



# On bounds for weighted norms for matrices and integral operators

Josip Pečarić <sup>a,\*</sup>, Ivan Perić <sup>b</sup>, Rajko Rokić <sup>c</sup>

<sup>a</sup>Faculty of Textile Technology, University of Zagreb, Zagreb, Croatia

<sup>b</sup>Faculty of Chemical Engineering and Technology, University of Zagreb, Zagreb, Croatia

<sup>c</sup>Faculty of Food Technology and Biotechnology, University of Zagreb, Zagreb, Croatia

Received 7 January 1999; accepted 4 October 2000

Submitted by G.P.H. Styan

---

## Abstract

In this note we consider inequalities of the form  $\|\mathbf{Ax}\|_{\omega,q} \leq \lambda \|\mathbf{Bx}\|_{v,p}$ , where  $\mathbf{A}$  and  $\mathbf{B}$  are matrices or integral operators,  $\mathbf{x}$  decreasing sequence or function and  $\omega$  and  $v$  are weights. Obtained results are generalizations of results of G. Bennett [Linear Algebra Appl. 82 (1986) 81] and P.E. Renaud [Bull. Aust. Math. Soc. 34 (1986) 225]. © 2001 Elsevier Science Inc. All rights reserved.

*Keywords:* Inequalities; Weights; Decreasing sequences and functions; Summation by parts in weighted  $\ell^p$  spaces; Integration by parts in weighted  $L^p$  spaces; Hilbert kernel; Riemann’s zeta function

---

## 1. Introduction

We shall be concerned with the spaces  $\ell^p$ ,  $0 < p < \infty$ , of sequences of real numbers satisfying

$$\|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty.$$

Bennett [1] considered inequality

$$\|\mathbf{Ax}\|_q \geq \lambda \|\mathbf{x}\|_p \tag{1}$$

---

\* Corresponding author.

for  $x \in \ell^p$  with  $x_1 \geq x_2 \geq \dots \geq 0$  and  $\mathbf{A}$  is a matrix with nonnegative entries, assumed to map  $\ell^p$  into  $\ell^q$ , and  $\lambda$  is a constant not depending on  $\mathbf{x}$ .

In this paper, we shall consider the inequality of the form

$$\|\mathbf{Ax}\|_{\omega,q} \geq \lambda \|\mathbf{x}\|_{v,p} \tag{2}$$

and inequality of the form

$$\|\mathbf{Ax}\|_{\omega,q} \leq \lambda \|\mathbf{Bx}\|_{v,p}, \tag{3}$$

where  $\|\mathbf{x}\|_{v,p}$  is defined by

$$\|\mathbf{x}\|_{v,p} = \left( \sum_{k=1}^{\infty} v_k x_k^p \right)^{1/p}$$

and  $\omega$  and  $\mathbf{v}$  are nonnegative weights.

In Section 2, we start with generalization of formula for summation by parts in  $\ell^p$  obtained by Bennett [1, Proposition 1]. In Section 3 we consider inequalities (2) and (3), while in Section 4 we show that result obtained by Bennett [1, Theorem 4] and Renaud [7] is also valid for sequences nonincreasing in mean. In Section 5 we consider integral analogues of such inequalities, proved in [2], but our proofs are simpler and in agreement with [1]. We also obtain integral analogues of the summation by parts in  $\ell^p$ .

## 2. Generalized formula for summation by parts

In this part the following elementary lemma will be needed (see [1]).

**Lemma 1.** *Let  $a, b, c \geq 0$  with  $a \geq b$ . If  $p > 1$ , then*

$$(a + c)^p - a^p > (b + c)^p - b^p, \tag{4}$$

*unless  $a = b$  or  $c = 0$ . If  $0 < p < 1$ , inequality in (4) is reversed.*

The following result is the generalization of Proposition 1 in [1].

**Proposition 1.** *Let  $a_1, a_2, \dots, a_n \geq 0$ ,  $v_1, v_2, \dots, v_n \geq 0$  and  $x_1 \geq x_2 \geq \dots \geq x_n \geq 0$ . If  $p \geq 1$  and  $0 < q \leq p$ , then*

$$\begin{aligned} & \left( \sum_{k=1}^n a_k x_k \right)^q \left[ \left( \sum_{i=1}^n v_i x_i^p \right)^{1/p} \right]^{p-q} \\ & \geq \sum_{r=1}^{n-1} \left( \sum_{i=1}^r a_i \right)^q \left( \sum_{j=1}^r v_j \right)^{1-(q/p)} (x_r^p - x_{r+1}^p) \end{aligned}$$

$$+ \left( \sum_{i=1}^n a_i \right)^q \left( \sum_{j=1}^n v_j \right)^{1-(q/p)} x_n^p. \tag{5}$$

If  $p \leq 1$  and  $q \geq p$ , the inequality in (5) is reversed. There is equality in (either version of) (5) if and only if at least one condition holds from each of the following pairs (1), (2) and (3), (4):

- (1)  $p = 1$ ,
- (2)  $x_u = \dots = x_v$ , where  $u$  is the smallest and  $v$  the largest value of  $k$  such that  $a_k x_k > 0$ .
- (3)  $q = p$ ,
- (4)  $(\sum_{j=1}^r v_j)^{-1} (\sum_{i=1}^r a_i)^p$  is constant for those values of  $r$ ,  $1 \leq r \leq n$ , satisfying  $x_r > x_{r+1}$ .

**Proof.** The idea of the proof is similar to that of Bennet’s. We prove the case  $p > 1$ ; the case  $0 < p \leq 1$  is similar. It is convenient to set  $x_{n+1} = 0$ ,  $s_r = a_1 + \dots + a_r$ , and to consider first the special case,  $q = p$ . Inequality (5) then reduces to

$$\left( \sum_{k=1}^n a_k x_k \right)^p \geq \sum_{r=1}^n s_r^p (x_r^p - x_{r+1}^p), \tag{6}$$

what was proved in [1] as a consequence of Lemma 1. Equality is valid in (6) if and only if either (1) or (2) is valid.

To prove the general case  $q \leq p$  we rewrite the right-hand side of (5) as

$$\begin{aligned} & \left( \sum_{k=1}^n a_k x_k \right)^q \left[ \left( \sum_{i=1}^n v_i x_i^p \right)^{1/p} \right]^{p-q} \\ &= \left[ \left( \sum_{k=1}^n a_k x_k \right)^p \right]^{q/p} \left( \sum_{i=1}^n v_i x_i^p \right)^{(p-q)/p} \\ &\geq \left[ \sum_{r=1}^n s_r^p (x_r^p - x_{r+1}^p) \right]^{q/p} \left[ \sum_{i=1}^n \left( \sum_{j=1}^i v_j \right) (x_i^p - x_{i+1}^p) \right]^{(p-q)/p}. \end{aligned} \tag{7}$$

Applying Hölder’s inequality with exponents  $p/q$  and  $p/(p - q)$ , we get

$$\begin{aligned} & \left( \sum_{k=1}^n a_k x_k \right)^q \left[ \left( \sum_{i=1}^n v_i x_i^p \right)^{1/p} \right]^{p-q} \\ &\geq \sum_{r=1}^n s_r^q (x_r^p - x_{r+1}^p)^{q/p} \left[ \left( \sum_{j=1}^r v_j \right) (x_r^p - x_{r+1}^p) \right]^{(p-q)/p} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{r=1}^{n-1} \left( \sum_{i=1}^r a_i \right)^q \left( \sum_{j=1}^r v_j \right)^{1-(q/p)} (x_r^p - x_{r+1}^p) \\
 &+ \left( \sum_{i=1}^n a_i \right)^q \left( \sum_{j=1}^n v_j \right)^{1-(q/p)} x_n^p.
 \end{aligned} \tag{8}$$

Equality in the last inequality is valid if and only if there is  $\mu \in \mathbf{R}$  such that

$$s_r^p (x_r^p - x_{r+1}^p) = \mu \left( \sum_{j=1}^r v_j \right) (x_r^p - x_{r+1}^p), \quad r = 1, \dots, n$$

and condition (4) follows.  $\square$

**Remark 1.** For  $v_1 = v_2 = \dots = v_n = 1$  we have Proposition 1 from [1].

### 3. Bounds for matrices

In this part our consideration is finite-dimensional (except Theorem 4). Infinite-dimensional case can be deduced from this in usual way. The following result is a generalization of Theorem 2 from [1].

**Theorem 1.** *Let  $x_1 \geq x_2 \geq \dots \geq x_n \geq 0$ ,  $p \geq 1$ ,  $0 < q \leq p$  and let  $\mathbf{A}$  be an  $m \times n$  matrix with nonnegative entries. Then*

$$\|\mathbf{Ax}\|_{\omega,q} \geq \lambda \|\mathbf{x}\|_{v,p}, \tag{9}$$

where

$$\lambda^q = \min_{1 \leq r \leq n} \left( \sum_{i=1}^r v_i \right)^{-(q/p)} \sum_{j=1}^m \omega_j \left( \sum_{k=1}^r a_{jk} \right)^q. \tag{10}$$

There is equality in (9) if  $x$  has the form

$$x_k = \begin{cases} x_1, & k \leq s, \\ 0, & k > s, \end{cases} \tag{11}$$

where  $s$  is any value of  $r$  at which the minimum in (10) occurs. If  $0 < p \leq 1$ ,  $p \leq q$  the inequality in (9) is reversed where  $\lambda$  is similarly defined with  $\max$  instead of  $\min$ .

**Proof.** The proof is similar to that of Bennett’s. We prove the case  $p \geq 1$ ,  $0 < q \leq p$ . We may assume, by homogeneity, that  $\|\mathbf{x}\|_{v,p} = 1$ . Applying Proposition 1, we have

$$\begin{aligned}
 \|\mathbf{Ax}\|_{\omega,q}^q &= \sum_{i=1}^m \omega_i \left( \sum_{j=1}^n a_{ij}x_j \right)^q \\
 &\geq \sum_{i=1}^m \omega_i \sum_{r=1}^n \left( \sum_{k=1}^r v_k \right)^{1-(q/p)} \left( \sum_{j=1}^r a_{ij} \right)^q (x_r^p - x_{r+1}^p) \\
 &= \sum_{r=1}^n \left( \sum_{k=1}^r v_k \right)^{-(q/p)} \sum_{i=1}^m \omega_i \left( \sum_{j=1}^r a_{ij} \right)^q \left( \sum_{k=1}^r v_k \right) (x_r^p - x_{r+1}^p) \\
 &\geq \lambda^q \sum_{r=1}^n \left( \sum_{k=1}^r v_k \right) (x_r^p - x_{r+1}^p) = \lambda^q \sum_{k=1}^n v_k x_k^p \\
 &= \lambda^q \|\mathbf{x}\|_{v,p}^p.
 \end{aligned}$$

Recalling that  $\|\mathbf{x}\|_{v,p} = 1$  we see that (9) follows by taking  $q$ th roots.

It is clear, by inspection, that equality holds in (9) whenever (11) is satisfied.  $\square$

**Remark 2.** For  $v_1 = \dots = v_n = 1 = \omega_1 = \dots = \omega_m$  we have Theorem 2 in [1].

The following theorem should be compared with Theorem 3.2 in [2].

**Theorem 2.** Let  $x_1 \geq x_2 \geq \dots \geq x_n \geq 0$ , let  $\mathbf{A}$  and  $\mathbf{B}$  be  $m \times n$  matrices with non-negative entries and  $0 < p \leq 1 \leq q < \infty$ . Then

$$\|\mathbf{Ax}\|_{\omega,q} \leq \lambda \|\mathbf{Bx}\|_{v,p}, \tag{12}$$

where

$$\lambda = \max_{1 \leq r \leq n} \frac{\left( \sum_{j=1}^m \omega_j \left( \sum_{k=1}^r a_{j,k} \right)^q \right)^{1/q}}{\left( \sum_{j=1}^m v_j \left( \sum_{k=1}^r b_{j,k} \right)^p \right)^{1/p}}. \tag{13}$$

There is an equality in (12) if  $\mathbf{x}$  has the form

$$x_k = \begin{cases} x_1, & k \leq s, \\ 0, & k > s, \end{cases} \tag{14}$$

where  $s$  is any value of  $r$  at which the maximum in (13) occurs.

**Proof.** Applying Abel’s identity and Minkowski inequality twice, we have

$$\|\mathbf{Ax}\|_{\omega,q} = \left( \sum_{j=1}^m \omega_j \left( \sum_{k=1}^n a_{j,k}x_k \right)^q \right)^{1/q}$$

$$\begin{aligned}
 &= \left( \sum_{j=1}^m \omega_j \left[ \sum_{r=1}^n \left( \sum_{k=1}^r a_{j,k} \right) (x_r - x_{r+1}) \right]^q \right)^{1/q} \\
 &\leq \sum_{r=1}^n \left( \sum_{j=1}^m \omega_j \left( \sum_{k=1}^r a_{j,k} \right)^q \right)^{1/q} (x_r - x_{r+1}) \\
 &\leq \lambda \sum_{r=1}^n \left( \sum_{j=1}^m v_j \left( \sum_{k=1}^r b_{j,k} \right)^p \right)^{1/p} (x_r - x_{r+1}) \\
 &\leq \lambda \left( \sum_{j=1}^m v_j \left[ \sum_{r=1}^n (x_r - x_{r+1}) \left( \sum_{k=1}^r b_{j,k} \right) \right]^p \right)^{1/p} \\
 &= \lambda \|\mathbf{B}\mathbf{x}\|_{v,p}. \quad \square
 \end{aligned}$$

In view of the discussion below Theorem 3 from [1, Theorem 2] gives the only possible case for general matrices.

The following theorem can be deduced from Proposition 1 (take  $w_k$ 's instead of  $a_k$ 's,  $x_k^q$ 's instead of  $x_k$ 's, 1 instead of  $q$  and  $p/q \leq 1$  instead of  $p$ ), but we choose to give an independent proof (for that reason compare [10, p. 176] and [8, p. 148]).

**Theorem 3.** *Let  $x_1 \geq x_2 \geq \dots \geq x_n \geq 0, 0 < p \leq q < \infty$ . Then*

$$\|\mathbf{x}\|_{\omega,q} \leq \lambda \|\mathbf{x}\|_{v,p}, \tag{15}$$

where

$$\lambda = \max_{1 \leq r \leq n} \frac{\left( \sum_{j=1}^r \omega_j \right)^{1/q}}{\left( \sum_{j=1}^r v_j \right)^{1/p}}. \tag{16}$$

There is an equality in (15) if  $\mathbf{x}$  has the form

$$x_k = \begin{cases} x_1, & k \leq s, \\ 0, & k > s, \end{cases} \tag{17}$$

where  $s$  is any value of  $r$  at which the maximum in (13) occurs.

**Proof.** Applying Abel's identity, inequality (6) with exponent  $q/p \geq 1$  we have

$$\begin{aligned}
 \|\mathbf{x}\|_{\omega,q}^q &= \sum_{k=1}^n x_k^q \omega_k = \sum_{k=1}^n \left( \sum_{r=1}^k \omega_r \right) (x_k^q - x_{k+1}^q) \\
 &\leq \left( \sum_{k=1}^n \left[ \left( \sum_{r=1}^k \omega_r \right)^{p/q} - \left( \sum_{r=1}^{k-1} \omega_r \right)^{p/q} \right] x_k^p \right)^{q/p}
 \end{aligned}$$

$$\begin{aligned}
 &= \left( \sum_{k=1}^n (x_k^p - x_{k+1}^p) \left( \sum_{r=1}^k \omega_r \right)^{p/q} \right)^{q/p} \\
 &\leq \lambda^q \left( \sum_{k=1}^n (x_k^p - x_{k+1}^p) \left( \sum_{r=1}^k v_r \right) \right)^{q/p} = \lambda^q \left( \sum_{k=1}^n x_k^p v_k \right)^{q/p} \\
 &= \lambda^q \|\mathbf{x}\|_{v,p}^q. \quad \square
 \end{aligned}$$

If  $0 < q < p < \infty$ , then by Hölder inequality we have sharp inequality

$$\left( \sum_{i=1}^n x_i^q \omega_i \right)^{1/q} \leq \left( \sum_{i=1}^n \omega_i^{r/q} v_i^{-(r/p)} \right)^{1/r} \left( \sum_{i=1}^n x_i^p v_i \right)^{1/p}, \tag{18}$$

where  $(x_i)$  is nonnegative sequence and  $1/r = 1/q - 1/p$ . Thus, inequality (16) is reversed Hölder type inequality.

To complete our discussion we give for  $0 < q < p < \infty$  inequality also of the type (18) but for decreasing sequences. In this case we found infinite-dimensional case more appropriate. For integral analogue compare [4,8,10]. We follow the idea from [10].

**Theorem 4.** Let  $\mathbf{x} = (x_i)$  be a decreasing nonnegative sequence,  $0 < q < p < \infty$  and  $1/r = 1/q - 1/p$ . If

$$\lambda^r = \sum_{i=1}^{\infty} \left( \sum_{j=1}^i \omega_j \right)^{r/q} \left( \sum_{j=1}^i v_j \right)^{-(r/q)} \quad v_{i+1} < \infty, \tag{19}$$

then there is  $C \in \mathbf{R}$  (not depending on  $\mathbf{x}$ ) such that

$$\|\mathbf{x}\|_{\omega,q} \leq C \|\mathbf{x}\|_{v,p}. \tag{20}$$

If  $\lambda < \infty$  and  $C$  is the best possible constant such that (20) holds, then

$$\left( \frac{q}{r} \right)^{1/p} \mu^{r/q} \lambda^{-(r/p)} \leq C \leq \left( \frac{r}{p} \right)^{1/r} \lambda, \tag{21}$$

where

$$\mu^r = \sum_{i=1}^{\infty} \left( \sum_{j=1}^i \omega_j \right)^{r/p} \left( \sum_{j=1}^{i+1} v_j \right)^{-(r/p)} \quad \omega_i < \infty. \tag{22}$$

**Proof.** To prove first implication and second inequality in (21) set

$$A_i = \sum_{k=i}^{\infty} \left( \sum_{j=1}^k \omega_j \right)^{r/p} \left( \sum_{j=1}^k v_j \right)^{-(r/q)} \quad v_{k+1},$$

and note that by Abel's identity

$$\sum_{i=1}^{\infty} \omega_i A_i = \sum_{i=1}^{\infty} (A_i - A_{i+1}) \sum_{k=1}^i \omega_k = \lambda^r.$$

Using Hölder's inequality (with exponents  $p/q$ ,  $r/q$ ) we have

$$\begin{aligned} \|\mathbf{x}\|_{\omega,q}^q &= \sum_{i=1}^{\infty} x_i^q \omega_i^{q/p} A_i^{-(q/r)} \omega_i^{1-(q/p)} A_i^{q/r} \\ &\leq \left( \sum_{i=1}^{\infty} x_i^p \omega_i A_i^{-(p/r)} \right)^{q/p} \left( \sum_{i=1}^{\infty} \omega_i A_i \right)^{q/r} \\ &= \lambda^q \left( \sum_{i=1}^{\infty} x_i^p \omega_i A_i^{-(p/r)} \right)^{q/p} \\ &= \lambda^q \left[ \sum_{i=1}^{\infty} \left( \sum_{j=1}^i \omega_j A_j^{-(p/r)} \right) (x_i^p - x_{i+1}^p) \right]^{q/p} \\ &\leq \lambda^q \left[ \sum_{i=1}^{\infty} A_i^{-(p/r)} \left( \sum_{j=1}^i \omega_j \right) (x_i^p - x_{i+1}^p) \right]^{q/p} \\ &\leq \lambda^q \left( \frac{r}{p} \right)^{q/r} \left[ \sum_{i=1}^{\infty} \left( \sum_{j=1}^i v_j \right) (x_i^p - x_{i+1}^p) \right]^{q/p} \\ &= \lambda^q \left( \frac{r}{p} \right)^{q/r} \left( \sum_{i=1}^{\infty} x_i^p v_i \right)^{q/p} = \lambda^q \left( \frac{r}{p} \right)^{q/r} \|\mathbf{x}\|_{v,p}^q, \end{aligned}$$

where the last inequality follows from

$$\begin{aligned} A_i^{-(p/r)} &\leq \left( \sum_{j=1}^i \omega_j \right)^{-1} \left[ \sum_{k=i}^{\infty} \left( \sum_{j=1}^k v_j \right)^{-(r/q)} v_{k+1} \right]^{-(p/r)} \\ &\leq \left( \frac{p}{r} \right)^{-(p/r)} \left( \sum_{j=1}^i \omega_j \right)^{-1} \\ &\quad \times \left[ \sum_{k=i}^{\infty} \left( \sum_{j=1}^k v_j \right)^{-(r/p)} - \left( \sum_{j=1}^{k+1} v_j \right)^{-(r/p)} \right]^{-(p/r)} \end{aligned}$$



$$= \left(\frac{r}{p}\right)^{p/r} \left(\sum_{j=1}^i \omega_j\right)^{-1} \sum_{j=1}^i v_j,$$

where some obvious estimations are used and elementary inequality

$$a^\alpha - b^\alpha \leq \alpha a^{\alpha-1}(a - b), \quad a < b, \alpha < 0$$

for  $\alpha = 1 - r/q$ ,  $a = \sum_{j=1}^k v_j$ ,  $b = \sum_{j=1}^{k+1} v_j$ .

Suppose now that  $\lambda < \infty$  and that (20) holds. First note that

$$\begin{aligned} \lambda^r &= \sum_{i=1}^{\infty} \left(\sum_{j=1}^i \omega_j\right)^{r/q} \left(\sum_{j=1}^i v_j\right)^{-(r/q)-1} v_{i+1} \left(\sum_{j=1}^i v_j\right) \\ &= \sum_{i=1}^{\infty} v_i \left[ \sum_{k=i}^{\infty} \left(\sum_{j=1}^k \omega_j\right)^{r/q} \left(\sum_{j=1}^k v_j\right)^{-(r/q)-1} v_{k+1} \right]. \end{aligned} \tag{23}$$

If  $x_i^p$  is defined by expression in square brackets in (23) for  $i = 1, 2, \dots$ , then (20) for the sequence  $\mathbf{x} = (x_i)$  gives

$$\begin{aligned} C^q \lambda^{r(q/p)} &\geq \sum_{i=1}^{\infty} \left[ \sum_{k=i}^{\infty} \left(\sum_{j=1}^k \omega_j\right)^{r/q} \left(\sum_{j=1}^k v_j\right)^{-(r/q)-1} v_{k+1} \right]^{q/p} \omega_i \\ &\geq \sum_{i=1}^{\infty} \left(\sum_{j=1}^i \omega_j\right)^{r/p} \left[ \sum_{k=i}^{\infty} \left(\sum_{j=1}^k v_j\right)^{-(r/q)-1} v_{k+1} \right]^{q/p} \omega_i \\ &\geq \left(\frac{q}{r}\right)^{q/p} \sum_{i=1}^{\infty} \left(\sum_{j=1}^i \omega_j\right)^{r/p} \left(\sum_{j=1}^i v_j\right)^{-(r/p)} \omega_i = \left(\frac{q}{r}\right)^{q/p} \mu^r, \end{aligned}$$

where the last inequality follows using again elementary inequality as above (now for  $\alpha = -r/q$ ). This shows the first inequality in (21) and that  $\mu < \infty$ .  $\square$

Note that from (21)  $\mu \leq (r/p)^{q/r^2} (q/r)^{-q/(rp)} \lambda$ . We also note that in the same manner we can prove that

$$\mu^r \geq \frac{q}{p} \sum_{i=1}^{\infty} \left(\sum_{j=1}^i \omega_j\right)^{r/q} \left(\sum_{j=1}^{i+1} v_j\right)^{-(r/q)} v_{i+1} \tag{24}$$

(in the first step take elementary inequality  $a^\alpha - b^\alpha \leq \alpha a^{\alpha-1}(a - b)$ ,  $\alpha > 1$ ,  $a > b$  with  $\alpha = r/q$ , and in the second step elementary inequality as in the proof of

Theorem 4 with  $\alpha = -r/p$ ). In this way, if  $\tilde{\lambda}^r$  denotes the sum in (24), we obtain weaker, but more symmetrical form of (21)

$$\left(\frac{q}{r}\right)^{1/p} \left(\frac{q}{p}\right)^{1/q} \tilde{\lambda}^{r/q} \lambda^{-(r/p)} \leq C \leq \left(\frac{r}{p}\right)^{1/r} \lambda. \tag{25}$$

Theorem 4 also implies that if  $\lambda < \infty$ , then identity operator  $\iota : \ell_v^p \mapsto \ell_\omega^q$  is bounded on the cone of nonnegative decreasing sequences.

**4. The Cesaro matrix**

An interesting application of (1) for Cesaro matrix  $\mathbf{C}$  was obtained in [1]. The same result has been obtained in [7].

**Theorem 5.** *Let  $p$  be fixed,  $1 < p < \infty$ . Then*

$$\|\mathbf{C}\mathbf{x}\|_p \geq \zeta(p)^{1/p} \|\mathbf{x}\|_p \tag{26}$$

for every  $x \in \ell^p$  satisfying  $x_1 \geq x_2 \geq \dots \geq 0$ , where  $\zeta(p)$  is Riemann’s zeta function. There is equality in (26) if and only if  $x_2 = x_3 = \dots = 0$ .

Here we note that inequality (26) holds under weaker assumption that  $(x_n)$  is decreasing in mean, that is

$$x_1 \geq \frac{x_1 + x_2}{2} \geq \dots \geq \frac{1}{n} \sum_{i=1}^n x_i \geq \dots \tag{27}$$

The proof of Renaud was based on the following two lemmas:

**Lemma 2.** *Let  $p > 1$  and let  $\mathbf{x} = (x_n), n = 1, 2, \dots$  be a nonincreasing sequence of nonnegative real numbers. Then*

$$\begin{aligned} & (x_1 + \dots + x_n)^p - (x_1^p + \dots + x_n^p) \\ & \geq \sum_{k=2}^n [k^p - (k-1)^p - 1] x_k^p, \quad n = 2, 3, \dots \end{aligned}$$

**Lemma 3.** *For  $n \geq 2$ , we have*

$$[n^p - (n-1)^p - 1] T_{n-1} \geq S_{n-1},$$

where

$$S_n = \sum_{k=1}^n \frac{1}{k^p} \quad \text{and} \quad T_n = \zeta(p) - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k^p}.$$

It was proved in [6] that Lemma 2 is valid for nonnegative sequences which are nonincreasing in mean. So, using idea of proof from [7] we can prove:

**Theorem 6.** *If  $p > 1$  and  $\mathbf{x} = (x_n) \in \ell^p$  is a nonincreasing in mean sequence of nonnegative real numbers, then (26) is valid. The constant  $\zeta(p)$  in (26) is best possible.*

### 5. Integral analogues

In this section we consider an integral version of Proposition 1 and a generalization of Theorem 7 from [1]. We give sharp lower and upper bounds on weighted Lebesgue spaces (to be more precise, on the cone of nonnegative decreasing functions in these spaces) for transformations,  $f = Kg$ , of the form

$$f(x) = \int_0^a K(x, y)g(y) dy, \quad 0 < x < a,$$

where  $K(x, y) \geq 0$  is measurable, and, in lower bound cases, for given  $p$  and  $q$  we assume that  $K$  maps  $L_v^p$  into  $L_q^\omega$ . As usual, for given  $p > 0$  and nonnegative weight  $v$  on  $(0, a)$ , for  $g \in L_v^p(0, a)$  we set

$$\|g\|_{v,p} = \left( \int_0^a g^p(x)v(x) dx \right)^{1/p}.$$

Also, for given weight  $v$  we set  $V(x) = \int_0^x v(t) dt, \quad x \in (0, a)$ .

We will need the following theorem (Theorem 2.1 in [2]):

**Theorem 7.** *Let  $-\infty < a < b \leq \infty$  and  $f \geq 0$  on  $(a, b)$  and  $g$  be continuous on  $(a, b)$ . Suppose  $f \uparrow$  on  $(a, b)$  and  $g \downarrow$  on  $(a, b)$  with  $\lim_{x \rightarrow b-} g(x) = 0$ . Then for any  $\gamma \in (0, 1]$*

$$\int_a^b f(x) d[-g(x)] \leq \left( \int_a^b f^\gamma(x) d[-g^\gamma(x)] \right)^{1/\gamma}. \tag{28}$$

*If  $1 \leq \gamma < \infty$ , the inequality in (28) is reversed.*

**Proposition 2.** *Suppose that  $p \geq 1, q \leq p, f$  is nonnegative on  $(0, a)$  and  $g$  is absolutely continuous nonincreasing on  $(0, a)$  such that  $g(a - 0) = 0$ . Then*

$$\begin{aligned} & \left( \int_0^a f(x)g(x) dx \right)^q \|g\|_{v,p}^{p-q} \\ & \geq \int_0^a \left( \int_0^x f(t) dt \right)^q v^{1-q/p}(x) d[-g^p(x)]. \end{aligned} \tag{29}$$

*For  $0 < p \leq 1, p \leq q$  the inequality in (29) is reversed.*

**Proof.** Applying integration by parts, Theorem 7 and Hölder inequality we have

$$\begin{aligned} & \left( \int_0^a f(x)g(x) \, dx \right)^q \left[ \left( \int_0^a g^p(x)v(x) \, dx \right)^{1/p} \right]^{p-q} \\ &= \left[ \left( \int_0^a \left( \int_0^x f(t) \, dt \right) d[-g(x)] \right)^p \right]^{q/p} \cdot \left( \int_0^a g^p(x)v(x) \, dx \right)^{1-(q/p)} \\ &\geq \left( \int_0^a \left( \int_0^x f(t) \, dt \right)^p d[-g^p(x)] \right)^{q/p} \cdot \left( \int_0^a V(x) d[-g^p(x)] \right)^{1-(q/p)} \\ &\geq \int_0^a \left( \int_0^x f(t) \, dt \right)^q V^{1-(q/p)}(x) d[-g^p(x)]. \end{aligned}$$

The proof for the case  $0 < p \leq 1$  is similar.  $\square$

The following theorem is integral analogue of the Theorem 1 (compare Theorem 3.2 in [2], Theorem 2.1 in [5] and [9]).

**Theorem 8.** Suppose that  $g \in L_v^p(0, a)$  is nonnegative and nonincreasing,  $p \geq 1$ ,  $q \leq p$ . Then

$$\|Kg\|_{\omega, q} \geq \lambda \|g\|_{v, p}, \quad (30)$$

where

$$\lambda = \inf_{0 < y < a} \frac{1}{V^{1/p}(y)} \left( \int_0^a \left( \int_0^y K(x, u) \, du \right)^q \omega(x) \, dx \right)^{1/q}. \quad (31)$$

The constant  $\lambda$  is the least possible. If  $0 < p \leq 1$ ,  $p \leq q$ , then the inequality in (30) is reversed with  $\lambda$  defined similarly with sup instead of inf.

**Proof.** Without loss of generality we can assume that  $g$  is absolutely continuous and that  $a$  is finite. Suppose also that  $g(a) = 0$ . Applying Proposition 2 with  $\|g\|_{v, p} = 1$  and Fubini theorem we have

$$\begin{aligned} \|Kg\|_{\omega, q}^q &= \int_0^a \left( \int_0^a K(x, y)g(y) \, dy \right)^q \omega(x) \, dx \\ &\geq \int_0^a \left[ \int_0^a \left( \int_0^y K(x, u) \, du \right)^q V^{1-(q/p)}(y) d[-g^p(y)] \right] \omega(x) \, dx \\ &= \int_0^a \left[ \int_0^a \left( \int_0^y K(x, u) \, du \right)^q \omega(x) \, dx \right] V^{1-(q/p)}(y) d[-g^p(y)] \\ &\geq \lambda^q \int_0^a V(y) d[-g^p(y)] = \lambda^q \int_0^a g^p(y)v(y) \, dy, \end{aligned}$$

which gives (30) for the case  $g(a) = 0$ . For the absolutely continuous nonincreasing function  $g$  such that  $g(a) > 0$  we define increasing sequence  $(g_n)$  of absolutely continuous nonincreasing functions by

$$g_n(x) = \begin{cases} g(x), & 0 < x < a - 1/n, \\ ng(a - 1/n)(a - x), & a - 1/n \leq x \leq a. \end{cases}$$

Applying proven case and using limiting procedure we obtain general assertion.

Sharpness of the constant can be obtained in standard way using characteristic function. Remaining case is similar.  $\square$

Using techniques from Theorem 8 and discrete case, one can easily prove the integral analogues of Theorems 2–4.

Although we can give applications of Theorem 8 for various kernels (fractional Riemann–Liouville  $K(x, y) = (x - y)^{\alpha-1}$ ,  $\alpha > 0$ ,  $y < x$ , extended Hilbert  $K(x, y) = (x + y)^{-\lambda}$ ,  $x, y \in \mathbf{R}$ ,  $\lambda > 0$ ; for Hardy kernel see [5]), in our opinion the most interesting one is transformation with Hilbert’s kernel  $K(x, y) = 1/(x + y)$  on weighted Lebesgue spaces, especially in view of the fact that the only bounded linear operator from  $L^p$  to  $L^q$  for  $0 < p < 1$ ,  $p < q \leq \infty$  is trivial one [3, p. 150] and interesting upper bound which appears in this case.

For given  $\beta \in \mathbf{R}$  we denote by  $L^p_\beta$  Lebesgue space with weight  $v(t) = t^\beta$ .

**Theorem 9.** *Let  $0 < p \leq 1$ ,  $p \leq q < \infty$ ,  $-1 < \beta$ ,  $0 < 1 + \alpha < q$ ,  $q/p = (1 + \alpha)/(1 + \beta)$  and let  $g$  be nonnegative decreasing function on  $(0, \infty)$  such that  $g \in L^p_\beta(0, \infty)$ . Then*

$$\|Kg\|_{\alpha,q} \leq (1 + \beta)^{1/p} \Gamma^{1/q}(1 + q) \zeta^{1/q}(q; \alpha) \|g\|_{\beta,p}, \tag{32}$$

where

$$\zeta(q; \alpha) = \sum_{n=0}^{\infty} \binom{\alpha + n + 1}{n} \frac{1}{(\alpha + n + 1)^{q+1}}. \tag{33}$$

For  $p \geq 1$ ,  $q \leq p$ , inequality (32) is reversed. There is equality in (32) if and only if  $g = A\chi_{[0,b]}$ , where  $A$  and  $b$  are nonnegative constants and  $\chi$  characteristic function.

**Proof.** Using Theorem 8 and simple transformations it is easy to see that

$$\begin{aligned} \lambda^q &= (\beta + 1)^{q/p} \int_0^\infty \frac{\log^q(1 + t)}{t^{\alpha+2}} dt \\ &= (\beta + 1)^{q/p} \int_0^\infty u^q e^{-(\alpha+1)u} (1 - e^{-u})^{-\alpha-2} du. \end{aligned}$$

Using binomial expansion of  $(1 - e^{-u})^{-\alpha-2}$  in power series in  $e^{-u}$  and integral representation of  $\Gamma$  function, inequality (32) follows.  $\square$

It is obvious that  $\zeta(q; 0) = \zeta(q)$ ,  $q > 1$ , where  $\zeta(q)$  is classical Riemann’s zeta function [11], and we see that Theorem 9 reduces for  $p = q > 1$ ,  $\alpha = \beta = 0$  to the Hilbert part of Corollary on page 97 in [1].

Appearing in Theorem 9 function  $\zeta(q; \alpha)$  as a generalization (in some sense very natural one) of classical Riemann’s zeta function, needs some attention. First recall that the Bernoulli numbers  $B_k^{(n)}$  of order  $n$ ,  $n = 0, 1, 2, \dots$  and degree  $k$ ,  $k = 0, 1, 2, \dots$  are defined by  $B_k^{(n)} = B_k^{(n)}(0)$ , where  $B_k^{(n)}(x)$  are the Bernoulli polynomials defined by expansion

$$\frac{t^n e^{xt}}{(e^t - 1)^n} = \sum_{k=0}^{\infty} B_k^{(n)}(x) \frac{t^k}{k!}. \tag{34}$$

In this context we find identity

$$B_k^{(n+1)}(x) = \frac{k!}{n!} \frac{d^{n-k}}{dx^{n-k}} [(x - 1)(x - 2) \cdots (x - n)], \quad 0 \leq k \leq n \tag{35}$$

more suitable. We also need recursion formula

$$(n + 1)B_{n+1-k}^{(n+2)} = kB_{n+1-k}^{(n+1)} - (n + 1)(n + 1 - k)B_{n-k}^{(n+1)}, \quad k \leq n. \tag{36}$$

Some properties of function  $\zeta(q; \alpha)$  are given in the following.

**Proposition 3.** *If  $0 < \alpha + 1 < q$ , then the function  $\zeta(q; \alpha)$  is well defined by (33). The following identities hold:*

$$\zeta(q; \alpha) = \frac{1}{\Gamma(q + 1)} \int_0^{\infty} \frac{\log^q(1 + t)}{t^{\alpha+2}} dt \tag{37}$$

$$(\alpha + 2)\zeta(q; \alpha + 1) = \zeta(q - 1; \alpha) - (\alpha + 1)\zeta(q; \alpha), \quad \alpha + 2 < q \tag{38}$$

$$\zeta(q; n) = \frac{1}{(n + 1)!} \sum_{k=0}^n \binom{n}{k} B_{n-k}^{(n+1)} \zeta(q - k), \tag{39}$$

$n + 1 < q, n = 0, 1, 2, \dots [6pt]$

**Proof.** Since the calculations and argumentations are elementary, we give just a sketch of the proof. That the function  $\zeta(q; \alpha)$  is well defined for  $0 < \alpha + 1 < q$  can be easily seen from the proof of Theorem 9 where also the first identity (integral representation) is contained. The second identity follows from the first using integration by parts twice (and trivial decomposition  $t^{-\alpha-2}(1 + t)^{-1} = t^{-\alpha-2} - t^{-\alpha-1}(1 + t)^{-1}$ ). The third identity follows from the second one using induction and recursion formula (36).  $\square$

## References

- [1] G. Bennett, Lower bounds for matrices, *Linear Algebra Appl.* 82 (1986) 81–98.
- [2] H. Heinig, L. Maligranda, Weighted inequalities for monotone and concave functions, *Studia Math.* 116 (2) (1995) 133–165.
- [3] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Math. 5, University of Campinas, Campinas SP, Brazil, 1989.
- [4] V.G. Maz'ja, *Sobolev Spaces*, Springer, Berlin, 1985.
- [5] E.A. Myasnikov, L.E. Persson, V.D. Stepanov, On the best constants in certain integral inequalities for monotone functions, *Acta Sci. Math. Szeged* 59 (1994) 613–624.
- [6] J. Pečarić, L.-E. Persson, On an inequality of Hardy–Littlewood–Pólya, *Math. Gazette* 79 (1995) 383–385.
- [7] P.E. Renaud, A reversed Hardy inequality, *Bull. Aust. Math. Soc.* 34 (1986) 225–232.
- [8] E. Sawyer, Boundedness of classical operators on classical Lorentz spaces, *Studia Math.* 96 (1990) 145–158.
- [9] V.D. Stepanov, Integral operators on the cone of monotone functions, *J. London Math. Soc.* 48 (3) (1994) 465–487.
- [10] V.D. Stepanov, The weighted Hardy's inequality for nonincreasing functions, *Trans. Amer. Math. Soc.* 338 (1993) 173–186.
- [11] E.C. Titchmarsh, *Theory of the Riemann Zeta-Function*, Oxford University Press, Oxford, 1951.