The Hahn–Banach theorem implies the Banach–Tarski paradox

by

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Abstract. Following an idea of Foreman and Wehrung, we prove the result announced in the title.

Recently Foreman and Wehrung [FW] have shown the existence of nonmeasurable sets from the Hahn–Banach theorem. Developing their idea, we show the Banach–Tarski duplication of the ball.

It suffices to prove the following lemma (see [W, §§1–3]).

LEMMA (ZF + Hahn–Banach). *If F, a free group of rank 2, acts freely on X (i.e., $xf \neq x$ for $f \neq e$, $x \in X$), then X is F-paradoxical.*

P r o o f. Let $\mathbf{B}$ be the free (boolean) product of the powerset algebras of all F-orbits. Find μ, a finitely additive measure on $\mathbf{B}$, $\mu(\mathbf{1}) = 1$ (by the Hahn–Banach theorem, [W, p. 214]). Fix $A_i \subseteq F$, $a_i \in F$, $i = 1, 2, 3, 4$, and a permutation σ of 1, 2, 3, 4, such that

(1) A_i's are pairwise disjoint,

(2) $a_i^{-1} A_i \cup A_{\sigma i} = F$, $i = 1, 2, 3, 4$.

(E.g. a_1 and a_2 are the free generators of F; $a_3 = a_1^{-1}$, $a_4 = a_2^{-1}$; $\sigma 1 = 3$, $\sigma 3 = 1$, $\sigma 2 = 4$, $\sigma 4 = 2$; A_i is the set of (reduced) words beginning with a_i.)

Define $X_i = \{x \in X: \mu\langle xA_i\rangle > \frac{1}{2}\}$. (Here, $\langle\ \rangle$ is the canonical embedding.) Let $Y_1 = X \setminus (X_1 a_1 \cup X_2 a_2)$, $Y_2 = X \setminus (X_3 a_3 \cup X_4 a_4)$. Clearly $X_1 a_1 \cup X_2 a_2 \cup Y_1 = X_3 a_3 \cup X_4 a_4 \cup Y_2 = X$. We show that Y_1, Y_2 and X_i's are pairwise disjoint. For X_i's this is due to (1). $Y_1 \cap Y_2 = \emptyset$ because $X = \bigcup_1^4 X_i a_i$. (Fix $x \in X$. $\langle xA_{\sigma i}\rangle$'s are pairwise disjoint by (1), so there is i with $\mu\langle xA_{\sigma i}\rangle < \frac{1}{2}$. By (2), $\langle xa_i^{-1} A_i\rangle \vee \langle xA_{\sigma i}\rangle = \mathbf{1}$, so $\mu\langle xa_i^{-1} A_i\rangle > \frac{1}{2}$, and $xa_i^{-1} \in X_i$, $x \in X_i a_i$.) Finally, $\bigcup_1^4 X_i \cap (Y_1 \cup Y_2) = \emptyset$ because $X_i \subseteq X_j a_j$ for $i \neq \sigma j$. (If $i \neq \sigma j$ then, by (1) and (2), $A_i \subseteq a_j^{-1} A_j$. So $x \in X_i$ yields $\mu\langle xA_i\rangle > \frac{1}{2}$, $\mu\langle xa_j^{-1} A_j\rangle > \frac{1}{2}$, $xa_j^{-1} \in X_j$.) ■

N o t e. 1. $\mu\mathbf{b} > \frac{1}{2}$, $\mu\mathbf{b} < \frac{1}{2}$ may be replaced by $\mathbf{b} \in \mathbf{D}$, $\mathbf{1} - \mathbf{b} \in \mathbf{D}$, provided $\mathbf{D} \subseteq \mathbf{B}$ is closed upwards and for any partition of $\mathbf{1}$ into nonzero $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$, at most one $\mathbf{b}_i \in \mathbf{D}$ and at least one $\mathbf{1} - \mathbf{b}_i \in \mathbf{D}$.

2. In fact, every $x \in X$ is in at least two sets $X_i a_i$ (because for at least two i,

$\mu\langle xA_{\sigma i}\rangle < \frac{1}{2}$, $\mu\langle xa_i^{-1}A_i\rangle > \frac{1}{2}$). So $2[X] \leqslant [X]$ in the semigroup of equidecomposability types (see [W, § 8]), i.e. X is F-paradoxical.

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References

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Ideals of the second category

by

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Abstract. We show that the intersection of less than h ideals of the second category is an ideal of the second category and there exists a family of d ideals of the second category which has an empty intersection.

Introduction. A family of infinite subsets of the set $N = \{0, 1, \ldots\}$ of all natural numbers is an *ideal* if it is closed under forming finite unions, taking infinite subsets and adding finite sets of natural numbers; we assume that the set of all natural numbers does not belong to an ideal.

If A and B are sets, then let $\langle A, B\rangle$ be the family of all infinite subsets of B containing A.

An ideal is of the *second category* if it is of the second category with respect to the topology on the set of all infinite sets of natural numbers generated by the sets $\langle x, N\setminus y\rangle$, where x and y are finite sets of natural numbers. This topology is called the *natural topology*.

The *Ellentuck topology* on the set of all infinite sets of natural numbers is generated by the sets $\langle x, V\rangle$, where x is a finite set of natural numbers and V is an infinite set of natural numbers. Let h denote the least cardinality among the cardinalities of families consisting of open and dense sets in the Ellentuck topology which have empty intersections. This definition of h is equivalent to that of Balcar and Simon [1], as shown in [4].

We prove that the intersection of less than h ideals of the second category is an ideal of the second category. This strengthens a result of Talagrand [6]. In Fremlin [3], p. 55, is was noticed that, in fact, Talagrand proved that the intersection of less than p ideals of the second category is an ideal of the second category, where p is a cardinal about which it is known that it is not greater than h (cf. Balcar and Simon [1]). There exists a model of ZFC in which p is less than h (see Dordal [2]).

It is known (see [5]) that the intersection of less than continuum many maximal ideals (they are always of the second category) is an ideal of the second category. We prove that without the assumption of maximality such a result cannot be proved in

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