J. Math. Anal. Appl. 418 (2014) 387–402

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Operator theoretic differences between Hardy and Dirichlet-type spaces ☆

José Ángel Peláez [a,*], Fernando Pérez-González [b], Jouni Rättyä [c]

[a] Departamento de Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain
[b] Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna, Tenerife, Spain
[c] University of Eastern Finland, P.O. Box 111, 80101 Joensuu, Finland

ARTICLE INFO

Article history:
Received 25 October 2013
Available online 27 March 2014
Submitted by Richard M. Aron

Keywords:
Operator theoretic differences
Hardy spaces
Spaces of Dirichlet type
Integral operators
Carleson measures

ABSTRACT

For $0 < p < \infty$, the Dirichlet-type space $\mathcal{D}^p_{p-1}$ consists of the analytic functions f in the unit disc $\mathbb{D}$ such that $\int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-1}\, dA(z) < \infty$. Motivated by operator theoretic differences between the Hardy space H^p and $\mathcal{D}^p_{p-1}$, the integral operator

$$T_g(f)(z) = \int_0^z f(\zeta) g'(\zeta)\, d\zeta, \quad z \in \mathbb{D},$$

acting from one of these spaces to another is studied. In particular, it is shown, on one hand, that $T_g : \mathcal{D}^p_{p-1} \to H^p$ is bounded if and only if $g \in \mathrm{BMOA}$ when $0 < p \leqslant 2$, and, on the other hand, that this equivalence is very far from being true if $p > 2$. Those symbols g such that $T_g : \mathcal{D}^p_{p-1} \to H^q$ is bounded (or compact) when $p < q$ are also characterized. Moreover, the best known sufficient L^∞-type condition for a positive Borel measure μ on $\mathbb{D}$ to be a p-Carleson measure for $\mathcal{D}^p_{p-1}$, $p > 2$, is significantly relaxed, and the established result is shown to be sharp in a very strong sense.

1. Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions in the unit disc $\mathbb{D} = \{z\colon |z| < 1\}$ of the complex plane $\mathbb{C}$. Let $\mathbb{T}$ be the boundary of $\mathbb{D}$. The *Carleson square* associated with an interval $I \subset \mathbb{T}$ is the set

☆ This research was supported in part by the Ramón y Cajal program of MICINN (Spain); by Ministerio de Educación y Ciencia, Spain, Projects MTM2011-25502 and MTM2011-26538; by La Junta de Andalucía, (FQM210) and (P09-FQM-4468); by Academy of Finland Project No. 268009, by Väisälä Foundation of Finnish Academy of Science and Letters, and by Faculty of Science and Forestry of University of Eastern Finland Project No. 930349.

* Corresponding author.
E-mail addresses: japelaez@uma.es (J.Á. Peláez), fernando.perez.gonzalez@ull.es (F. Pérez-González), jouni.rattya@uef.fi (J. Rättyä).

http://dx.doi.org/10.1016/j.jmaa.2014.03.058

$S(I) = \{re^{it}\colon e^{it} \in I,\ 1-|I| \leqslant r < 1\}$, where $|E|$ denotes the normalized Lebesgue measure of the set $E \subset \mathbb{T}$. For our purposes it is also convenient to define for each $a \in \mathbb{D} \setminus \{0\}$ the interval $I_a = \{e^{i\theta}\colon |\arg(ae^{-i\theta})| \leqslant \pi(1-|a|)\}$, and denote $S(a) = S(I_a)$. For $0 < p \leqslant \infty$, the *Hardy space* H^p consists of the functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p} = \lim_{r \to 1^-} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

and

$$M_\infty(r, f) = \max_{0 \leqslant \theta \leqslant 2\pi} \left|f\left(re^{i\theta}\right)\right|.$$

For the theory of the Hardy spaces, see [9,12].

For $0 < p < \infty$ and $-1 < \alpha < \infty$, the *Dirichlet space* $\mathcal{D}^p_\alpha$ consists of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{\mathcal{D}^p_\alpha} = \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1 - |z|^2\right)^\alpha dA(z) + \left|f(0)\right|^p < \infty,$$

where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$.

The purpose of this study is to underline operator theoretic differences between the closely related spaces $\mathcal{D}^p_{p-1}$ and H^p. Before going to that, it is appropriate to recall inclusion relations between these spaces. The classical Littlewood–Paley formula implies $D^2_1 = H^2$. Moreover, it is well known [10,17] that

$$\mathcal{D}^p_{p-1} \subsetneq H^p, \quad 0 < p < 2, \tag{1.1}$$

and

$$H^p \subsetneq \mathcal{D}^p_{p-1}, \quad 2 < p < \infty. \tag{1.2}$$

It is also worth mentioning that there are no inclusion relations between $\mathcal{D}^p_{p-1}$ and $\mathcal{D}^q_{q-1}$ when $p \neq q$ [14].

A natural way to illustrate differences between two given spaces is to consider classical operators acting on them. For example, if $0 < p < 2$, then the behavior of the *composition operator* $C_\varphi(f) = f \circ \varphi$ reveals that $\mathcal{D}^p_{p-1}$ is in a sense a much smaller space than H^p. Namely, it follows from Littlewood's subordination theorem that $C_\varphi : H^p \to H^p$ is bounded for each $0 < p < \infty$ and all analytic self-maps φ of $\mathbb{D}$, but in contrast to this, there are symbols φ which induce unbounded operators $C_\varphi : \mathcal{D}^p_{p-1} \to \mathcal{D}^p_{p-1}$ when $0 < p < 2$ [8, Theorem 1.1(b)]. As in the case of Hardy spaces, any composition operator $C_\varphi : \mathcal{D}^p_{p-1} \to \mathcal{D}^p_{p-1}$ is bounded when $2 \leqslant p < \infty$.

There are operators which do not distinguish between $\mathcal{D}^p_{p-1}$ and H^p. For a given $g \in \mathcal{H}(\mathbb{D})$, the *generalized Hilbert operator* $\mathcal{H}_g$ is defined by

$$\mathcal{H}_g(f)(z) = \int_0^1 f(t) g'(tz)\, dt, \tag{1.3}$$

for any $f \in \mathcal{H}(\mathbb{D})$ such that $\int_0^1 |f(t)|\,dt < \infty$. If $1 < p < \infty$, then $\mathcal{H}_g : \mathcal{D}^p_{p-1} \to \mathcal{D}^p_{p-1}$ is bounded (compact) if and only if $\mathcal{H}_g : H^p \to \mathcal{D}^p_{p-1}$ is bounded (compact) by [11]. Moreover, the same condition, depending on g and p, describes the boundedness (compactness) of the operators $\mathcal{H}_g : \mathcal{D}^p_{p-1} \to H^p$ and $\mathcal{H}_g : H^p \to H^p$ when $1 < p \leqslant 2$. As far as we know, the problem of characterizing the symbols g for which $\mathcal{H}_g : \mathcal{D}^p_{p-1} \to H^p$ and $\mathcal{H}_g : H^p \to H^p$ are bounded when $2 < p < \infty$ remains unsolved.

We shall next study operator theoretic differences between $\mathcal{D}^p_{p-1}$ and H^p by considering the integral operator

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta, \quad z \in \mathbb{D}.$$

The bilinear operator $(f,g) \to \int fg'$ was introduced by Calderón in harmonic analysis in the 60's [5]. After his research on commutators of singular integral operators, this bilinear form and its different variations, usually called "paraproducts", have been extensively studied and they have become a fundamental tool in harmonic analysis. Pommerenke was probably one of the first complex function theorists to consider the operator T_g. He used it in late 70's to study the space BMOA, which consists of the functions in the Hardy space H^1 that have *bounded mean oscillation* on the boundary $\mathbb{T}$ [20]. The space BMOA can be equipped with several different equivalent norms [12], here we shall use the one given by

$$\|g\|^2_{\mathrm{BMOA}} = \sup_{a\in\mathbb{D}} \frac{\int_{S(a)} |g'(z)|^2(1-|z|^2)\,dA(z)}{1-|a|} + \left|g(0)\right|^2.$$

Two decades later, in late 90's, the pioneering works by Aleman and Siskakis [2,3] lead to an abundant research activity on the operator T_g. In particular, the analytic symbols g such that $T_g : H^p \to H^q$ is bounded were characterized by Aleman, Cima and Siskakis [1,2]. Their result in the case $p = q$ says that $T_g : H^p \to H^p$ is bounded if and only if $g \in \mathrm{BMOA}$. Our first result shows that whenever $0 < p \leqslant 2$, the domain space H^p can be replaced by $\mathcal{D}^p_{p-1}$.

Theorem 1. *Let $0 < p \leqslant 2$ and $g \in \mathcal{H}(\mathbb{D})$. Then the following are equivalent:*

(i) $T_g : \mathcal{D}^p_{p-1} \to H^p$ *is bounded;*
(ii) $T_g : H^p \to H^p$ *is bounded;*
(iii) $g \in \mathrm{BMOA}$.

The implication (ii) ⇒ (i) is a direct consequence of (1.1), so our contribution here consists of showing (i) ⇒ (iii). The proof of the implication (ii) ⇒ (iii) in [1,2] relies on several powerful properties of BMOA and H^p such as the conformal invariance of BMOA. Our proof is based on a circle of ideas developed in [19, Chapter 4], and does not rely on these properties. Instead, the Fefferman–Stein formula [22], which states that

$$\|f\|^p_{H^p} \asymp \int_{\mathbb{T}} S_f^p(\zeta)\,|d\zeta| + \left|f(0)\right|^p, \tag{1.4}$$

plays an important role in the reasoning. Here, $|d\zeta|$ denotes the arclength measure on $\mathbb{T}$, and S_f denotes the usual square function, also called the Lusin area function,

$$S_f(\zeta) = \left(\int\limits_{\Gamma_\sigma(\zeta)} \left|f'(z)\right|^2 dA(z) \right)^{1/2}, \quad \zeta \in \mathbb{T}, \tag{1.5}$$

where $\Gamma_\sigma(\zeta)$ denotes a nontangential approach region (a Stolz angle) with vertex at ζ and of aperture σ.

We also show that the statement in Theorem 1 drastically fails for $p > 2$. In order to give the precise statement, we need to fix the notation. The *disc algebra* $\mathcal{A}$ is the space of all analytic functions in $\mathbb{D}$ that admit a continuous extension to the closed unit disc $\overline{\mathbb{D}}$. For $0 < \alpha \leqslant 1$, the *Lipschitz space* $\Lambda(\alpha)$ consists of the functions $g \in \mathcal{H}(\mathbb{D})$, having a non-tangential limit $g(e^{i\theta})$ almost everywhere on $\mathbb{T}$, such that

$$\sup_{\theta\in[0,2\pi],\, 0<t<1} \frac{|g(e^{i(\theta+t)}) - g(e^{i\theta})|}{t^\alpha} < \infty.$$

The "little oh" counterpart of this space is denoted by $\lambda(\alpha)$. The following chain of strict inclusions is known:

$$\lambda(\alpha) \subsetneq \Lambda(\alpha) \subsetneq \mathcal{A} \subsetneq H^\infty \subsetneq \mathrm{BMOA} \subsetneq \mathcal{B}, \quad 0 < \alpha \leqslant 1.$$

Here, as usual, $\mathcal{B}$ stands for the *Bloch space* which consists of the functions $f \in \mathcal{H}(\mathbb{D})$ such that $\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) + |f(0)| < \infty$.

Theorem 2. *Let $2 < p < \infty$ and $g \in \mathcal{H}(\mathbb{D})$.*

(i) *If $T_g : \mathcal{D}^p_{p-1} \to H^p$ is bounded, then $g \in \mathrm{BMOA}$.*
(ii) *There exist $g \in \mathcal{A}$ and $f \in \mathcal{D}^p_{p-1}$ such that $T_g(f) \notin H^p$.*

Part (ii) shows that $\mathcal{D}^p_{p-1}$ is in a sense a much larger space than H^p when $p > 2$, since we may choose the inducing symbol g to be as smooth as admitting a continuous extension to the boundary, but still a suitably chosen $f \in \mathcal{D}^p_{p-1}$ establishes $T_g(f) \notin H^p$. In contrast to this, when the inducing index of the domain space is strictly smaller than the one of the target space, that is $p < q$, then T_g does not distinguish between $\mathcal{D}^p_{p-1}$ and H^p.

Theorem 3. *Let $0 < p < q < \infty$ and $g \in \mathcal{H}(\mathbb{D})$.*

(a) *If $\frac{1}{p} - \frac{1}{q} \leqslant 1$, then the following are equivalent:*
  (i) *$T_g : \mathcal{D}^p_{p-1} \to H^q$ is bounded;*
  (ii) *$T_g : H^p \to H^q$ is bounded;*
  (iii) *$g \in \Lambda(\frac{1}{p} - \frac{1}{q})$.*
(b) *If $\frac{1}{p} - \frac{1}{q} > 1$, then $T_g : \mathcal{D}^p_{p-1} \to H^q$ is bounded if and only if g is constant.*

Part (a) allows us to deduce a strengthened version of the classical result of Hardy–Littlewood which states that a primitive of each function $f \in H^p$, $0 < p < 1$, belongs to $H^{\frac{p}{1-p}}$.

Proposition 4. *Let p, p_1 and p_2 be positive numbers such that $p < 1 < p_2$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. If $f \in \mathcal{H}(\mathbb{D})$ such that $f = f_1 \cdot f_2$ where $f_1 \in \mathcal{D}^{p_1}_{p_1-1}$ and $f_2 \in \mathcal{H}(\mathbb{D})$ satisfies $|f_2(z)| = \mathrm{O}(\frac{1}{(1-|z|)^{1/p_2}})$, then f is the derivative of a function in $H^{\frac{p}{1-p}}$.*

The statement in Proposition 4 with H^{p_1} in place of $\mathcal{D}^{p_1}_{p_1-1}$ was proved by Aleman and Cima [1, p. 158]. The strict inclusions (1.1) and (1.2) show that their result is better when $p_1 < 2$, which is contrary to the case $p_1 > 2$.

An important ingredient in the proofs of both Theorems 1 and 3 is the following result on a Hörmander-type maximal function

$$M(\varphi)(z) = \sup_{I:\ z\in S(I)} \frac{1}{|I|}\int_I |\varphi(\zeta)|\frac{|d\zeta|}{2\pi}, \quad z\in\mathbb{D},$$

defined for each 2π-periodic function $\varphi(e^{i\theta})\in L^1(\mathbb{T})$.

Theorem A. *Let $0<p\leqslant q<\infty$ and $0<\alpha<\infty$ such that $p\alpha>1$. Let μ be a positive Borel measure on $\mathbb{D}$. Then there exists a positive constant $C>0$ such that*

$$\big\|\big[\big(M(f)^{\frac{1}{\alpha}}\big)\big]^{\alpha}\big\|_{L^q(\mu)} \leqslant C\|f\|_{L^p(\mathbb{T})}, \quad \textit{for all } f\in L^p(\mathbb{T}),$$

if and only if $\sup_{I\subset\mathbb{T}} \frac{\mu(S(I))}{|I|^{\frac{q}{p}}}<\infty$.

Moreover,

$$\big\|\big[M\big((\cdot)^{\frac{1}{\alpha}}\big)\big]^{\alpha}\big\|^q \overset{\text{def}}{=} \sup_{\|f\|_{L^p(\mathbb{T})}=1}\big\|\big[\big(M(f)^{\frac{1}{\alpha}}\big)\big]^{\alpha}\big\|^q_{L^q(\mu)} \asymp \sup_{I\subset\mathbb{T}}\frac{\mu(S(I))}{|I|^{\frac{q}{p}}}.$$

This result follows by the well-known works by Carleson [6,7], and hence the measures μ for which $\mu(S(I))\leqslant C|I|^{\frac{q}{p}}$ are known as $\frac{q}{p}$-*Carleson measures*.

For further references, see either [9, Section 9.5], or the proof of [19, Theorem 2.1] for a similar result. Theorem A has been used to characterize the so-called q-Carleson measures for Hardy spaces. Recall that, for a given Banach space (or a complete metric space) X of analytic functions in $\mathbb{D}$, a positive Borel measure μ on $\mathbb{D}$ is called a q-*Carleson measure for* X if the identity operator $I_d: X\to L^q(\mu)$ is bounded. Nowadays these measures are a standard tool in the operator theory in spaces of analytic functions in $\mathbb{D}$.

Let us now turn back to the two remaining cases that are not covered by Theorems 1 and 2. They are the ones in which the operator T_g acts from either H^p or $\mathcal{D}^p_{p-1}$ to $\mathcal{D}^p_{p-1}$. It is easy to see that, in terms of the language of the previous paragraph, $T_g: H^p\to\mathcal{D}^q_{q-1}$ is bounded if and only if $\mu_{g,q} = |g'(z)|^q(1-|z|^2)^{q-1}\,dA(z)$ is a q-Carleson measure for H^p. Therefore, in this case the symbols g that induce bounded operators get characterized by [9, Theorem 9.5], when $q\geqslant p$, and [18] if $q<p$. Analogously, it follows that $T_g:\mathcal{D}^p_{p-1}\to\mathcal{D}^q_{q-1}$ is bounded if and only if $\mu_{g,q}$ is a q-Carleson measure for $\mathcal{D}^p_{p-1}$. Unfortunately, as far as we know, the existing literature does not offer a characterization of these measures, for the full range of parameter values, in terms of a condition depending on μ only. It is known that they coincide with q-Carleson measures for H^p and can therefore be described by the condition

$$\sup_{I\subset\mathbb{T}}\frac{\mu(S(I))}{|I|^{q/p}}<\infty, \tag{1.6}$$

provided $q>p$ [16, Theorem 1(a)]. This statement remains valid also in the diagonal case $q=p$, if $p\leqslant 2$, but fails for $p>2$ [15,21]. In more general terms, the p-Carleson measures for $\mathcal{D}^p_\alpha$ are known excepting the case $\alpha=p-1$ for $p>2$ [4,21]. This corresponds to the diagonal case $q=p>2$ which interests us in particular. It is known in this case that μ being a 1-Carleson measure is a necessary but not a sufficient condition for μ to be a p-Carleson measure for $\mathcal{D}^p_{p-1}$ [15], and that the more restrictive condition

$$\sup_{I\subset\mathbb{T}}\frac{\mu(S(I))}{|I|(\log\frac{e}{|I|})^{-p/2}}<\infty$$

is a sufficient condition for $I_d:\mathcal{D}^p_{p-1}\to L^p(\mu)$ to be bounded [13]. Our next result shows that this best known sufficient condition can be relaxed by one logarithmic factor.

Theorem 5. *Let $2 < p < \infty$, and let μ be a positive Borel measure on $\mathbb{D}$. If*

$$\sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{|I|(\log \frac{e}{|I|})^{-p/2+1}} < \infty, \tag{1.7}$$

then μ is a p-Carleson measure for $\mathcal{D}^p_{p-1}$.

We shall see in Proposition 12 that the statement in Theorem 5 is sharp in a very strong sense.

The remaining part of the paper is organized as follows. In Section 2 we state and prove some preliminary results. Theorems 1 and 3 and their expected analogues for compact operators as well as Proposition 4 are proved in Section 3. In Section 4 we shall deal with the growth of integral means of functions $f \in \mathcal{D}^p_{p-1}$, $p > 2$, and we shall prove Theorem 2.

Before proceeding further, a word about notation to be used. We shall write $\|T\|_{(X,Y)}$ for the norm of an operator $T : X \to Y$, and if no confusion arises with regards to X and Y, we shall simply write $\|T\|$. Moreover, for two real-valued functions E_1, E_2 we write $E_1 \asymp E_2$ or $E_1 \lesssim E_2$, if there exists a positive constant k, independent of the argument, such that $\frac{1}{k}E_1 \leqslant E_2 \leqslant kE_1$ or $E_1 \leqslant kE_2$, respectively.

2. Preliminaries

We begin with a straightforward but useful estimate that will be used in proofs of Theorems 1 and 3.

Lemma 6. *Let $0 < q, p < \infty$ and $g \in \mathcal{H}(\mathbb{D})$. If $T_g : \mathcal{D}^p_{p-1} \to H^q$ is bounded, then*

$$M_\infty(r, g') \lesssim \frac{\|T_g\|_{(\mathcal{D}^p_{p-1}, H^q)}}{(1-r)^{1-\frac{1}{p}+\frac{1}{q}}}, \quad 0 \leqslant r < 1. \tag{2.1}$$

Proof. The functions

$$F_{a,p,\gamma}(z) = \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{\frac{1+\gamma}{p}}, \quad 0 < \gamma < \infty,\ a \in \mathbb{D},$$

satisfy

$$|F_{a,p,\gamma}(z)| \asymp 1, \quad z \in S(a), \tag{2.2}$$

and a calculation shows that

$$\|F_{a,p,\gamma}\|^p_{\mathcal{D}^p_{p-1}} \asymp 1 - |a|, \quad a \in \mathbb{D}.$$

Since $T_g : \mathcal{D}^p_{p-1} \to H^q$ is bounded by the assumption, the well known relations $M_\infty(r, f) \lesssim M_q(\frac{1+r}{2}, f)(1-r)^{-\frac{1}{q}}$ and $M_q(r, f') \lesssim M_q(\frac{1+r}{2}, f)(1-r)^{-1}$, valid for all $f \in \mathcal{H}(\mathbb{D})$ (see [9, Chapter 5]), yield

$$\begin{aligned}
|g'(a)| = |(T_g(F_{a,p,\gamma}))'(a)| &\lesssim \frac{M_q(\frac{1+|a|}{2}, (T_g(F_{a,p,\gamma}))')}{(1-|a|)^{\frac{1}{q}}} \\
&\lesssim \frac{M_q(\frac{3+|a|}{4}, T_g(F_{a,p,\gamma}))}{(1-|a|)^{1+\frac{1}{q}}} \lesssim \frac{\|T_g(F_{a,p,\gamma})\|_{H^q}}{(1-|a|)^{1+\frac{1}{q}}} \\
&\lesssim \frac{\|T_g\|_{(\mathcal{D}^p_{p-1}, H^q)} \|F_{a,p,\gamma}\|_{\mathcal{D}^p_{p-1}}}{(1-|a|)^{1+\frac{1}{q}}} \lesssim \frac{\|T_g\|_{(\mathcal{D}^p_{p-1}, H^q)}}{(1-|a|)^{1+\frac{1}{q}-\frac{1}{p}}}, \quad a \in \mathbb{D},
\end{aligned}$$

and the assertion follows. □

We next recall some suitable reformulations of Lipschitz spaces $\Lambda(\alpha)$ [9].

Lemma B. *Let $0<\alpha\leqslant 1$ and $g\in\mathcal{H}(\mathbb{D})$. Then the following are equivalent:*

(i) $g\in\Lambda(\alpha)$;
(ii) $M_\infty(r,g')=\mathrm{O}(\frac{1}{(1-r)^{1-\alpha}})$, $r\to 1^-$;
(iii) *The measure* $d\mu_g(z)=|g'(z)|^2(1-|z|^2)\,dA(z)$ *satisfies the condition*

$$\sup_{I\subset\mathbb{T}}\frac{\mu_g(S(I))}{|I|^{2\alpha+1}}<\infty.$$

We shall also need the following result [16, Theorem 1(i)].

Theorem C. *Let $0<p<q<\infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then μ is a q-Carleson measure for $\mathcal{D}^p_{p-1}$ if and only if μ is a $\frac{q}{p}$-Carleson measure. Moreover,*

$$\|I_{d(\mathcal{D}^p_{p-1},L^q(\mu))}\|^q\asymp\sup_{I\subset\mathbb{T}}\frac{\mu(S(I))}{|I|^{\frac{q}{p}}}.$$

3. Integral operators from Dirichlet to Hardy spaces

Proof of Theorem 1. It is known that $T_g:H^p\to H^p$ is bounded if and only if $g\in\mathrm{BMOA}$ [1], and therefore (ii) and (iii) are equivalent. Moreover, since $\mathcal{D}^p_{p-1}\subset H^p$ for $0<p\leqslant 2$, (ii) implies (i). To complete the proof we shall show that $g\in\mathrm{BMOA}$, whenever $T_g:\mathcal{D}^p_{p-1}\to H^p$ is bounded. To see this, note first that $\|g\|_{\mathcal{B}}\lesssim\|T_g\|_{(\mathcal{D}^p_{p-1},H^p)}$ by Lemma 6, and thus $g\in\mathcal{B}$. Let now $1<\alpha,\beta<\infty$ such that $\beta/\alpha=p/2<1$, and let α' and β' be the conjugate indices of α and β, respectively. Assume for a moment that g' is continuous on $\overline{\mathbb{D}}$. Then (2.2), Fubini's theorem and Hölder's inequality yield

$$\begin{aligned}\int\limits_{S(a)}\big|g'(z)\big|^2\big(1-|z|^2\big)\,dA(z)&\asymp\int\limits_{\mathbb{T}}\Bigg(\int\limits_{S(a)\cap\Gamma_\sigma(\zeta)}\big|g'(z)\big|^2\big|F_{a,p,\gamma}(z)\big|^2\,dA(z)\Bigg)^{\frac{1}{\alpha}+\frac{1}{\alpha'}}|d\zeta|\\&\leqslant\Bigg(\int\limits_{\mathbb{T}}\Bigg(\int\limits_{\Gamma_\sigma(\zeta)}\big|g'(z)\big|^2\big|F_{a,p,\gamma}(z)\big|^2\,dA(z)\Bigg)^{\frac{\beta}{\alpha}}|d\zeta|\Bigg)^{\frac{1}{\beta}}\\&\quad\cdot\Bigg(\int\limits_{\mathbb{T}}\Bigg(\int\limits_{\Gamma_\sigma(\zeta)\cap S(a)}\big|g'(z)\big|^2\,dA(z)\Bigg)^{\frac{\beta'}{\alpha'}}|d\zeta|\Bigg)^{\frac{1}{\beta'}}\\&\asymp\big\|T_g(F_{a,p,\gamma})\big\|_{H^p}^{\frac{p}{\beta}}\big\|S_g(\chi_{S(a)})\big\|_{L^{\frac{\beta'}{\alpha'}}(\mathbb{T})}^{\frac{1}{\alpha'}},\quad a\in\mathbb{D},\end{aligned}\tag{3.1}$$

where

$$S_g(\varphi)(\zeta)=\int\limits_{\Gamma_\sigma(\zeta)}\big|\varphi(z)\big|^2\big|g'(z)\big|^2\,dA(z),\quad\zeta\in\mathbb{T},$$

for any bounded function φ in $\mathbb{D}$. Now $(\frac{\beta'}{\alpha'})'=\frac{\beta(\alpha-1)}{\alpha-\beta}>1$, and hence by duality

$$\big\|S_g(\chi_{S(a)})\big\|_{L^{\frac{\beta'}{\alpha'}}(\mathbb{T})}=\sup\Bigg|\int\limits_{\mathbb{T}}h(\zeta)S_g(\chi_{S(a)})(\zeta)\,|d\zeta|\Bigg|,\tag{3.2}$$

where the supremum is taken on all h such that $\|h\|_{L^{\frac{\beta(\alpha-1)}{\alpha-\beta}}(\mathbb{T})} \leqslant 1$. To estimate the right hand side, we shall write $I(z)$ for the arc $\{\zeta \in \mathbb{T}\colon\ z \in \Gamma_\sigma(\zeta)\}$ with $|I(z)| \asymp 1-|z|$. Then Fubini's theorem, Hölder's inequality and Theorem A yield

$$\begin{aligned}
\left|\int_{\mathbb{T}} h(\zeta) S_g(\chi_{S(a)})(\zeta)\,|d\zeta|\right| &\leqslant \int_{\mathbb{T}} \big|h(\zeta)\big| \int_{\Gamma_\sigma(\zeta)\cap S(a)} \big|g'(z)\big|^2\,dA(z)\,|d\zeta| \\
&\asymp \int_{S(a)} \big|g'(z)\big|^2\big(1-|z|^2\big)\left(\frac{1}{1-|z|^2}\int_{I(z)} \big|h(\zeta)\big|\,|d\zeta|\right) dA(z) \\
&\lesssim \int_{S(a)} \big|g'(z)\big|^2\big(1-|z|^2\big) M\big(|h|\big)(z)\,dA(z) \\
&\leqslant \left(\int_{S(a)} \big|g'(z)\big|^2\big(1-|z|^2\big)\,dA(z)\right)^{\frac{\alpha'}{\beta'}} \\
&\quad \cdot \left(\int_{\mathbb{D}} M\big(|h|\big)^{(\frac{\beta'}{\alpha'})'}\big|g'(z)\big|^2\big(1-|z|^2\big)\,dA(z)\right)^{1-\frac{\alpha'}{\beta'}} \\
&\lesssim \left(\int_{S(a)} \big|g'(z)\big|^2\big(1-|z|^2\big)\,dA(z)\right)^{\frac{\alpha'}{\beta'}} \\
&\quad \cdot \left(\sup_{b\in\mathbb{D}} \frac{\int_{S(b)} |g'(z)|^2(1-|z|^2)\,dA(z)}{1-|b|}\right)^{1-\frac{\alpha'}{\beta'}} \|h\|_{L^{(\frac{\beta'}{\alpha'})'}(\mathbb{T})}. \qquad (3.3)
\end{aligned}$$

Since any dilated function $g_r(z) = g(rz)$, $0<r<1$, is analytic in $D(0,\frac{1}{r})$, we deduce by replacing g by g_r in (3.1)–(3.3) that

$$\begin{aligned}
\int_{S(a)} \big|g_r'(z)\big|^2\big(1-|z|^2\big)\,dA(z) &\lesssim \big\|T_{g_r}(F_{a,p,\gamma})\big\|_{H^p}^{\frac{p}{\beta}}\left(\int_{S(a)} \big|g_r'(z)\big|^2\big(1-|z|^2\big)\,dA(z)\right)^{\frac{1}{\beta'}} \\
&\quad \cdot \left(\sup_{b\in\mathbb{D}} \frac{\int_{S(b)} \big|g_r'(z)\big|^2(1-|z|^2)\,dA(z)}{1-|b|}\right)^{\frac{1}{\alpha'}(1-\frac{\alpha'}{\beta'})}. \qquad (3.4)
\end{aligned}$$

We claim that there exists $\gamma > 0$ and a constant $C = C(p,\gamma) > 0$ such that

$$\sup_{0<r<1} \big\|T_{g_r}(F_{a,p,\gamma})\big\|_{H^p}^p \leqslant C\|T_g\|_{(\mathcal{D}^p_{p-1},H^p)}^p\big(1-|a|\big), \quad a \in \mathbb{D}, \qquad (3.5)$$

the proof of which is postponed for a moment. Now this combined with (3.4) and Fatou's lemma yield

$$\sup_{a\in\mathbb{D}} \frac{\int_{S(a)} |g'(z)|^2(1-|z|^2)\,dA(z)}{1-|a|} \lesssim \|T_g\|^2_{(\mathcal{D}^p_{p-1},H^p)},$$

and so $g \in \mathrm{BMOA}$.

It remains to prove (3.5). To see this fix $\gamma > p$. Recall that

$$\big\|T_{g_r}(F_{a,p,\gamma})\big\|_{H^p}^p \asymp \int_{\mathbb{T}}\left(\int_{\Gamma_\sigma(\zeta)} r^2\big|g'(rz)\big|^2\big|F_{a,p,\gamma}(z)\big|^2\,dA(z)\right)^{p/2}|d\zeta|.$$

If $|a| < \frac{1}{2}$, then

$$\|T_{g_r}(F_{a,p,\gamma})\|_{H^p}^p \lesssim (1-|a|)^{\gamma+1} \int_{\mathbb{T}} \left(\int_{\Gamma_\sigma(\zeta)} r^2 |g'(rz)|^2 \, dA(z) \right)^{\frac{p}{2}} |d\zeta|$$
$$\asymp (1-|a|)^{\gamma+1} \|g_r - g(0)\|_{H^p}^p \leqslant (1-|a|) \|g - g(0)\|_{H^p}^p$$
$$= (1-|a|) \|T_g(1)\|_{H^p}^p \lesssim (1-|a|) \|T_g\|_{(\mathcal{D}_{p-1}^p, H^p)}^p.$$

Let now $\frac{1}{2} \leqslant |a| < \frac{1}{2-r}$. Then $|1-\bar{a}rz| \leqslant 2|1-\bar{a}z|$ for all $z \in \mathbb{D}$, and hence

$$\|T_{g_r}(F_{a,p,\gamma})\|_{H^p}^p \lesssim \|(T_g(F_{a,p,\gamma}))_r\|_{H^p}^p \leqslant \|T_g(F_{a,p,\gamma})\|_{H^p}^p$$
$$\leqslant \|T_g\|_{(\mathcal{D}_{p-1}^p, H^p)}^p \|F_{a,p,\gamma}\|_{\mathcal{D}_{p-1}^p}^p \asymp \|T_g\|_{(\mathcal{D}_{p-1}^p, H^p)}^p (1-|a|).$$

In the remaining case $\frac{1}{2-r} \leqslant |a| < 1$ we have $r \leqslant 2 - \frac{1}{|a|} \leqslant |a|$. Now $\gamma > p$, and hence

$$\|T_{g_r}(F_a)\|_{H^p}^p \lesssim M_\infty^p(r, g')(1-|a|)^{\gamma+1} \int_{\mathbb{T}} \left(\int_{\Gamma_\sigma(\zeta)} \frac{dA(z)}{|1-\bar{a}z|^{\frac{2(\gamma+1)}{p}}} \right)^{p/2} |d\zeta|$$
$$\lesssim M_\infty^p(|a|, g')(1-|a|)^{\gamma+1} \left\| \frac{1}{(1-\bar{a}z)^{\frac{\gamma+1}{p}-1}} \right\|_{H^p}^p$$
$$\asymp \left(M_\infty(|a|, g')(1-|a|)\right)^p (1-|a|) \leqslant \|g\|_{\mathcal{B}}^p (1-|a|)$$
$$\lesssim \|T_g\|_{(\mathcal{D}_{p-1}^p, H^p)}^p (1-|a|).$$

By combining these three separate cases, we deduce (3.5). □

Next, we shall prove Theorem 3 by using similar ideas to those employed in the proof of Theorem 1.

Proof of Theorem 3. It is known that (ii) and (iii) are equivalent [1]. Further, Lemma 6 and Lemma B give (i) ⇒ (iii) and (b). Moreover, if $0 < p \leqslant 2$, then $\mathcal{D}_{p-1}^p \subset H^p$ and hence, in this case, (ii) implies (i). To complete the proof, we show that (iii) implies (i) when $2 < p < \infty$. Since $q > 2$, $L^{q/2}(\mathbb{T})$ can be identified with the dual of $L^{\frac{q}{q-2}}(\mathbb{T})$, that is, $L^{q/2}(\mathbb{T}) \simeq (L^{\frac{q}{q-2}}(\mathbb{T}))^\star$. Therefore, $T_g : \mathcal{D}_{p-1}^p \to H^q$ is bounded if and only if

$$\left| \int_{\mathbb{T}} h(\zeta) \left(\int_{\Gamma_\sigma(\zeta)} |f(z)|^2 |g'(z)|^2 \, dA(z) \right) |d\zeta| \right| \lesssim \|h\|_{L^{\frac{q}{q-2}}(\mathbb{T})} \|f\|_{\mathcal{D}_{p-1}^p}^2$$

for all $h \in L^{\frac{q}{q-2}}(\mathbb{T})$ and $f \in \mathcal{D}_{p-1}^p$. To see this, we use first Fubini's theorem to obtain

$$\left| \int_{\mathbb{T}} h(\zeta) \left(\int_{\Gamma_\sigma(\zeta)} |f(z)|^2 |g'(z)|^2 \, dA(z) \right) |d\zeta| \right| \leqslant \int_{\mathbb{D}} |f(z)|^2 |g'(z)|^2 \left(\int_{I(z)} |h(\zeta)| \, |d\zeta| \right) dA(z)$$
$$\lesssim \int_{\mathbb{D}} |f(z)|^2 M(|h|)(z) |g'(z)|^2 (1-|z|^2) \, dA(z).$$

Next, we estimate the last integral upwards by Hölder's inequality with exponent $x = 1 + p(\frac{1}{2} - \frac{1}{q})$ and its conjugate $x' = 1 + \frac{2q}{p(q-2)}$,

$$\left(\int_{\mathbb{D}} \left|f(z)\right|^{2+p-\frac{2p}{q}} \left|g'(z)\right|^2 \left(1-|z|^2\right) dA(z)\right)^{\frac{2q}{(2+p)q-2p}}$$

$$\cdot \left(\int_{\mathbb{D}} \left(M\left(|h|\right)(z)\right)^{1+\frac{2q}{p(q-2)}} \left|g'(z)\right|^2 \left(1-|z|^2\right) dA(z)\right)^{\frac{1}{1+\frac{2q}{p(q-2)}}}.$$

Since $|g'(z)|^2(1-|z|^2)\,dA(z)$ is a $(2(\frac{1}{p}-\frac{1}{q})+1)$-Carleson measure by Lemma B, $(2+p-\frac{2p}{q})/p = 2(\frac{1}{p}-\frac{1}{q})+1$ and $\frac{1+\frac{2q}{p(q-2)}}{\frac{q}{q-2}} = 2(\frac{1}{p}-\frac{1}{q})+1$, by using Theorem C and Theorem A, we get

$$\left|\int_{\mathbb{T}} h(\zeta)\left(\int_{\Gamma_\sigma(\zeta)} \left|f(z)\right|^2\left|g'(z)\right|^2 dA(z)\right) d|\zeta|\right| \lesssim \|f\|^2_{\mathcal{D}^p_{p-1}} \|h\|_{L^{\frac{q}{q-2}}(\mathbb{T})},$$

and thus $T_g : \mathcal{D}^p_{p-1} \to H^q$ is bounded. □

Proof of Proposition 4. Let F_2 be such that $F_2' = f_2$. Then $|F_2'(z)| = \mathrm{O}(\frac{1}{(1-|z|)^{1/p_2}})$ by the assumption, and hence $F_2 \in \Lambda(1-\frac{1}{p_2})$ by Lemma B. Now Theorem 3 implies that the integral operator $T_{F_2} : \mathcal{D}^{p_1}_{p_1-1} \to H^{\frac{p}{1-p}}$ is bounded, and since $f_1 \in \mathcal{D}^{p_1}_{p_1-1}$ by the assumption, we deduce $T_{F_2}(f_1)(z) = \int_0^z F_2'(\zeta) f_1(\zeta)\, d\zeta = \int_0^z f(\zeta)\, d\zeta \in H^{\frac{p}{1-p}}$, which gives the assertion. □

We finish this section by proving the expected versions of Theorems 1 and 3 for compact operators. The next auxiliary result is standard, and therefore its proof is omitted.

Lemma 7. *Let $0 < p, q < \infty$ and $g \in \mathcal{H}(\mathbb{D})$. Then the following are equivalent:*

(i) *$T_g : \mathcal{D}^p_{p-1} \to H^q$ is compact;*
(ii) *For any sequence of analytic functions $\{f_n\}_{n=1}^\infty$ in $\mathbb{D}$ that converges uniformly to 0 on compact subsets of $\mathbb{D}$ and satisfies $\sup_{n\in\mathbb{N}} \|f_n\|_{\mathcal{D}^p_{p-1}} < \infty$, we have $\lim_{n\to\infty} \|T_g(f_n)\|_{H^q} = 0$.*

Obviously the statement in this lemma remains valid if $\mathcal{D}^p_{p-1}$ is replaced by H^p.

The space VMOA consists of the functions in the Hardy space H^1 that have *vanishing mean oscillation* on the boundary $\mathbb{T}$. It is known that this space is the closure of polynomials in BMOA and is characterized by the condition

$$\lim_{|a|\to 1^-} \frac{\int_{S(a)} |g'(z)|^2(1-|z|^2)\, dA(z)}{1-|a|} = 0.$$

Theorem 8. *Let $0 < p \leqslant 2$ and $g \in \mathcal{H}(\mathbb{D})$. Then the following are equivalent:*

(i) *$T_g : \mathcal{D}^p_{p-1} \to H^p$ is compact;*
(ii) *$T_g : H^p \to H^p$ is compact;*
(iii) *$g \in$ VMOA.*

Proof. It is known that (ii) and (iii) are equivalent by [1]. Moreover, by bearing in mind Lemma 7 and (1.1), we see that (ii) implies (i). It remains to show that $g \in$ VMOA, whenever $T_g : \mathcal{D}^p_{p-1} \to H^p$ is compact. Since the proof of this implication is similar to its counterpart in the proof of Theorem 1, we only show in detail those steps that are significantly different. First observe, that $g \in$ BMOA by Theorem 1. Let $f_{a,p,\gamma} = \frac{F_{a,p,\gamma}}{(1-|a|)^{1/p}}$, where $\gamma > 0$ and $F_{a,p,\gamma}$ are the functions defined in the proof of Lemma 6. It is clear that $\|f_{a,p,\gamma}\|_{\mathcal{D}^p_{p-1}} \asymp 1$ and $f_{a,p} \to 0$, as $|a| \to 1^-$, uniformly in compact subsets of $\mathbb{D}$. Therefore $\|T_g(f_{a,p,\gamma})\|_{H^p} \to 0$, as $|a| \to 1^-$, by Lemma 7. Now, let $1 < \alpha, \beta < \infty$ such that $\beta/\alpha = p/2 < 1$. Arguing as in (3.1), we deduce

$$\frac{1}{(1-|a|)^{\frac{2}{p}}} \int\limits_{S(a)} \left|g'(z)\right|^2 \left(1-|z|^2\right) dA(z) \lesssim \left\|T_g(f_{a,p,\gamma})\right\|_{H^p}^{\frac{p}{\beta}} \left\|S_g(\chi_{S(a)} f_{a,p,\gamma})\right\|_{L^{\frac{\beta'}{\alpha'}}(\mathbb{T})}^{\frac{1}{\alpha'}}$$

for all $a \in \mathbb{D}$. Following the reasoning in the proof of Theorem 1 and bearing in mind that $g \in \mathrm{BMOA}$, we obtain

$$\frac{\int_{S(a)} |g'(z)|^2(1-|z|^2)\, dA(z)}{(1-|a|)^{\frac{2}{p}}} \lesssim \left\|T_g(f_{a,p,\gamma})\right\|_{H^p}^{\frac{p}{\beta}} \frac{\left(\int_{S(a)} |g'(z)|^2(1-|z|^2)\, dA(z)\right)^{\frac{\alpha'}{\beta'}\cdot\frac{1}{\alpha'}}}{(1-|a|)^{\frac{2}{p}\cdot\frac{1}{\alpha'}}},$$

which is equivalent to

$$\frac{\int_{S(a)} |g'(z)|^2(1-|z|^2)\, dA(z)}{(1-|a|)} \lesssim \left\|T_g(f_{a,p,\gamma})\right\|_{H^p}^{p}.$$

Therefore $g \in \mathrm{VMOA}$. □

It is known that the "little oh" analogue of Lemma B is valid. This together with appropriate modifications in the proofs of Lemma 6 and Theorem 3 give the next result.

Theorem 9. *Let $0 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} \leqslant 1$, and $g \in \mathcal{H}(\mathbb{D})$. The following are equivalent:*

(i) $T_g : \mathcal{D}^p_{p-1} \to H^q$ *is compact;*
(ii) $T_g : H^p \to H^q$ *is compact;*
(iii) $g \in \lambda(\frac{1}{p} - \frac{1}{q})$.

4. Growth of integral means of functions in $\mathcal{D}^p_{p-1}$

In this section we shall prove sharp estimates for the growth of $M_p(r,f)$ when $f \in \mathcal{D}^p_{p-1}$ and $2 < p < \infty$. If $f \in \mathcal{D}^p_{p-1}$ and $0 < p < 2$, then $M_p(r,f)$ is uniformly bounded due to (1.1).

Lemma 10. *Let $2 < p < \infty$ and $\Phi : [0,1) \to (1,\infty)$ be a differentiable increasing unbounded function such that $\frac{\Phi'(r)}{\Phi(r)}(1-r)$ is decreasing. Then the following hold:*

(i) $M_p(r,f) = \mathrm{o}((\log \frac{e}{1-r})^{\frac{1}{2}-\frac{1}{p}})$, *as $r \to 1^-$, for all $f \in \mathcal{D}^p_{p-1}$;*
(ii) *there exists $f \in \mathcal{D}^p_{p-1}$ such that*

$$M_q(r,f) \gtrsim \left(\log \frac{e}{1-r}\right)^{\frac{1}{2}} \left(\frac{\Phi'(r)}{\Phi^2(r)}(1-r)\right)^{\frac{1}{p}}, \quad 0 < r < 1, \tag{4.1}$$

for any fixed $0 < q < \infty$.

Part (i) is essentially known, but we include a proof for the sake of completeness. Part (ii), apart from showing that (i) is sharp in a very strong sense, will be used to prove Theorem 2(ii) and the sharpness of Theorem 5. It is also worth noticing that each function

$$\Phi_{N,\alpha}(r)=\left(\log_N\frac{\exp_N 2}{1-r}\right)^{\alpha},\quad N\in\mathbb{N}=\{1,2,\dots\},\ 0<\alpha<\infty,\tag{4.2}$$

satisfies both hypotheses on the auxiliary function Φ in Lemma 10. Here, as usual, $\log_n x=\log(\log_{n-1}x)$, $\log_1 x=\log x$, $\exp_n x=\exp(\exp_{n-1}x)$ and $\exp_1 x=e^x$. We remark that $\exp_N 2$ is a normalization factor and the key point is the extremely slow growth of the iterated logarithm.

Proof of Lemma 10. (i) First observe that [13, Theorem 1.4] yields

$$\mathcal{D}^p_{p-1}\subset A^p_{v_{\frac{p}{2}}},\quad \|f\|^p_{\mathcal{D}^p_{p-1}}\gtrsim\|f\|^p_{A^p_{v_{\frac{p}{2}}}},\quad f\in\mathcal{H}(\mathbb{D}),\tag{4.3}$$

where $A^p_{v_{\frac{p}{2}}}$ denotes the weighted Bergman space induced by the rapidly increasing weight $v_{\frac{p}{2}}(z)=(1-|z|)^{-1}(\log\frac{e}{1-|z|})^{-\frac{p}{2}}$, $z\in\mathbb{D}$, see [19, Section 1.2]. Therefore,

$$\begin{aligned}\|f\|^p_{\mathcal{D}^p_{p-1}}&\gtrsim\|f\|^p_{A^p_{v_{\frac{p}{2}}}}\geqslant\int_r^1 sM_p^p(s,f)v_{\frac{p}{2}}(s)\,ds\geqslant M_p^p(r,f)\int_r^1 sv_{\frac{p}{2}}(s)\,ds\\&\asymp M_p^p(r,f)\left(\log\frac{e}{1-r}\right)^{1-\frac{p}{2}},\quad 0<r<1,\end{aligned}$$

and (i) follows. (ii) Let Φ be as in the lemma. Consider the lacunary series

$$f(z)=\sum_{k=1}^{\infty}\left(\frac{h(r_k)-h(r_{k-1})}{\Phi(r_k)}\right)^{\frac{1}{p}}z^{2^k},\quad r_k=1-2^{-k},\ k\in\mathbb{N},\tag{4.4}$$

where $h(r)=\log\Phi(r)$ is a positive function such that $h'(r)(1-r)$ is decreasing by the assumptions. By [15, Proposition 3.2],

$$\begin{aligned}\|f\|^p_{\mathcal{D}^p_{p-1}}&\lesssim\sum_{k=1}^{\infty}\left(\frac{h(r_k)-h(r_{k-1})}{\Phi(r_k)}\right)\\&=\sum_{k=1}^{\infty}\frac{\int_{r_{k-1}}^{r_k}h'(t)\,dt}{\Phi(r_k)}\leqslant\int_0^1\frac{h'(t)}{\Phi(t)}\,dt=\Phi(0)^{-1}<1,\end{aligned}$$

and thus $f\in\mathcal{D}^p_{p-1}$.

On the other hand,

$$\begin{aligned}M_2^2(r_N,f)&=\sum_{k=1}^{\infty}\left(\frac{h(r_k)-h(r_{k-1})}{\Phi(r_k)}\right)^{\frac{2}{p}}r_N^{2^{k+1}}\\&\geqslant\sum_{k=1}^{N}\left(\frac{h(r_k)-h(r_{k-1})}{\Phi(r_k)}\right)^{\frac{2}{p}}r_N^{2^{k+1}}\\&\geqslant\frac{r_N^{2^{N+1}}}{(\Phi(r_N))^{\frac{2}{p}}}\sum_{k=1}^{N}\left(\int_{r_{k-1}}^{r_k}h'(s)(1-s)\frac{ds}{1-s}\right)^{\frac{2}{p}}\\&\geqslant\frac{r_N^{2^{N+1}}(\log 2)^{\frac{2}{p}}}{(\Phi(r_N))^{\frac{2}{p}}}\sum_{k=1}^{N}\big(h'(r_k)(1-r_k)\big)^{\frac{2}{p}}\end{aligned}$$

$$\gtrsim \frac{1}{(\Phi(r_N))^{\frac{2}{p}}}\big(h'(r_N)(1-r_N)\big)^{\frac{2}{p}}N.$$

Let $r \in [\frac{1}{2},1)$ be given, and choose $N \in \mathbb{N}$ such that $r_N \leqslant r < r_{N+1}$. Then [22, Theorem 8.20 in p. 215, Vol. I] yields

$$\begin{aligned} M_q^2(r,f) \asymp M_2^2(r,f) \geqslant M_2^2(r_N,f) &\gtrsim \frac{1}{(\Phi(r_N))^{\frac{2}{p}}}\big(h'(r_N)(1-r_N)\big)^{\frac{2}{p}}N \\ &\gtrsim \frac{1}{(\Phi(r))^{\frac{2}{p}}}\big(h'(r)(1-r)\big)^{\frac{2}{p}}\log\frac{e}{1-r} \\ &\asymp \left(\log\frac{e}{1-r}\right)\left(\frac{\Phi'(r)}{\Phi^2(r)}(1-r)\right)^{\frac{2}{p}}, \end{aligned}$$

which finishes the proof. □

With these preparations we are ready to prove Theorem 2.

Proof of Theorem 2. (i) If $T_g : \mathcal{D}^p_{p-1} \to H^p$ is bounded, then $T_g : H^p \to H^p$ is bounded because $H^p \subsetneq \mathcal{D}^p_{p-1}$ for $2 < p < \infty$ by (1.2), and hence $g \in \mathrm{BMOA}$.

(ii) In this part we use ideas from the proof of [15, Theorem 2.1]. Take a function Φ as in Lemma 10 and let $f \in \mathcal{D}^p_{p-1}$ be the lacunary series associated with Φ via (4.4).

By using [22, Theorem 8.25, Chap. V, Vol. I], we find two constants $A > 0$ and $B > 0$ such that for every $r \in (0,1)$ the set

$$E_r = \{t \in [0,2\pi]: \ |f(re^{it})| > BM_2(r,f)\} \tag{4.5}$$

has the Lebesgue measure greater than or equal to A. Let now g be a lacunary series. By using [22, Lemma 6.5, Chap. V, Vol. I] we find a constant $C_1 > 0$ such that

$$\int_{E_r} |g'(re^{it})|^2\,dt \geqslant C_1 A M_2^2(r,g') = C_2 M_2^2(r,g'), \quad 0 < r < 1, \tag{4.6}$$

where $C_2 = C_1 A$. Bearing in mind the definition (4.5) of the sets E_r and using (4.6), we obtain

$$\begin{aligned} \|T_g(f)\|^2_{H^p} \geqslant \|T_g(f)\|^2_{H^2} &\gtrsim \int_{\mathbb{D}} |f(z)|^2|g'(z)|^2\big(1-|z|^2\big)\,dA(z) \\ &\geqslant \int_0^1 r(1-r)\int_{E_r}|f(re^{it})|^2|g'(re^{it})|^2\,dt\,dr \\ &\geqslant B^2\int_0^1 r(1-r)M_2^2(r,f)\int_{E_r}|g'(re^{it})|^2\,dt\,dr \\ &\geqslant B^2C_2\int_0^1 r(1-r)M_2^2(r,f)M_2^2(r,g')\,dr \\ &\geqslant B^2C_2C\int_0^1 r(1-r)\left(\log\frac{e}{1-r}\right)\left(\frac{\Phi'(r)}{\Phi^2(r)}(1-r)\right)^{\frac{2}{p}}M_2^2(r,g')\,dr. \end{aligned} \tag{4.7}$$

Choose now $\Phi(r) = (\log \frac{e}{1-r})^\varepsilon$, where $0 < \varepsilon < \frac{p}{2} - 1$, so that

$$\left(\log \frac{e}{1-r}\right)\left(\frac{\Phi'(r)}{\Phi^2(r)}(1-r)\right)^{\frac{2}{p}} \asymp \left(\log \frac{e}{1-r}\right)^{1-\frac{2}{p}(1+\varepsilon)}.$$

Further, let

$$g(z) = \sum_{j=0}^{\infty} \frac{1}{(j+1)(\log j + 1)^\alpha} z^{2^{2^j}}, \quad 1 < \alpha < \infty.$$

Then, clearly, $g \in \mathcal{A}$. Moreover, since $\omega(r) = (1-r)(\log \frac{e}{1-r})^{1-\frac{2}{p}(1+\varepsilon)}$ is a so-called regular weight [19, Section 1.2], we deduce

$$\int_0^1 r^{2n+1}\omega(r) \asymp n^{-1}\omega\left(1-n^{-1}\right), \quad n \in \mathbb{N},$$

by [19, Lemma 1.3 and (1.1)]. This together with (4.7) yields

$$\begin{aligned}
\left\|T_g(f)\right\|_{H^p}^2 &\gtrsim \int_0^1 r(1-r)\left(\log \frac{e}{1-r}\right)^{1-\frac{2}{p}(1+\varepsilon)} M_2^2\left(r, g'\right) dr \\
&\asymp \sum_{j=1}^{\infty} \frac{2^{2^{j+1}}}{(j+1)^2(\log j + 1)^{2\alpha}}\left(\int_0^1 r^{2^{2^j+1}-1}(1-r)\left(\log \frac{e}{1-r}\right)^{1-\frac{2}{p}(1+\varepsilon)} dr\right) \\
&\asymp \sum_{j=1}^{\infty} \frac{2^{(j+1)(1-\frac{2}{p}(1+\varepsilon))}}{(j+1)^2(\log j + 1)^{2\alpha}} = \infty,
\end{aligned}$$

and finishes the proof. □

5. Carleson measures for the Dirichlet space $\mathcal{D}^p_{p-1}$

The statement in Theorem 5 follows directly by (4.3) and [19, Theorem 2.1] with $\omega = v_{p/2}$. Next, we show that this result is sharp in a very strong sense. For this purpose, the following lemma is needed.

Lemma 11. *Let $2 < p < \infty$, and let $\Phi : [0,1) \to (0,\infty)$ be a differentiable increasing function such that*

$$\frac{\Phi(r)}{(\log \frac{e}{1-r})^{\frac{p}{2}-1}} \to 0, \quad r \to 1^-, \tag{5.1}$$

and

$$m = -\liminf_{r \to 1^-} \frac{\Phi'(r)}{\Phi(r)}(1-r)\log \frac{e}{1-r} > 1 - \frac{p}{2}. \tag{5.2}$$

Then

$$\int_r^1 \frac{\Phi(s)\, ds}{(1-s)(\log \frac{e}{1-s})^{\frac{p}{2}}} \lesssim \frac{\Phi(r)}{(\log \frac{e}{1-r})^{\frac{p}{2}-1}}, \quad r \in (0,1).$$

Proof. By the Bernoulli–l'Hôpital theorem,

$$\limsup_{r\to 1^-} \frac{\int_r^1 \frac{\Phi(s)\,ds}{(1-s)(\log\frac{e}{1-r})^{\frac{p}{2}}}}{\frac{\Phi(r)}{(\log\frac{e}{1-r})^{\frac{p}{2}-1}}} \leqslant \left(m+\frac{p}{2}-1\right)^{-1} \in (0,\infty),$$

and the assertion follows. □

If $\Phi_c(r) = (\log\frac{e}{1-r})^c$ and $c > 0$, then

$$\frac{\Phi_c'(r)}{\Phi_c(r)}(1-r)\log\frac{e}{1-r} = c, \quad 0 < r < 1,$$

and thus Φ_c satisfies both (5.1) and (5.2) if $c < \frac{p}{2} - 1$. Further, each function $\Phi_n(r) = \log_n \frac{\exp_n(2)}{1-r}$, $n \in \mathbb{N}$, satisfies

$$\frac{\Phi_n'(r)}{\Phi_n(r)}(1-r)\log\frac{e}{1-r} \to 0, \quad r \to 1,$$

and hence satisfies all hypotheses of the next result.

Proposition 12. *Let $2 < p < \infty$, and let $\Phi : [0,1) \to (1,\infty)$ be a differentiable increasing unbounded function such that $\frac{\Phi'(r)}{\Phi(r)}(1-r)$ is decreasing and (5.1) and (5.2) are satisfied. Then there exists a positive Borel measure μ on $\mathbb{D}$ such that*

$$\sup_{I\subset\mathbb{T}} \frac{\mu(S(I))}{|I|(\log\frac{e}{|I|})^{-p/2+1}\Phi(1-|I|)} < \infty, \tag{5.3}$$

but μ is not a p-Carleson measure for $\mathcal{D}^p_{p-1}$.

Proof. The radial measure

$$d\mu(z) = \frac{\Phi(|z|)\,dA(z)}{(1-|z|)(\log\frac{e}{1-|z|})^{p/2}}, \quad z \in \mathbb{D},$$

satisfies (5.3) by Lemma 11. To see that μ is not a p-Carleson measure for $\mathcal{D}^p_{p-1}$, consider the lacunary series associated with Φ via (4.4). By Lemma 10, $f \in \mathcal{D}^p_{p-1}$ and

$$\begin{aligned}\|f\|^p_{L^p(\mu)} &= \int_0^1 \frac{M_p^p(r,f)\Phi(r)}{(1-r)(\log\frac{e}{1-r})^{p/2}} r\,dr \\ &\gtrsim \int_0^1 \frac{r\Phi'(r)}{\Phi(r)}\,dr \gtrsim \lim_{t\to 1-} \log\Phi(t) = \infty,\end{aligned}$$

which finishes the proof. □

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