, we have

$$\|[b, T_4]f\|_p \le C_p \|b\|_{\mathrm{BMO}} \|f\|_p, \ q' \le p < \infty; \tag{3.22}$$
$$\|[b, T_4^*]f\|_p \le C_p \|b\|_{\mathrm{BMO}} \|f\|_p, \ 1 < p \le q. \tag{3.23}$$

Proof We only need to prove $[b, T_4]$. Because

$$T_4 = (-\triangle_{H^n} + V)^{-1} \nabla^2 = (-\triangle_{H^n} + V)^{-1} \triangle_{H^n} \triangle_{H^n}^{-1} \triangle_{H^n}^2$$
$$= (I - (-\triangle_{H^n} + V)V) \frac{\nabla_{H_n}^2}{\triangle_{H^n}} = (I - T_1) \frac{\nabla_{H_n}^2}{\triangle_{H^n}}, \tag{3.24}$$

we can get $[b, T_4] = [b, I - T_1] \frac{\nabla_{H_n}^2}{\triangle_{H^n}} - (I - T_1)[b, \frac{\nabla_{H_n}^2}{\triangle_{H^n}}]$. Hence, the boundedness of $[b, T_4]$ follows from that of $[b, I - T_1]$ and $(I - T_1)$.

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