HARMONIC SCHWARZ LEMMAS: CHEN. KALAJ–VUORINEN. PAVLOVIĆ. HEINZ

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Abstract. We put together several forms of Schwarz lemma for harmonic functions and add some remarks. I am grateful to David Kalaj and Matti Vuorinen, who pointed out to me Chen's paper [1].

Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$.

Theorem 1 (Heinz [2]). *If u is a real valued function harmonic in $\mathbb{D}$ such that $|u| < 1$ in $\mathbb{D}$ and $u(0) = 0$, then*

$$|u(z)| \leq \frac{4}{\pi} \arctan |z|, \quad \in \mathbb{D}. \tag{1}$$

Given a Banach space X, we can apply (1) to the functions Λu, $\Lambda \in X^\star$ (dual of X), where u is X-valued harmonic function, to extend Theorem 1 to vector valued functions.

A generalization of Heinz's theorem were proved in [4, Theorem 3.6.1]:

Theorem 2. *If $f : \mathbb{D} \mapsto \mathbb{D}$ is a harmonic function, then*

$$\Big| f(z) - \frac{1 - |z|^2}{1 + |z|^2} f(0) \Big| \leq \frac{4}{\pi} \arctan |z|. \tag{2}$$

The following variant of Schwarz lemma was proved by Kalaj and Vuorinen.

Theorem 3 ([3]). *Under the hypotheses of Theorem 1 (except $u(0) = 0$), we have*

$$|\nabla f(z)| \leq \frac{4}{\pi} \frac{1 - |u(z)|^2}{1 - |z|^2}. \tag{3}$$

There is another form, which is a consequence of the inequality

$$|\nabla u(0)| \leq 2u(0), \quad \text{where } u > 0 \text{ in } \mathbb{D}. \tag{4}$$

Theorem 4. *Under the hypotheses of Theorem 1, we have*

$$|\nabla u(z)| \leq 2 \frac{1 - |u(z)|}{1 - |z|^2}. \tag{5}$$

It is easy to verify that (3) and (5) are incomparable.

Both (3) and (5) are deduced from the case $z = 0$ by an application of this case to the function $u \circ \sigma_z$, where

$$\sigma_z(w) = \frac{z - w}{1 - \bar{z} w}. \tag{6}$$

I did not analyze Kalaj–Vuorinen's proof, but probably it gives the following improvement of Theorem 3.

Theorem 5 (Chen [1])**.** *Under the hypotheses of Theorem 3, we have*

$$|\nabla u(z)| \le \frac{4}{\pi}\frac{\cos\left(\frac{\pi}{2}u(z)\right)}{1-|z|^2}. \tag{7}$$

Let d_h denotes the hyperbolic metric on $\mathbb{D}$,

$$d_h(a,b) = \inf_\gamma \int_\gamma \frac{|dz|}{1-|z|^2}, \tag{8}$$

where the infimum is taken over all "admissible" curves joining a and b. We have

$$d_h(a,0) = \frac{1}{2}\log\frac{1+|a|}{1-|b|}, \tag{9}$$

and since d_h is Möbius invariant, we also have

$$d_h(a,b) = \frac{1}{2}\log\frac{1+|\sigma_a(b)|}{1-|\sigma_a(b)|}. \tag{10}$$

Writing (7) as

$$\frac{|du|}{\cos(\pi u/2)} \le \frac{4}{\pi}\frac{|dz|}{1-|z|^2}, \tag{11}$$

we obtain, after a careful computation,

$$d_h(\tan(\pi u(a)/4, \tan(\pi u(b)/4) \le d_h(a,b), \tag{12}$$

or, what is the same,

$$\left|\frac{\tan(\pi u(a)/4) - \tan(\pi u(b)/4)}{1-\tan(\pi u(a)/4)\tan(\pi u(b)/4)}\right| \le \left|\frac{a-b}{1-\bar{a}b}\right|, \quad a,b\in\mathbb{D}. \tag{13}$$

Inequality (12) can be expressed as:

Theorem 6. *Under the hypotheses of Theorem 3, the function*

$$K(z) = \tan\left(\frac{\pi}{4}u(z)\right) \tag{14}$$

is a contraction from $(\mathbb{D}, d_h)$ *into* $((-1,1), d_h)$.

Choose $b=0$ to obtain Theorem 1. Also, Chen's theorem can be easily deduced from (13).

Before presenting Chen's proof of Theorem 5 we also note the following:

- Inequality (7) improves (3) (which is implicit in [3]). Namely, we have

  $$1-\cos 2t = 2\sin^2(t) \ge 2\left(\frac{1/\sqrt{2}}{\pi/4}t\right)^2, \quad |t| < \pi/4.$$

  Now let $t = \pi x/4$ to obtain $1-\cos(\pi x/2) \ge x^2$.
- Also, it is easy to see that (7) improves (5).
- If

  $$u(z) = \frac{2}{\pi}\arctan\frac{2y}{1-x^2-y^2} \quad \text{(example from [4] [3])} \tag{15}$$

  then a straightforward but tedious computation (using polar coordinates) shows that equality in (7) holds for all z.

Proof of Theorem 3. Let $f = u+iv$, where v is harmonic conjugate to u, $v(0) = 0$. Then the function $\tan(\pi z/4)$ maps the strip $|\Re z| < 1$ onto $\mathbb{D}$. Hence, the holomorphic function

$$g(z) = \tan\left(\frac{\pi}{4} f(z)\right) \tag{16}$$

maps $\mathbb{D}$ into $\mathbb{D}$. By the classical lemma, we have

$$\frac{\pi}{4}\frac{1}{|\cos^2(\pi f(0)/4)|}|f'(0)| \le 1 - |\tan(\pi f(0)/4)|^2, \tag{17}$$

and hence, because $|f'(0)| = |\nabla u(0)|$,

$$\frac{\pi}{4}|\nabla u(0)| \le |\cos w|^2 - |\sin w|^2, \quad \text{where } w = \pi f(0)/4\ . \tag{18}$$

Taking $w = x + iy$, we get

$$\begin{aligned} 4|\cos w|^2 - 4|\sin w|^2 &= e^{2y} + e^{-2y} + 2\Re\cos(2x) \\ &\quad - \left(e^{2y} + e^{-2y} - 2\Re\cos(2x)\right) \\ &= 4\cos(2x) = 4\cos(\pi u(0)/2). \end{aligned}$$

It follows that

$$|\nabla u(0)| \le \frac{4}{\pi}\cos\left(\frac{\pi}{2}u(0)\right). \tag{19}$$

The desired inequality now follows by considering the functions $u \circ \sigma_z$, $z \in \mathbb{D}$. □

Remark 7. The reader can prove the following. Let $f : \mathbb{D} \mapsto X$ be a harmonic function (X is a Banach space), then the function

$$K(z) = \tan\left(\frac{\pi}{2}\|f(z)\|\right)$$

is a contraction from $(\mathbb{D}, d_h)$ into $[0, 1), d_h)$.

References

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