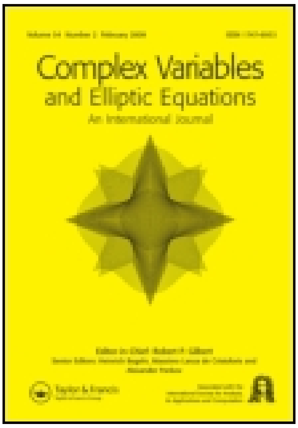

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Power matrices for Faber polynomials and conformal welding

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Power matrices for Faber polynomials and conformal welding†

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The power matrix is the matrix of the coefficients of the power series at 0 of powers of an analytic function. Composition corresponds to matrix multiplication. We generalize the power matrix by replacing power series with Faber polynomial expansions. We show that composition corresponds to multiplication of the generalized power matrices, for both simply and doubly connected domains. We apply this to give some matrix product formulas for the coefficients of conformal welding maps of an analytic homeomorphism of an analytic curve. In particular, in some sense one can solve for the coefficients of the conformal welding maps in terms of the generalized power matrix of the analytic homeomorphism.

Keywords: Faber polynomials; power matrix; conformal welding; univalent functions

AMS Subject Classifications: 30C55; 30C35; 30B99

1. Introduction

1.1. *Results*

Jabotinsky [1] generalized the power matrices of Schur to include the inverse powers of maps which are analytic in a neighbourhood of 0. In doing so, he was able to illuminate some algebraic structure of the Grunsky matrices and Faber polynomials. In this article, we generalize the power matrices still further to functions which are analytic in a doubly connected neighbourhood of an arbitrary analytic curve. The generalization uses Faber polynomials on the interior and exterior of the domain and image curve rather than positive and negative powers of z.

We show that the product of generalized power matrices corresponds to composition of functions between such doubly connected domains (Theorem 2.5). In contrast to the case of the power matrix, the sums involved in the product are infinite, and thus convergence is an issue. However, Faber polynomials are well-suited to the problem.

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As an application, we show that a product formula holds for conformal welding, in the case that the resultant curve is analytic (Corollary 3.4). Furthermore, given an analytic one-to-one map ϕ: $S^1 \to S^1$, in a certain sense one can solve for the coefficients of the welding maps F and G using matrix operations (Theorem 3.10). Note, however, that these operations involve infinite sums.

Although the proofs in this article are elementary, we felt that since the matrix product formulas are so simple and unexpected, they deserve to be recorded.

1.2. *Motivation*

The matrix algebra here has an interesting motivation that we would like to briefly explain.

A version of the conformal welding theorem appears in conformal field theory, where it is sometimes referred to as the 'sewing equation'. It arises when joining the Riemann surfaces together. This version is as follows. Let $\mathbb{D} = \{z\colon |z| < 1\}$ and let F_1: $\mathbb{D} \to \mathbb{C}$ be a one-to-one holomorphic map such that $F_1(0) = 0$. Denote $\mathbb{D}^* = \{z\colon |z| > 1\} \cup \{\infty\}$ and let $G_1\colon \mathbb{D}^* \to \overline{\mathbb{C}}$ be a one-to-one meromorphic map with a simple pole at ∞ and no other poles. We also assume that both maps have analytic extensions to a neighbourhood of the closure of their respective domains. The 'geometric sewing equation' [2] is

$$G = F \circ G_1 \circ F_1^{-1} \tag{1.1}$$

where F: $\overline{G_1(\mathbb{D}^*)}^c \to \mathbb{C}$ and G: $\overline{F_1(\mathbb{D})}^c \to \overline{\mathbb{C}}$, and $\cdot^c$ denotes the complement in the Riemann sphere. For fixed F_1 and G_1, the pair of maps F and G is called a solution to the sewing equation.

Given F_1 and G_1 such a pair of maps F and G is guaranteed to exist by the conformal welding theorem (Remark 3.3), which says that given a quasisymmetric homeomorphism ϕ between quasicircles Q_1 and Q_2, there are holomorphic maps F and G, on the interior of Q_2 and the exterior of Q_1, respectively, such that $F^{-1} \circ G = \phi$. Applying this theorem with $\phi = G_1 \circ F_1^{-1}$ demonstrates the existence of a solution to the above geometric sewing equation.

However, Huang [2] proved the following surprising fact: there is a unique solution to the sewing equation even in the setting of formal power series. The proof (necessarily) is entirely algebraic, and in this setting, Huang refers to Equation (1.1) as the 'algebraic sewing equation'. Given the existence of an algebraic proof of a theorem whose content is so near that of the conformal welding theorem, one is led to ask the following questions. (1) Is there an algebraic formula for the coefficients of F and G in terms of the coefficients of ϕ? (2) More generally, is there an algebraic structure to conformal welding?

This article provides some answers to these questions. Our notion of 'algebraic' differs from Huang's, in that we deal with matrix products rather than formal power series. Since the matrix products necessarily involve infinite sums, convergence becomes a crucial assumption. Thus, our theorems involve stronger analytic assumptions, but result in a much simpler algebraic structure.

Remark 1.1 Huang's proof of the existence of a solution to the algebraic sewing equation relies in a fundamental way on the fact that the map ϕ is already factored as a pair of formal series $G_1 \circ F_1^{-1}$. This assumption is probably necessary to get a proof in the formal power series setting. It is an interesting question to what extent the results of this article can be formulated and proved for formal power series.

The power matrix of an analytic diffeomorphism ϕ of S^1 is just the expression in the basis $\{z^n: n \in \mathbb{Z}\}$ for the operator given as composition by ϕ in an appropriate function space. Nag and Sullivan [3] showed that composition by ϕ is a bounded operator on a certain Hilbert space $H^{1/2}$, and its algebraic properties were investigated in the context of the so-called period map. Algebraic relations between the components of this composition operator and the Grunsky matrices were given by Takhtajan and Teo [4] in their monograph, also in the context of the Kirillov–Yuriev–Nag–Sullivan period map. Their proof of the invertibility of one of the blocks of the composition operator allows us to find the aforementioned formula for the conformal welding maps in terms of the analytic diffeomeorphism ϕ. In this connection, we also mention a generalization by Teo [5] of Faber polynomials and identities for the generalized Grunsky matrices to Hilbert spaces of differentials of arbitrary order on planar domains.

2. Generalized power matrices

2.1. *Faber polynomials and power matrices*

Let F be a function holomorphic in a neighbourhood of 0, such that $F(0) = 0$ and $F'(0) \neq 0$. The power matrix of F is the matrix $[F]$ with coefficients $[F]_n^m$ defined by

$$F(z)^n = \sum_{m=n}^{\infty} [F]_m^n z^m. \tag{2.1}$$

The upper index refers to the row and the lower to the column.

If F_1 and F_2 are both of this type, it is easily shown that $[F_1 \circ F_2] = [F_1][F_2]$. Similarly, if G is meromorphic in neighbourhood of ∞, with a simple pole at ∞, the power matrix $[G]$ is given by

$$G(z)^n = \sum_{m=-\infty}^{n} [G]_m^n z^m. \tag{2.2}$$

For functions G_1, G_2 of this type we also have that $[G_1 \circ G_2] = [G_1][G_2]$.

Now we define Faber polynomials.

Definition 2.1 Let ω be a Jordan curve in $\mathbb{C}$, not containing 0 or ∞. Let F be the unique mapping from $\mathbb{D}$ to the interior of ω such that $F(0) = 0$ and $F'(0)$ is real, and let G be the unique conformal mapping from $\mathbb{D}^*$ to the exterior of ω such that $G(\infty) = \infty$ and $G'(\infty)$ is positive real (by $G'(\infty)$ we mean $\lim_{z\to\infty} G(z)/z$). For $n \geq 0$ define the Faber polynomials $\Phi_n(z)$ by

$$\Phi_n(z) = \sum_{m=0}^{n} [G^{-1}]_m^n z^m. \tag{2.3}$$

For $n \leq 0$ define the Faber polynomials by

$$\Phi_n(z) = \sum_{m=n}^{0} [F^{-1}]_m^n z^m.$$

Note that $\Phi_0(z) = 1$ according to both formulas. There are also several alternate but equivalent definitions of Faber polynomials using generating functions [1,6,7].

Remark 2.2 (Notation) The Faber polynomials obviously depend on the curve ω. If we want to emphasize the dependence on ω, we will write $\Phi_n(\omega)$.

Faber polynomials have the following useful property: for $n \geq 0$ we have

$$\Phi_n(G(z)) = z^n + \sum_{k=0}^{\infty} d_k z^{-k} \tag{2.4}$$

for some coefficients d_k, and for $n \leq 0$ we have for some coefficients d_{-k}

$$\Phi_n(F(z)) = z^n + \sum_{k=0}^{\infty} d_{-k} z^k.$$

This property immediately follows from the facts that $[G^{-1}][G]$ and $[F^{-1}][F]$ are the identity (this approach to the Faber polynomials is due to Jabotinsky [1]).

2.2. *Faber–Laurent series*

In this section we define a simultaneous generalization of the Laurent series and Faber series. This is a series expansion of a function holomorphic in a doubly connected neighbourhood of an analytic curve. Our exposition mainly follows Tietz [8]. See also Teo [5], which contains generalizations of Faber polynomials suitable for holomorphic differentials of arbitrary order, treated in the context of certain natural weighted L^2 spaces.

Let ω be a simple closed analytic curve in $\mathbb{C}$ not containing 0, positively oriented. Let Ω denote the open, simply connected domain inside the curve, and let Ω^* denote the open simply connected domain in the Riemann sphere lying in the exterior of the curve. Let f be analytic on a doubly connected domain A containing the curve ω. Given such an analytic f, let f_+ and f_- be analytic functions on Ω and Ω^*, respectively, such that $f = f_+ + f_-$. There is a unique such decomposition up to a constant term, by the Plemelj–Sokhotsk jump formula. Namely, we can set

$$f_+(z) = \frac{1}{2\pi i} \int_\omega \frac{f(\zeta)}{\zeta - z} d\zeta$$

for $z \in \Omega$ and

$$f_-(z) = -\frac{1}{2\pi i} \int_\omega \frac{f(\zeta)}{\zeta - z} d\zeta$$

for $z \in \Omega^*$. It follows that $f = f_+ + f_-$ on $\partial\Omega = \omega$ in the obvious limiting sense. In fact, by moving the curve of integration, we can see that f_+ extends to an analytic function on $\Omega \cup A$, and f_- extends to an analytic function on $\Omega^* \cup A$. Note that using these integral formulas results in the specific choice of constant $f_-(\infty) = 0$.

Now let f be analytic on Ω, and let F and G be as in Definition 2.1. Since ω is an analytic curve, f, F and G all have analytic extensions to an open neighbourhood of the closure of their domains. In the following, we will assume that the doubly connected neighbourhood A of $\partial\Omega = \omega$ is chosen to be small enough such that F, G and f are all analytic on A. Now consider the function $f \circ G$. This is analytic in a neighbourhood of $\partial\mathbb{D}$, and thus has a Laurent series expansion

$$f \circ G(z) = \sum_{n=-\infty}^{\infty} a_n z^n \tag{2.5}$$

so

$$f(z) = \sum_{n=-\infty}^{\infty} a_n G^{-1}(z)^n.$$

This sum converges locally uniformly on A.

We can see that $(G^{-1})^n_+ = 0$ for $n < 0$ in the following way. The Laurent series of $G^n(z)$ in a punctured neighbourhood of infinity contains only non-positive powers of z. Applying the Cauchy integral to $(G^{-1})^n$, the contour can be deformed to a circle $|z| = R$ where R is large enough so that it is within the radius of convergence of the Laurent series of $(G^{-1})^n$. (Note that we have used the fact that the image of G does not contain 0). Therefore the Cauchy integral is zero.

Thus, applying a Cauchy integral to both sides, we see that

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} a_n (G^{-1})^n_+(z) \\ &= \sum_{n=0}^{\infty} a_n \Phi_n(z), \end{aligned} \tag{2.6}$$

where in the last step we have used the definition of Faber polynomials, and the fact that $f = f_+$. It is easy to see that this series converges locally uniformly in Ω.

There is a simple integral formula for the coefficients of the Faber series. It follows from Equation (2.5) that

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f \circ G(\zeta)}{\zeta^{n+1}} \mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_{\omega} \frac{f(z)(G^{-1})'(z)}{G^{-1}(z)^{n+1}} \mathrm{d}\zeta. \end{aligned} \tag{2.7}$$

The same argument shows that a function g which is holomorphic in Ω^* and satisfies $g(\infty) = 0$ has a locally uniformly convergent Faber series

$$\begin{aligned} g(z) &= \sum_{n=-\infty}^{\infty} b_n (F^{-1})^n_-(z) \\ &= \sum_{n=1}^{\infty} b_n \Phi_{-n}(z). \end{aligned} \tag{2.8}$$

By locally uniformly we mean that the series composed with $1/z$ converges locally uniformly to $g(1/z)$. In particular, the series converges uniformly on compact sets not containing ∞. Note that the Cauchy integral of the zeroth-order term is zero. The integral formula for the coefficients of g is

$$b_n = \frac{1}{2\pi i}\int_{\omega} \frac{g(z)(F^{-1})'(z)}{F^{-1}(z)^{-n+1}}\,\mathrm{d}\zeta. \tag{2.9}$$

Remark 2.3 If f is analytic on a simply connected domain D containing Ω, the Faber series (2.6) converges locally uniformly to f on D. This can easily be seen by deforming the contour of integration within D. A similar statement holds for the Faber series (2.8) of a function analytic on a simply connected domain E containing Ω^*.

We are now prepared to define the Faber–Laurent series.

Definition 2.4 Let ω be an analytic Jordan curve not containing 0 or ∞, and denote the domains inside and outside ω by Ω and Ω^*, respectively. Let f be an analytic function on a doubly connected domain A containing $\partial\Omega = \omega$. The Faber–Laurent series of f in A is

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \Phi_n(z) \tag{2.10}$$

where

$$\sum_{n=0}^{\infty} c_n \Phi_n(z) \quad \text{and} \quad \sum_{n=1}^{\infty} c_{-n}\Phi_{-n}(z)$$

are the Faber series of f_+ and f_-, respectively.

It is understood that the doubly infinite sum (2.10) is the sum of the positive series and negative series, after each is summed separately.

We have shown the following theorem.

THEOREM 2.5 *Let f be analytic on a doubly connected domain A not containing* 0 *or* ∞ *and ω be an analytic Jordan curve in A. Let Φ_n be the Faber polynomials of ω for $n \in \mathbb{Z}$. The Faber–Laurent series of f converges locally uniformly in A in the sense that*

$$\sum_{n=0}^{\infty} c_n \Phi_n(z) \quad \textit{and} \quad \sum_{n=1}^{\infty} c_{-n}\Phi_{-n}(z)$$

each converge locally uniformly on A, and the total sum is f.

Remark 2.6 As in Remark 2.3, it is easily shown that the positive series converges locally uniformly on the union of A and the interior of ω, and the negative series converges locally uniformly on the union of A and the exterior of ω.

Unlike the case of power series and Laurent series, it is not possible to say anything about absolute convergence of the Faber–Laurent series.

2.3. *Generalized power matrices*

We can now define a generalization of the power matrix. It applies to the composition of Faber series on doubly connected domains.

Definition 2.7 Let ω be an analytic Jordan curve not containing 0 or ∞. Let f be analytic on a neighbourhood of ω. Let β be another analytic Jordan curve in the plane. The Faber–Laurent matrix of f corresponding to ω and β is the matrix $[f]_{\omega,\beta}$ whose entry $[f]_n^m$ in the mth row and nth column, $m, n \in \mathbb{Z}$ is given by

$$\Phi_m(\beta)(f(z)) = \sum_{n=-\infty}^{\infty} [f]_n^m \Phi_n(\omega)(z),$$

where the right-hand side is the Faber–Laurent expansion of $\Phi_m(\beta)(f(z))$.

The Faber–Laurent matrices also respect the composition of functions $f \circ g$, provided that the image of the curve under g is in the domain of f.

THEOREM 2.8 *Let α, γ and β be analytic Jordan curves not containing* 0 *or* ∞. *Let f be analytic on an open doubly connected neighbourhood A of α, and g be analytic on a doubly connected neighbourhood Γ of γ. Assume further that $f(\alpha)$ is contained in Γ. Then the Faber–Laurent matrix of* $[g \circ f]$ *satisfies*

$$[g \circ f]_{\alpha,\beta} = [g]_{\gamma,\beta}[f]_{\alpha,\gamma}.$$

Proof The series

$$\Phi_n(\beta)(g(z)) = \sum_{k=-\infty}^{\infty} [g]_k^n \Phi_k(\gamma)(z)$$

converges uniformly on Γ by Remark 2.3. Since $f(\alpha) \subset \Gamma$, the series

$$\Phi_n(\beta)(g(f(z))) = \sum_{k=-\infty}^{\infty} [g]_k^n \Phi_k(\gamma)(f(z))$$

converges uniformly on α (we have suppressed the subscripts γ and β in the Faber–Laurent series). Thus, for $m \geq 0$, letting G be the normalized map from $\mathbb{D}^*$ to the exterior of α,

$$\begin{aligned}[g \circ f]_m^n &= \frac{1}{2\pi i}\int_\alpha \frac{\Phi_n(\beta)(g(f(\zeta)))(G^{-1})'(\zeta)}{G^{-1}(\zeta)^{m+1}}\,\mathrm{d}\zeta \\ &= \sum_{k=-\infty}^{\infty} [g]_k^n \frac{1}{2\pi i}\int_\alpha \frac{\Phi_k(\gamma)(f(z))(G^{-1})'(\zeta)}{G^{-1}(\zeta)^{m+1}}\,\mathrm{d}\zeta \\ &= \sum_{k=-\infty}^{\infty} [g]_k^n [f]_m^k\end{aligned}$$

using Equation (2.7). The same argument using the normalized map F from the $\mathbb{D}$ to the interior of α and Equation (2.9) proves the claim for $m < 0$. ■

Remark 2.9 It immediately follows from Theorem 2.5 that $[f^{-1}]_{\omega,\alpha} = [f]_{\alpha,\omega}^{-1}$ in the sense that $[f]_{\alpha,\omega}[f^{-1}]_{\omega,\alpha}$ and $[f^{-1}]_{\omega,\alpha}[f]_{\alpha,\omega}$ both converge to 1 on the diagonal and 0 off the diagonal.

We now outline two special cases.

Definition 2.10 Let D and Ω be planar, simply connected domains containing 0 but not ∞, with analytic boundary curves. Let $\Phi_n(\partial D)$ and $\Phi_n(\partial\Omega)$ denote the Faber polynomials of ∂D and $\partial\Omega$ for $n \geq 0$. Let f: $D \to \Omega$ be holomorphic. Denote by $[f]_{D,\Omega}$ the matrix whose entry $[f]^n_m$ in the nth row and mth column, $m, n \geq 0$ is given by

$$\Phi_n(\partial\Omega)(f(z)) = \sum_{m=0}^{\infty} [f]^n_m \Phi_m(\partial D)(z).$$

Again the sum on the right-hand side converges locally uniformly on D. Of course, this is just one block of the matrix of Definition 2.7.

In the above definition, it is not really necessary to assume that the image of f lies in Ω, since $\Phi_n(\Omega)$ are polynomials. Nevertheless, the matrices obtained are more useful if that is the case, since we then have the multiplicative property.

COROLLARY 2.11 *Let D and Ω be simply connected domains containing zero and bounded by analytic curves. Let f: $D \to \Omega$ and g: $\Omega \to E$ be holomorphic. Then*

$$[g \circ f]_{D,E} = [g]_{\Omega,E}[f]_{D,\Omega}.$$

The sums in the product on the right-hand side converge.

Remark 2.12 Similarly, one could formulate the above definition and corollary for domains containing ∞ but not 0.

Remark 2.13 If D and Ω are the unit disc $\mathbb{D}$, then one obtains the power matrix (for positive powers only) of f. However, in the case of the power matrix, one need not assume that f and g are analytic on $\mathbb{D}$, or that f maps into $\mathbb{D}$ for Corollary 2.11 to hold. In fact, the product formula holds even for formal power series [9].

3. An application to conformal welding

In this section, we use the generalized power matrices to give product formulas for the conformal welding maps associated to an analytic curve in the plane. We also give a formula for the coefficients of the conformal welding maps corresponding to a homeomorphism of the circle ϕ: $S^1 \to S^1$ in terms of the coefficients of the Laurent series of powers of ϕ.

3.1. *Conformal welding*

We quickly recall the conformal welding theorem.

Definition 3.1 Let Q_1 and Q_2 be quasicircles in $\mathbb{C}$. An orientation-preserving homeomorphism h: $Q_1 \to Q_2$ (with the relative topology) is a quasisymmetry if there exists a quasiconformal map $\tilde{h}$: $D_1 \to D_2$, where D_1 and D_2 are the bounded interiors of Q_1 and Q_2 respectively, such that $\tilde{h}$ has a continuous extension to Q_1 equal to h.

Of course, quasisymmetries can be (and usually are) defined using a direct analytic condition; however we adopt the above definition for brevity. For the same reason we will not concern ourselves with the quasiconformal or quasisymmetric constants.

THEOREM 3.2 (Conformal welding theorem) *Let ϕ: $Q_1 \to Q_2$ be a quasisymmetry. Let D_i and D_i^* denote the interior and exterior (including the point at infinity) of Q_i for $i = 1$, 2. Fix $a \in \mathbb{C} \setminus \{0\}$. There exists a one-to-one analytic map F: $D_2 \to \mathbb{C}$ with a quasiconformal extensions to $\mathbb{C}$ satisfying $F(0) = 0$, and a one-to-one map $G\colon D_1^* \to \mathbb{C}$, analytic except for a simple pole at ∞, satisfying $g'(\infty) = a$ and possessing a quasiconformal extension to $\overline{\mathbb{C}}$, such that*

$$F^{-1} \circ G = \phi.$$

The proof, which uses the existence and uniqueness of solutions to the Beltrami equation, can be found for example in [10]. In this article, we will only be concerned with the case that the curves Q_1 and Q_2 are analytic, and the map ϕ is analytic. In that case the uniformization theorem suffices.

Remark 3.3 Theorem 3.2 demonstrates the existence of a solution to the sewing equation (1.1) in the more general case that the maps f and g have not necessarily analytic but rather quasiconformal extensions. In conformal field theory, it is customary to use maps with analytic extensions, and to use the uniformization theorem rather than the conformal welding theorem, as in [2]. The weaker assumption that f and g are quasisymmetries allows one to draw connections with the Teichmüller theory [11].

3.2. *Matrix product formulas for conformal welding*

We now outline some consequences for the generalized power matrices of the conformal welding maps.

COROLLARY 3.4 *Let Q_1, Q_2 and Q_3 be analytic Jordan curves in the plane, none of which contains* 0, *and let ϕ: $Q_1 \to Q_2$ be analytic and one-to-one. Let F: $D_2 \to \mathbb{C}$ and $G\colon D_1^* \to \mathbb{C}^*$ be the conformal welding maps associated to Q_1 and Q_2 as in Theorem* 3.2, *where D_2 and D_1^* are the interior of Q_2 and the exterior of Q_1, respectively. The generalized power matrices satisfy the matrix equation*

$$[F]_{Q_2,Q_3}[\phi]_{Q_1,Q_2} = [G]_{Q_1,Q_3}.$$

This follows immediately from Theorem 2.5.

It is clear from Theorem 2.5 that the product formula holds for any choice of curve Q_3. Note that in the case that $Q_1 = Q_2 = S^1$, and one further chooses $Q_3 = S^1$, Corollary 3.4 relates the power series of F and G to the Laurent series of the powers of ϕ. This immediate consequence is worth spelling out explicitly.

COROLLARY 3.5 *Let ϕ: $S^1 \to S^1$ be an analytic homeomorphism. Let F and G be the conformal welding maps as in Theorem* 3.2. *Let* $[F]$ *and* $[G]$ *denote the standard power matrices of F and G, respectively, as in Equations* (2.1) *and* (2.2). *Let* $[\phi]$ *be the matrix whose entry $[\phi]_n^m$ is the nth coefficient of the Laurent series of ϕ^m. Then* $[F][\phi] = [G]$, *and each entry in the matrix product on the left-hand side converges.*

Another natural choice is $Q_3 = \partial F(D_2) = \partial G(D_1^*)$. In that case we could also rearrange the product formula.

Corollary 3.6 *Let Q_i, $i = 1, 2$, ϕ, F and G be as in Corollary 3.4. Let $Q_3 = \partial F(D_2) = \partial G(D_1^*)$ where D_2 is the interior of Q_1 and D_1^* is the exterior of Q_1. The generalized power matrices satisfy the matrix equation*

$$[F^{-1}]_{Q_3,Q_2}[G]_{Q_1,Q_3} = [\phi]_{Q_1,Q_2}.$$

Remark 3.7 In the conformal welding theorem, it is also possible to factor a quasisymmetry ϕ: $Q_1 \to Q_2$ via two maps $\tilde{F}: D_1 \to \mathbb{C}$ and $\tilde{G}: D_2^* \to \overline{\mathbb{C}}$ so that $\phi = \tilde{G}^{-1} \circ \tilde{F}$ on Q_1. One may obtain this factorization by applying the conformal welding theorem as stated to ϕ^{-1} and reversing the roles of Q_1 and Q_2. Of course, Theorem 2.5 implies matrix product formulas for the alternate factorization.

It may be possible to extend the various matrix product formulas to maps with less regular boundary behaviour, by treating the matrix as the matrix of an operator in the appropriate function space. For example, the conformal welding matrix product formula might be extended to quasisymmetries by using an appropriate representation of the corresponding composition operators. We formulated the results in the analytic setting so as to avoid distracting particulars.

Finally, we show that in some sense one can 'solve for' the coefficients of the power series of F and G in terms of the coefficients of ϕ. From now on, we assume that $Q_1 = Q_2 = S^1$. Let ϕ: $S^1 \to S^1$ be an analytic one-to-one map. Let $[\phi]_{S^1,S^1}$ denote the generalized power matrix of ϕ. Let (ϕ_{++}) denote the block of $[\phi]_{S^1,S^1}$ consisting of the entries $[\phi]_n^m$ with $m > 0$ and $n > 0$. Let (ϕ_{+-}) denote the block with $m > 0$ and $n \leq 0$. We need the fact that the block (ϕ_{++}) is invertible. This was established by Takhtajan and Teo in the course of a proof of identities relating $[\phi]$ to the generalized Grunsky matrices, though the result is not explicitly stated [4, Part II, Proposition 5.1]. We include the proof here for the convenience of the reader.

Lemma 3.8 *Let ϕ: $S^1 \to S^1$ be an analytic one-to-one map. The block (ϕ_{++}) of the power matrix is invertible, in the sense that there is an infinite matrix B such that for all $i, j \in \mathbb{N}$ the i, jth entries of $B(\phi_{++})$ and $(\phi_{++})B$ converge to*

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Some preliminaries are required. In this case we do need to use a specific representation of composition by quasisymmetry: convergence of the sum is more easily handled in a certain Hilbert space, which we now describe. We refer the reader to [3] for details and proofs. Let

$$\mathcal{H} = \left\{ \sum_{n=-\infty, n\neq 0}^{\infty} a_n e^{in\theta} : \sum_{n=-\infty, n\neq 0}^{\infty} |n||a_n|^2 < \infty, \quad \bar{a}_n = a_{-n} \right\}.$$

It can be shown that such series converge almost everywhere on S^1 with respect to Lebesgue measure, and thus this is a set of real-valued functions, defined up to sets of the Lebesgue measure zero. $\mathcal{H}$ is a Hilbert space, as is the complexification

$$\mathcal{H}^{\mathbb{C}} = \left\{ \sum_{n=-\infty, n\neq 0}^{\infty} a_n e^{in\theta} : \sum_{n=-\infty, n\neq 0}^{\infty} |n||a_n|^2 < \infty \right\}.$$

The Hilbert space $\mathcal{H}^{\mathbb{C}}$ can be identified with the set of complex-valued harmonic functions on $\mathbb{D}$ with finite Dirichlet energy. Alternatively, the positive part of the series can be identified with the Dirichlet space

$$\mathcal{D} = \left\{ f\colon \mathbb{D} \to \mathbb{C}\colon \iint_{\mathbb{D}} |f'|^2 \mathrm{d}A < \infty,\ \ f(0) = 0 \right\},$$

where integration is with respect to the standard Euclidean area measure, and the negative part of the series can be identified with the Dirichlet space on the exterior of the disc $\mathbb{D}^*$

$$\mathcal{D}^* = \left\{ f\colon \mathbb{D}^* \to \mathbb{C}\colon \iint_{\mathbb{D}} |f'|^2 \mathrm{d}A < \infty,\ \ f(\infty) = 0 \right\}.$$

This identification is obtained by replacing $e^{in\theta}$ by z^n.

For a quasisymmetric mapping $\phi\colon S^1 \to S^1$, consider the operator

$$\begin{aligned} \hat{\mathcal{C}}_\phi &: \mathcal{H} \to \mathcal{H} \\ h &\mapsto h \circ \phi - \int_{S^1} h \circ \phi(e^{i\theta}) \mathrm{d}\theta. \end{aligned}$$

The main fact we need for the proof of the lemma is that $\hat{\mathcal{C}}_\phi$ is a bounded operator, and thus so is its complex linear extension to $\mathcal{H}^{\mathbb{C}}$. Denoting by $\mathcal{H}^{\mathbb{C}}_+$ the set of series in $\mathcal{H}^{\mathbb{C}}$ with only positive powers of $e^{in\theta}$ and by $P_+\colon \mathcal{H}^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}$ the natural projection, we have that $P_+\hat{\mathcal{C}}_\phi$ is bounded. The matrix of its restriction to $\mathcal{H}^{\mathbb{C}}_+$ in the basis $\{z^k\colon k \in \mathbb{Z} \setminus \{0\}\}$ is just (ϕ_{++}). Finally, we need the fact that the matrix of the restriction of $P_+\hat{\mathcal{C}}_{\phi^{-1}}$ to $\mathcal{H}^{\mathbb{C}}_+$ is the conjugate transpose $\overline{(\phi_{++})}^T$. This follows from the fact that $\hat{\mathcal{C}}_\phi$ preserves a certain symplectic form and thus satisfies certain relations between its blocks [4, p. 99].

With these facts in hand we may now prove the lemma.

Proof of Lemma 3.8 Let F and G be the conformal welding maps in Theorem 3.2, with normalization $G'(\infty) = 1$ and such that factorization $\phi = F^{-1} \circ G$ (to obtain this, we apply the Theorem to ϕ^{-1}). Denote the conformal welding curve $\gamma = F(\partial\mathbb{D}) = G(\partial\mathbb{D}^*)$.

Let $[G]$ denote the ordinary power matrix of G (Equation (2.2)). For $n > 0$ let

$$\Phi^0_n(z) = \Phi_n(\gamma)(z) - \Phi_n(\gamma)(0) = \sum_{m=1}^{n} [G^{-1}]^n_m z^m, \tag{3.1}$$

where the final equality follows from Equation (2.3). Now define the matrix B^n_m for $n > 0$ and $m > 0$ by

$$\Phi^0_n(F(z)) = \sum_{m=1}^{\infty} B^n_m z^m.$$

For fixed n, $\Phi^0_n(F(z)) \in \mathcal{D}$. This is because $F \in \mathcal{D}$, since it has finite area by the existence of a quasiconformal extension to the sphere such that ∞ is in the complement of $F(\overline{\mathbb{D}})$, and Φ^0_n is a polynomial. Treating $\Phi^0_n(F(z))$ as an element of $\mathcal{H}^{\mathbb{C}}_+$, since $P_+\hat{\mathcal{C}}_\phi$ is bounded we have that

$$P_+\hat{\mathcal{C}}_\phi(\Phi_n^0(F(z))) = P_+\hat{\mathcal{C}}_\phi\left(\sum_{m=1}^{\infty} B_m^n z^m\right)$$

is in $\mathcal{H}_+^{\mathbb{C}}$. Evaluating the above equation in the $\{z^k\colon k \in \mathbb{Z}\setminus\{0\}\}$ basis, we have that

$$\sum_{m=1}^{\infty} B_m^n (\phi_{++})_k^m$$

converges for any $n > 0$ and $k > 0$.

We now show that B is an inverse of (ϕ_{++}). This follows from Equation (3.1) since

$$\hat{\mathcal{C}}_\phi(\Phi_n^0 \circ F)(z) = \Phi_n^0(F \circ \phi(z)) = \Phi_n^0(G(z)),$$

so $P_+\hat{\mathcal{C}}_\phi(\Phi_n^0 \circ F)(z) = z^n$ by Equation (2.4). Thus B is a left inverse of (ϕ_{++}). Since the matrix of $\hat{\mathcal{C}}_{\phi^{-1}}$ is $\overline{(\phi_{++})}^T$, applying the above argument to $\hat{\mathcal{C}}_{\phi^{-1}}$ implies that $\overline{(\phi_{++})}^T$ has a left inverse. Thus (ϕ_{++}) has a right inverse. ■

Remark 3.9 Note that this shows that the matrix of $P_+\hat{\mathcal{C}}_\phi$ is invertible for a quasisymmetry $\phi\colon S^1 \to S^1$. Takhtajan and Teo [4] also showed that the inverse B is a block of the generalized Grunsky matrix.

We may now 'solve for' the coefficients of F and G in terms of those of ϕ.

THEOREM 3.10 *Let $\phi\colon S^1 \to S^1$ be analytic and one-to-one. The coefficients of the power series of F and G can be written in terms of $(\phi_{++})^{-1}$, (ϕ_{+-}) and $G'(\infty)$ as follows. Let $(F)_+$ denote the row vector $(F_1, F_2, \ldots)$ of coefficients of the power series $F(z) = F_1 z + F_2\ z^2 + \cdots$ of F. Let $(G)_-$ denote the row vector $(\ldots, G_{-2}, G_{-1}, G_0)$ of coefficients of the power series $G(z) = G_1 z + G_0 + G_{-1} z^{-1} + \cdots$ of G and let $(G)_+ = (G_1, 0, 0, \ldots)$. Then*

$$(F)_+ = (G)_+(\phi_{++})^{-1}$$

and

$$(G)_- = (F)_+(\phi_{+-}).$$

The infinite sums in the second equation converge.

Note that the first equation has no infinite sums, and for fixed ϕ, the coefficients of $(F)_+$ and $(G)_-$ so obtained only depend on G_1.

Proof By Corollary 3.4,

$$(F)_+(\phi_{++}) = (G)_+ \quad \text{and} \quad (F)_+(\phi_{+-}) = (G)_-$$

and the infinite sums in the second equation converge. The claim now follows from Lemma 3.8. ■

Note that this does not give a finite procedure for obtaining the coefficients of F and G from ϕ (or even an obvious approximation method). This is because when inverting the block (ϕ_{++}), infinite operations are necessary to obtain any single coefficient of $(\phi_{++})^{-1}$. Similarly, once $(\phi_{++})^{-1}$ is obtained, each coefficient of $(G)_-$ involves an infinite sum. Nevertheless, the above formula is of interest.

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