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ON THE COEFFICIENTS OF SEVERAL CLASSES OF BI-UNIVALENT FUNCTIONS*

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Abstract In this paper, we investigate the bounds of the coefficients of several classes of bi-univalent functions. The results presented in this paper improve or generalize the recent works of other authors.

Key words coefficient; univalent function; bi-univalent function; subordination

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1 Introduction

By $\mathcal{A}$ we denote the class of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $D = \{z : |z| < 1\}$. Also let S denote the subclass of functions in $\mathcal{A}$ which are univalent in D. A function f in S is said to be starlike of order $\alpha, 0 \le \alpha < 1$, and is denoted by $S^*(\alpha)$ if $\mathrm{Re}\{zf'(z)/f(z)\} > \alpha, z \in D$, and is said to be convex of order $\alpha, 0 \le \alpha < 1$, and is denoted by $K(\alpha)$ if $\mathrm{Re}\{1 + zf''(z)/f'(z)\} > \alpha, z \in D$. Mocanu [1] studied linear combinations of the representations of convex and starlike functions and defined the class of α-convex functions. In [2], it was shown that if

$$\mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0$$

for $z \in D$, then f is in the class of starlike functions $S^*(0)$ for α real and is in the class of convex functions $K(0)$ for $\alpha \ge 1$.

A function $f(z) \in \mathcal{A}$ is said to be in the class of strongly starlike functions of order α, denoted by $S_s^*(\alpha)$, if it satisfies the following inequality

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (|z| < 1, 0 \le \alpha < 1)$$

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and is said to be in the class of strongly convex functions of order α, denoted by $K_c(\alpha)$, if it satisfies the following inequality

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (|z|<1, 0\le\alpha<1).$$

Further, we say that $f(z)\in\mathcal{A}$ is α-starlike in D if $f(z)$ satisfies $f(z)f'(z)(1+zf''(z)/f'(z))\neq 0$ $(0<|z|<1)$ and

$$\mathrm{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\right\}>0.$$

For such α-starlike functions, Lewandowski, Miller and Zlotkiewicz [3] proved that all α-starlike functions are univalent and starlike for all $\alpha(-\infty<\alpha<+\infty)$.

An analytic function f is subordinate to an analytic function g, written $f(z)\prec g(z)$, provided there is an analytic function $\phi(z)$ defined in D with $\phi(0)=0$ and $|\phi(z)|<1$ so that $f(z)=g(\phi(z))$. In particular, if g is univalent in D then $f(z)\prec g(z)$ if and only if $f(0)=g(0)$ and $f(D)\subset g(D)$.

Let us denote

$$\mathcal{M}(\lambda,\varphi)=\left\{f\in\mathcal{A}:(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\varphi\right\},$$

where $0\le\lambda\le1$, φ is an analytic function with positive real part in the open unit disk D, $\varphi(0)=1,\varphi'(0)>0$ and $\varphi(D)$ is symmetric with respect to the real axis. For special choices for the number λ and function φ, we can obtain several important classes of analytic functions. For example, $\mathcal{M}(0,[1+(1-2\alpha)z]/(1-z))$ is the class $S^*(\alpha)$, $\mathcal{M}(1,[1+(1-2\alpha)z]/(1-z))$ is the class $K(\alpha)$, $\mathcal{M}(0,((1+z)/(1-z))^{\alpha})$ is the class $S_s^*(\alpha)$, $\mathcal{M}(1,((1+z)/(1-z))^{\alpha})$ is the class $K_c(\alpha)$. These classes of analytic functions were well investigated by many authors [1–9].

For each $f\in S$, the Koebe one-quarter theorem [10] ensures the image of D under f contains a disk of radius $1/4$. Thus every univalent function $f\in S$ has an inverse f^{-1} satisfying

$$f^{-1}(f(z))=z \quad (|z|<1)$$

and

$$f(f^{-1}(w))=w \quad \left(|w|<r_0(f), r_0(f)\ge\frac{1}{4}\right).$$

A function $f\in\mathcal{A}$ is said to be bi-univalent in D if $f(z)$ is univalent in $|z|<1$ and $g(w)=f^{-1}(w)$ is univalent in $|w|<1$. If f given by (1.1) is bi-univalent, then a computation shows that $g=f^{-1}$ has the expansion

$$g(w)=w-a_2w^2+(2a_2^2-a_3)w^3+\cdots. \tag{1.2}$$

Let Σ denote the class of all bi-univalent functions defined in the unit disk D. Lewin [11] investigated the class Σ and showed that $|a_2|<1.51$ for function $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n\in\Sigma$. Subsequently, Brannan and Clunie [12] conjectured that $|a_2|\le\sqrt{2}$ for $f\in\Sigma$. Netanyahu [13] showed that $\max|a_2|=4/3$ if $f(z)\in\Sigma$. Brannan and Taha [14] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions [15–18]. Rosihan et al [18] obtained estimates on the initial coefficients for bi-starlike functions of Ma-Minda type and bi-convex functions of Ma-Minda type. In the same paper several related classes were also considered and a connection to earlier known results were made.

In this paper, by using the method different from that used by other authors, we also investigate the bounds of initial coefficients for the subclasses of bi-univalent functions considered by Rosihan et al and get more accurate estimation than that given in [15–18].

2 Coefficient Estimates

In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk D, satisfying $\varphi(0)=1, \varphi'(0)>0$, and $\varphi(D)$ is symmetric with respect to the real axis. Such a function has a Taylor series of the form

$$\varphi(z)=1+B_1z+B_2z^2+b_3z^3+\cdots\ (B_1>0). \tag{2.1}$$

Suppose that $u(z)$ and $v(z)$ are analytic in the unit disk D with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(z)|<1$, and suppose that

$$u(z)=b_1z+\sum_{n=2}^{\infty}b_nz^n,\quad v(z)=c_1z+\sum_{n=2}^{\infty}c_nz^n\ (|z|<1). \tag{2.2}$$

It is well known that

$$|b_1|\le 1,\quad |b_2|\le 1-|b_1|^2,\quad |c_1|\le 1,\quad |c_2|\le 1-|c_1|^2. \tag{2.3}$$

By a simple calculation, we have

$$\varphi(u(z))=1+B_1b_1z+(B_1b_2+B_2b_1^2)z^2+\cdots,\quad |z|<1 \tag{2.4}$$

and

$$\varphi(v(w))=1+B_1c_1w+(B_1c_2+B_2c_1^2)w^2+\cdots,\quad |w|<1. \tag{2.5}$$

Definition 2.1 [18] A function $f\in\Sigma$ is said to be in the class $\mathcal{H}_\Sigma(\varphi)$ if and only if

$$f'(z)\prec\varphi(z),\quad g'(w)\prec\varphi(w),$$

where $g(w)=f^{-1}(w)$.

Theorem 2.1 If f given by (1.1) is in the class $\mathcal{H}_\Sigma(\varphi)$, then

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{4B_1+|3B_1^2-4B_2|}} \tag{2.6}$$

and

$$|a_3|\le\begin{cases}\left(1-\dfrac{4}{3B_1}\right)\dfrac{B_1^3}{4B_1+|3B_1^2-4B_2|}+\dfrac{1}{3}B_1, & \text{if } B_1\ge\dfrac{4}{3},\\ \dfrac{1}{3}B_1, & \text{if } B_1<\dfrac{4}{3}.\end{cases} \tag{2.7}$$

Proof Let $f \in \mathcal{H}_\Sigma(\varphi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : D \to D$ given by (2.2) such that

$$f'(z) = \varphi(u(z)), \quad g'(w) = \varphi(v(w)). \tag{2.8}$$

Since

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + \cdots, \quad g'(w) = 1 - 2a_2 w + 3(2a_2^2 - a_3)w^2 + \cdots, \tag{2.9}$$

it follows from (2.4), (2.5), (2.8) and (2.9) that

$$2a_2 = B_1 b_1, \tag{2.10}$$

$$3a_3 = B_1 b_2 + B_2 b_1^2, \tag{2.11}$$

$$-2a_2 = B_1 c_1, \tag{2.12}$$

$$3(2a_2^2 - a_3) = B_1 c_2 + B_2 c_1^2. \tag{2.13}$$

From (2.10) and (2.12), we get

$$b_1 = -c_1. \tag{2.14}$$

By adding (2.13) to (2.11), further computations using (2.10) and (2.14) lead to

$$6(B_1^2 - 8B_2)a_2^2 = B_1^3(b_2 + c_2), \tag{2.15}$$

(2.14), (2.15), together with (2.3), give that

$$|6(B_1^2 - 8B_2)a_2^2| \le 2B_1^3(1 - |b_1|^2). \tag{2.16}$$

From (2.10) and (2.16) we get

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{4B_1 + |3B_1^2 - 4B_2|}}. \tag{2.17}$$

By subtracting (2.13) from (2.11), in view of (2.14), we have

$$6a_3 = 6a_2^2 + B_1(b_2 - c_2). \tag{2.18}$$

From (2.3), (2.10), (2.14) and (2.18), it follows that

$$\begin{aligned}
|a_3| &\le |a_2|^2 + \frac{1}{6}B_1(|b_2| + |c_2|) \\
&\le |a_2|^2 + \frac{1}{6}B_1(1 - |b_1|^2 + 1 - |c_1|^2) \\
&= \left(1 - \frac{4}{3B_1}\right)|a_2|^2 + \frac{1}{3}B_1 \\
&\le \begin{cases} \left(1 - \frac{4}{3B_1}\right)\frac{B_1^3}{4B_1 + |3B_1^2 - 4B_2|} + \frac{1}{3}B_1, & \text{if } B_1 \ge \frac{4}{3}, \\ \frac{1}{3}B_1, & \text{if } B_1 < \frac{4}{3}. \end{cases}
\end{aligned}$$

□

Remark 2.1 The bounds on $|a_2|$ and $|a_3|$ given by (2.6) and (2.7) are smaller than that given in Theorem 2.1 in [18]. If let

$$\phi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+2\alpha^2z^2+\cdots\quad(0<\alpha\le 1),$$

then inequalities (2.6) and (2.7) become

$$|a_2|\le\frac{\sqrt{2}\alpha}{\sqrt{2+\alpha}},\quad |a_3|\le\begin{cases}\dfrac{8\alpha^2}{6+3\alpha}, & \text{if } \dfrac{2}{3}\le\alpha\le 1,\\ \dfrac{2\alpha}{3}, & \text{if } 0<\alpha<\dfrac{2}{3}.\end{cases}\tag{2.19}$$

The bound on $|a_3|$ given in (2.19) is more accurate than that given by Theorem 1 in [16].

If let

$$\phi(z)=\frac{1+(1-2\alpha)z}{1-z}=1+2(1-\alpha)z+2(1-\alpha)z^2+\cdots\quad(0\le\alpha<1),$$

then inequalities (2.6) and (2.7) become

$$|a_2|\le\frac{\sqrt{2}(1-\alpha)}{\sqrt{2+|1-3\alpha|}},\quad |a_3|\le\begin{cases}\dfrac{8-12\alpha}{9}, & \text{if } 0\le\alpha\le\dfrac{1}{3},\\ \dfrac{2(1-\alpha)}{3}, & \text{if } \dfrac{1}{3}<\alpha<1.\end{cases}\tag{2.20}$$

The bounds on $|a_2|,|a_3|$ given in (2.20) are more accurate than that given by Theorem 2 in [16].

Definition 2.2 [18] A function $f\in\Sigma$ is said to be in the class $\mathcal{ST}_\Sigma(\alpha,\varphi),\alpha\ge 0$, if the following subordination hold

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\prec\varphi(z),\quad \frac{wg'(w)}{g(w)}+\alpha\frac{w^2g''(w)}{g(w)}\prec\varphi(w),$$

where $g(w)=f^{-1}(w)$.

Theorem 2.2 Let f given by (1.1) be in the class $\mathcal{ST}_\Sigma(\alpha,\varphi)$. Then

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{|(1+4\alpha)B_1^2-(1+2\alpha)^2B_2|+(1+2\alpha)^2B_1}}\tag{2.21}$$

and

$$|a_3|\le\begin{cases}\dfrac{B_1}{1+4\alpha}, & \text{if } |B_2|\le B_1,\\ \dfrac{|(1+4\alpha)B_1^2-(1+2\alpha)^2B_2|B_1+(1+2\alpha)^2B_1|B_2|}{(1+4\alpha)(|(1+4\alpha)B_1^2-(1+2\alpha)^2B_2|+(1+2\alpha)^2B_1)}, & \text{if } |B_2|>B_1.\end{cases}\tag{2.22}$$

Proof Let $f\in\mathcal{ST}_\Sigma(\alpha,\varphi)$. Then there are analytic functions $u,v:D\to D$ given by (2.2) such that

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}=\varphi(u(z)),\quad \frac{wg'(w)}{g(w)}+\alpha\frac{w^2g''(w)}{g(w)}=\varphi(v(w)),\tag{2.23}$$

where $g(w)=f^{-1}(w)$. Since

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}=1+(1+2\alpha)a_2z+(2(1+3\alpha)a_3-(1+2\alpha)a_2^2)z^2+\cdots$$

and

$$\frac{wg'(w)}{g(w)} + \alpha\frac{w^2g''(w)}{g(w)} = 1 - (1+2\alpha)a_2w + ((3+10\alpha)a_2^2 - 2(1+3\alpha)a_3)w^2 + \cdots,$$

it follows from (2.4), (2.5) and (2.23) that

$$(1+2\alpha)a_2 = B_1b_1, \tag{2.24}$$

$$2(1+3\alpha)a_3 - (1+2\alpha)a_2^2 = B_1b_2 + B_2b_1^2, \tag{2.25}$$

$$-(1+2\alpha)a_2 = B_1c_1, \tag{2.26}$$

$$(3+10\alpha)a_2^2 - 2(1+3\alpha)a_3 = B_1c_2 + B_2c_1^2. \tag{2.27}$$

From (2.24) and (2.26) we get

$$c_1 = -b_1. \tag{2.28}$$

Equations (2.25), (2.26) and (2.28) lead to

$$[(2+8\alpha)B_1^2 - 2(1+2\alpha)^2B_2]a_2^2 = B_1^3(b_2+c_2).$$

Then, in view of (2.3), (2.24) and (2.28), we have

$$\begin{aligned}&|(2+8\alpha)B_1^2 - 2(1+2\alpha)^2B_2||a_2|^2\\ \le\ & B_1^3(|b_2|+|c_2|) \le 2B_1^3(1-|b_1|^2) = 2B_1^3 - 2(1+2\alpha)^2B_1|a_2|^2.\end{aligned}$$

Thus, we get

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{|(1+4\alpha)B_1^2 - (1+2\alpha)^2B_2| + (1+2\alpha)^2B_1}}.$$

Next, from (2.25) and (2.27), we have

$$4(1+3\alpha)(1+4\alpha)a_3 = (3+10\alpha)B_1b_2 + (1+2\alpha)B_1c_2 + 4(1+3\alpha)B_2b_1^2.$$

Then, in view of (2.3), we have

$$\begin{aligned}4(1+3\alpha)(1+4\alpha)|a_3| &\le (3+10\alpha)B_1|b_2| + (1+2\alpha)B_1|c_2| + 4(1+3\alpha)|B_2||b_1|^2\\ &= 4(1+3\alpha)B_1 - 4(1+3\alpha)B_1|b_1|^2 + 4(1+3\alpha)|B_2||b_1|^2.\end{aligned}$$

Since

$$|b_1|^2 = \frac{(1+2\alpha)^2}{B_1^2}|a_2|^2 \le \frac{(1+2\alpha)^2B_1}{|(1+4\alpha)B_1^2 - (1+2\alpha)^2B_2| + (1+2\alpha)^2B_1},$$

it follows that

$$|a_3| \le \begin{cases}\dfrac{B_1}{1+4\alpha}, & \text{if } |B_2| \le B_1,\\[2ex] \dfrac{|(1+4\alpha)B_1^2 - (1+2\alpha)^2B_2|B_1 + (1+2\alpha)^2B_1|B_2|}{(1+4\alpha)(|(1+4\alpha)B_1^2 - (1+2\alpha)^2B_2| + (1+2\alpha)^2B_1)}, & \text{if } |B_2| > B_1.\end{cases}$$

□

Remark 2.2 The bounds on $|a_2|$ and $|a_3|$ given by (2.21) and (2.22) are smaller than that given in Theorem 2.2 in [18]. If let $\alpha = 0$ and if let

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots \quad (0 < \gamma \le 1),$$

then inequalities (2.21) and (2.22) become

$$|a_2| \le \frac{2\gamma}{\sqrt{1+\gamma}}, \quad |a_3| \le 2\gamma. \tag{2.29}$$

The bound on $|a_3|$ given in (2.29) is more accurate than that given by Theorem 2.1 in [14].

If let $\alpha = 0$ and let

$$\phi(z) = \frac{1+(1-2\gamma)z}{1-z} = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + \cdots \quad (0 \le \gamma < 1),$$

then inequalities (2.21) and (2.22) become

$$|a_2| \le \frac{2(1-\gamma)}{\sqrt{1+|1-2\gamma|}}, \quad |a_3| \le 2(1-\gamma). \tag{2.30}$$

The bound on $|a_2|$ given in (2.30) is more accurate than that given by Theorem 3.1 in [14].

Definition 2.3 [18] A function $f \in \Sigma$ is said to be in the class $\mathcal{M}_\Sigma(\alpha, \varphi), \alpha \ge 0$, if the following subordination hold:

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z), \quad (1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \varphi(w),$$

where $g(w) = f^{-1}(w)$.

Theorem 2.3 Let f given by (1.1) be in the class $\mathcal{M}_\Sigma(\alpha, \varphi), \alpha \ge 0$. Then

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{(1+\alpha)|B_1^2 - (1+\alpha)B_2| + (1+\alpha)^2 B_1}} \tag{2.31}$$

and

$$|a_3| \le \begin{cases} \dfrac{B_1}{1+\alpha}, & \text{if } |B_2| \le B_1, \\ \dfrac{|B_1^2 - (1+\alpha)B_2|B_1 + (1+\alpha)B_1|B_2|}{(1+\alpha)(|B_1^2 - (1+\alpha)B_2| + (1+\alpha)B_1)}, & \text{if } |B_2| > B_1. \end{cases} \tag{2.32}$$

Proof Let $f \in \mathcal{M}_\Sigma(\alpha, \varphi), \alpha \ge 0$. Then there are analytic functions $u, v : D \to D$ given by (2.2) such that

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi(u(z)) \tag{2.33}$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) = \varphi(v(w)), \tag{2.34}$$

where $g(w) = f^{-1}(w)$. Since

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + (1+\alpha)a_2 z + (2(1+2\alpha)a_3 - (1+3\alpha)a_2^2)z^2 + \cdots$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)=1-(1+\alpha)a_2w+((3+5\alpha)a_2^2-2(1+2\alpha)a_3)w^2+\cdots,$$

it follows from (2.4), (2.5), (2.33) and (2.34) that

$$(1+\alpha)a_2=B_1b_1, \tag{2.35}$$

$$2(1+2\alpha)a_3-(1+3\alpha)a_2^2=B_1b_2+B_2b_1^2, \tag{2.36}$$

$$-(1+\alpha)a_2=B_1c_1, \tag{2.37}$$

$$(3+5\alpha)a_2^2-2(1+2\alpha)a_3=B_1c_2+B_2c_1^2. \tag{2.38}$$

From (2.35) and (2.37) we get

$$c_1=-b_1. \tag{2.39}$$

Equations (2.35),(2.36), (2.38) and (2.39) lead to

$$2(1+\alpha)[B_1^2-(1+\alpha)B_2]a_2^2=B_1^3(b_2+c_2).$$

Then, in view of (2.3), (2.35) and (2.39), we have

$$\begin{aligned}&2(1+\alpha)|B_1^2-(1+\alpha)B_2||a_2|^2\\ \le\ &B_1^3(|b_2|+|c_2|)\le 2B_1^3(1-|b_1|^2)=2B_1^3-2(1+\alpha)^2B_1|a_2|^2.\end{aligned}$$

Thus, we get

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{(1+\alpha)|B_1^2-(1+\alpha)B_2|+(1+\alpha)^2B_1}}.$$

Next, from (2.36) and (2.38), we have

$$4(1+2\alpha)(1+\alpha)a_3=(3+5\alpha)B_1b_2+(1+3\alpha)B_1c_2+4(1+2\alpha)B_2b_1^2.$$

Then, in view of (2.3), we have

$$4(1+2\alpha)(1+\alpha)|a_3|\le 4(1+2\alpha)B_1+4(1+2\alpha)[|B_2|-B_1]|b_1|^2.$$

Notice that

$$|b_1|=\frac{(1+\alpha)^2}{B_1^2}|a_2|^2\le\frac{(1+\alpha)B_1}{|B_1^2-(1+\alpha)B_2|+(1+\alpha)B_1},$$

we get

$$|a_3|\le\begin{cases}\dfrac{B_1}{1+\alpha}, & \text{if } |B_2|\le B_1,\\[2mm] \dfrac{|B_1^2-(1+\alpha)B_2|B_1+(1+\alpha)B_1|B_2|}{(1+\alpha)(|B_1^2-(1+\alpha)B_2|+(1+\alpha)B_1)}, & \text{if } |B_2|>B_1.\end{cases}$$

□

Remark 2.3 The bounds on $|a_2|$ and $|a_3|$ given by (2.31) and (2.32) are smaller than that given in Theorem 2.3 in [18].

Definition 2.4 [18] A function $f \in \Sigma$ is said to be in the class $\mathcal{L}_\Sigma(\alpha,\varphi), \alpha \geq 0$, if the following subordination hold

$$\left(\frac{zf'(z)}{f(z)}\right)^\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z), \quad \left(\frac{wg'(w)}{g(w)}\right)^\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)^{1+\alpha} \prec \varphi(w),$$

where $g(w) = f^{-1}(w)$.

Theorem 2.4 Let f given by (1.1) be in the class $\mathcal{L}_\Sigma(\alpha,\varphi), 0 \leq \alpha < \frac{3}{2}$. Then

$$|a_2| \leq \frac{B_1\sqrt{2B_1}}{\sqrt{|(\alpha^2-3\alpha+4)B_1^2-2(2-\alpha)^2B_2|+2(2-\alpha)^2B_1}} \tag{2.40}$$

and

$$|a_3| \leq \begin{cases} \dfrac{2B_1}{\alpha^2-3\alpha+4}, & \text{if } |B_2| \leq B_1, \\ \dfrac{2|(\alpha^2-3\alpha+4)B_1^2-2(2-\alpha)^2B_2|B_1+4(2-\alpha)^2B_1|B_2|}{(\alpha^2-3\alpha+4)(|(\alpha^2-3\alpha+4)B_1^2-2(2-\alpha)^2B_2|+2(2-\alpha)^2B_1)}, & \text{if } |B_2| > B_1. \end{cases} \tag{2.41}$$

Proof Let $f \in \mathcal{L}_\Sigma(\alpha,\varphi), \alpha \geq 0$. Then there are analytic functions $u, v : D \to D$ given by (2.2) such that

$$\left(\frac{zf'(z)}{f(z)}\right)^\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = \varphi(u(z)) \tag{2.42}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha} = \varphi(v(w)), \tag{2.43}$$

where $g(w) = f^{-1}(w)$. Since

$$\begin{aligned}&\left(\frac{zf'(z)}{f(z)}\right)^\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\\ &= 1+(2-\alpha)a_2z+\left(2(3-2\alpha)a_3+\frac{(\alpha-2)^2-3(4-3\alpha)}{2}a_2^2\right)z^2+\cdots\end{aligned}$$

and

$$\begin{aligned}&\left(\frac{wg'(w)}{g(w)}\right)^\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha}\\ &= 1-(2-\alpha)a_2w+\left(8(1-\alpha)+\frac{1}{2}\alpha(\alpha+5)\right)a_2^2-2(3-2\alpha)a_3)w^2+\cdots,\end{aligned}$$

it follows from (2.4), (2.5), (2.42) and (2.43) that

$$(2-\alpha)a_2 = B_1b_1, \tag{2.44}$$

$$2(3-2\alpha)a_3+\frac{(\alpha-2)^2-3(4-3\alpha)}{2}a_2^2 = B_1b_2+B_2b_1^2, \tag{2.45}$$

$$-(2-\alpha)a_2 = B_1c_1, \tag{2.46}$$

$$(8(1-\alpha)+\frac{1}{2}\alpha(\alpha+5))a_2^2-2(3-2\alpha)a_3 = B_1c_2+B_2c_1^2. \tag{2.47}$$

From (2.44) and (2.46) we get

$$c_1 = -b_1. \tag{2.48}$$

Equations (2.44), (2.45), (2.47) and (2.48) lead to

$$((\alpha^2 - 3\alpha + 4)B_1^2 - 2(2-\alpha)^2 B_2)a_2^2 = B_1^3(b_2 + c_2).$$

Then, in view of (2.3), (2.44) and (2.48), we have

$$\begin{aligned}&|(\alpha^2 - 3\alpha + 4)B_1^2 - 2(2-\alpha)^2 B_2||a_2|^2\\ &\le B_1^3(|b_2| + |c_2|) \le 2B_1^3(1 - |b_1|^2) = 2B_1^3 - 2(2-\alpha)^2 B_1|a_2|^2.\end{aligned}$$

Thus, we get

$$|a_2| \le \frac{B_1\sqrt{2B_1}}{\sqrt{|(\alpha^2 - 3\alpha + 4)B_1^2 - 2(2-\alpha)^2 B_2| + 2(2-\alpha)^2 B_1}}.$$

Next, from (2.45), (2.47) and (2.48), we have

$$8(3-2\alpha)(\alpha^2 - 3\alpha + 4)a_3 = 2(\alpha^2 - 11\alpha + 16)B_1 b_2 - 2(\alpha^2 + 5\alpha - 8)B_1 c_2 + 16(3-2\alpha)B_2 b_1^2.$$

Then, in view of (2.3), we have

$$8(3-2\alpha)(\alpha^2 - 3\alpha + 4)|a_3| \le 16(3-2\alpha)B_1 + 16(3-2\alpha)[|B_2| - B_1]|b_1|^2.$$

Notice that

$$|b_1|^2 = \frac{(2-\alpha)^2}{B_1^2}|a_2|^2 \le \frac{2(2-\alpha)^2 B_1}{|(\alpha^2 - 3\alpha + 4)B_1^2 - 2(2-\alpha)^2 B_2| + 2(2-\alpha)^2 B_1},$$

we get

$$|a_3| \le \begin{cases} \dfrac{2B_1}{\alpha^2 - 3\alpha + 4}, & \text{if } |B_2| \le B_1,\\[2ex] \dfrac{2|(\alpha^2 - 3\alpha + 4)B_1^2 - 2(2-\alpha)^2 B_2|B_1 + 4(2-\alpha)^2 B_1|B_2|}{(\alpha^2 - 3\alpha + 4)(|(\alpha^2 - 3\alpha + 4)B_1^2 - 2(2-\alpha)^2 B_2| + 2(2-\alpha)^2 B_1)}, & \text{if } |B_2| > B_1.\end{cases}$$

□

Remark 2.4 The bounds on $|a_2|$ and $|a_3|$ given by (2.40) and (2.41) are smaller than that given in Theorem 2.4 in [18].

Definition 2.5 A function $f \in \Sigma$ is said to be in the class $\mathcal{B}_\Sigma(\lambda, \varphi)$, $\lambda \ge 0$, if the following subordination hold

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \varphi(z), \quad (1-\lambda)\frac{g(w)}{z} + \lambda g'(w) \prec \varphi(w),$$

where $g(w) = f^{-1}(w)$.

Theorem 2.5 Let f given by (1.1) be in the class $\mathcal{B}_\Sigma(\lambda, \varphi)$, $\lambda \ge 0$. Then

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{|(2\lambda+1)B_1^2 - (\lambda+1)^2 B_2| + (\lambda+1)^2 B_1}} \tag{2.49}$$

and

$$|a_3| \le \begin{cases} \dfrac{B_1}{2\lambda+1}, & \text{if } B_1 \le \dfrac{(\lambda+1)^2}{2\lambda+1},\\[2ex] \dfrac{|(2\lambda+1)B_1^2 - (\lambda+1)^2 B_2|B_1 + (2\lambda+1)B_1^3}{(2\lambda+1)(|2\lambda+1)B_1^2 - (\lambda+1)^2 B_2| + (\lambda+1)^2 B_1)}, & \text{if } B_1 > \dfrac{(\lambda+1)^2}{2\lambda+1}.\end{cases} \tag{2.50}$$

Proof Let $f \in \mathcal{B}_\Sigma(\lambda,\varphi), \lambda \geq 0$. Then there are analytic functions $u, v : D \to D$ given by (2.2) such that

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = \varphi(u(z)) \tag{2.51}$$

and

$$(1-\lambda)\frac{g(w)}{z} + \lambda g'(w) = \varphi(v(w)), \tag{2.52}$$

where $g(w) = f^{-1}(w)$. Since

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = 1 + (1+\lambda)a_2 z + (1+2\lambda)a_3 z^2 + \cdots$$

and

$$(1-\lambda)\frac{g(w)}{z} + \lambda g'(w) = 1 - (1+\lambda)a_2 w + (1+2\lambda)(2a_2^2 - a_3)w^2 + \cdots,$$

it follows from (2.4), (2.5), (2.51) and (2.52) that

$$(1+\lambda)a_2 = B_1 b_1, \tag{2.53}$$

$$(2\lambda+1)a_3 = B_1 b_2 + B_2 b_1^2, \tag{2.54}$$

$$-(1+\lambda)a_2 = B_1 c_1, \tag{2.55}$$

$$(2\lambda+1)(2a_2^2 - a_3) = B_1 c_2 + B_2 c_1^2. \tag{2.56}$$

From (2.53) and (2.55) we get

$$c_1 = -b_1. \tag{2.57}$$

Equations (2.53), (2.54), (2.56) and (2.57) lead to

$$(2(2\lambda+1)B_1^2 - 2(1+\lambda)^2 B_2)a_2^2 = B_1^3(b_2 + c_2).$$

Then, in view of (2.3), (2.53) and (2.57), we have

$$\begin{aligned}&|2(2\lambda+1)B_1^2 - 2(1+\lambda)^2 B_2||a_2|^2\\ \leq{}& B_1^3(|b_2| + |c_2|) \leq 2B_1^3(1-|b_1|^2) = 2B_1^3 - 2(1+\lambda)^2 B_1|a_2|^2.\end{aligned}$$

Thus, we get

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|(2\lambda+1)B_1^2 - (\lambda+1)^2 B_2| + (\lambda+1)^2 B_1}}.$$

Next, from (2.54), (2.56) and (2.57), we have

$$2(2\lambda+1)a_3 = 2(2\lambda+1)a_2^2 + B_1(b_2 - c_2).$$

Then, in view of (2.3) and (2.57), we have

$$\begin{aligned}2(2\lambda+1)|a_3| &\leq 2(2\lambda+1)|a_2|^2 + B_1(|b_2| + |c_2|)\\ &\leq 2(2\lambda+1)|a_2|^2 + 2B_1(1-|b_1|^2).\end{aligned}$$

It follows from (2.53) that

$$(2\lambda+1)B_1|a_3| \leq [(2\lambda+1)B_1 - (\lambda+1)^2]|a_2|^2 + B_1^2.$$

Notice that (2.49), we have

$$|a_3| \le \begin{cases} \dfrac{B_1}{2\lambda+1}, & \text{if } B_1 \le \dfrac{(\lambda+1)^2}{2\lambda+1}, \\ \dfrac{|(2\lambda+1)B_1^2-(\lambda+1)^2B_2|B_1+(2\lambda+1)B_1^3}{(2\lambda+1)(|2\lambda+1)B_1^2-(\lambda+1)^2B_2|+(\lambda+1)^2B_1)}, & \text{if } B_1 > \dfrac{(\lambda+1)^2}{2\lambda+1}. \end{cases}$$

□

Remark 2.5 If let

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1+2\alpha z+2\alpha^2z^2+\cdots \quad (0<\alpha\le 1),$$

then inequalities (2.49) and (2.50) become

$$|a_2| \le \frac{2\alpha}{\sqrt{(\lambda+1)^2+|1+2\lambda-\lambda^2|\alpha}} \tag{2.58}$$

and

$$|a_3| \le \begin{cases} \dfrac{2\alpha}{2\lambda+1}, & \text{if } 0<\alpha\le \dfrac{(\lambda+1)^2}{2(2\lambda+1)}, \\ \dfrac{4(2\lambda+1)\alpha^2+2\alpha^2|1+2\lambda-\lambda^2|}{(2\lambda+1)(|1+2\lambda-\lambda^2|\alpha+(\lambda+1)^2)}, & \text{if } \dfrac{(\lambda+1)^2}{2(2\lambda+1)}<\alpha\le 1. \end{cases} \tag{2.59}$$

The bounds on $|a_2|$ and $|a_3|$ given in (2.58) and (2.59) are more accurate than that given by Theorem 2.2 in [17].

If let

$$\phi(z) = \frac{1+(1-2\alpha)z}{1-z} = 1+2(1-\alpha)z+2(1-\alpha)z^2+\cdots \quad (0\le\alpha<1),$$

then inequalities (2.49) and (2.50) become

$$|a_2| \le \frac{2(1-\alpha)}{\sqrt{|2(2\lambda+1)(1-\alpha)-(\lambda+1)^2|+(\lambda+1)^2}} \tag{2.60}$$

and

$$|a_3| \le \begin{cases} \dfrac{2(1-\alpha)}{2\lambda+1}, & \text{if } \dfrac{1+2\lambda-\lambda^2}{2(2\lambda+1)}\le\alpha<1, \\ \dfrac{4(2\lambda+1)(1-\alpha)-(\lambda+1)^2}{(2\lambda+1)^2}, & \text{if } 0\le\alpha<\dfrac{1+2\lambda-\lambda^2}{2(2\lambda+1)}. \end{cases} \tag{2.61}$$

The bounds on $|a_2|$ and $|a_3|$ given in (2.60) and (2.61) are more accurate than that given by Theorem 3.2 in [17].

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