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Stochastic integral representations of second quantization operators

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Abstract

We give a necessary and sufficient condition for the second quantization operator $\Gamma(h)$ of a bounded operator h on $L^2(\mathbb{R}_+)$, or for its differential second quantization operator $\lambda(h)$, to have a representation as a quantum stochastic integral. This condition is exactly that h writes as the sum of a Hilbert–Schmidt operator and a multiplication operator. We then explore several extensions of this result. We also examine the famous counterexample due to Journé and Meyer and explain its representability defect.

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0. Introduction

Second quantization operators and differential second quantization operators on Fock spaces are the most basic operators that appear in the quantum theory of fields, after creation and annihilation operators. On the other hand, on Fock spaces of the form $\Phi_{\mathscr{K}} = \Gamma(L^2(\mathbb{R}_+, \mathscr{K}))$, where $\mathscr{K}$ is a separable Hilbert space, an effective theory of quantum stochastic integration is now well developed and has found numerous applications (such as the ergodic properties of dissipative quantum systems). One of the basic questions in that context is to characterize the operators

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on $\Phi_{\mathscr{K}}$ which can be represented as a quantum stochastic integral. Many articles have been devoted to that problem (see for example [At1,Coq,P-S]), yet it is far from being closed.

We study here two particular families of operators: the second quantization operators $\Gamma(h)$ and the differential second quantization operators $\lambda(h)$, for a bounded operator h on $L^2(\mathbb{R}_+, \mathscr{K})$. Rather surprisingly we find a necessary and sufficient condition for $\Gamma(h)$ and $\Gamma(h^*)$ to be represented as quantum stochastic integrals on the set of exponential vectors or on the set of finite particle vectors:

Theorem 1. *Let h be a bounded operator on $L^2(\mathbb{R}_+)$. The following properties are equivalent*:

1. *$\Gamma(h)$ and $\Gamma(h^*)$ have a quantum stochastic integral representation on the set $\mathscr{E}(L^2(\mathbb{R}_+))$ of exponential vectors,*
2. *$\Gamma(h)$ and $\Gamma(h^*)$ have a quantum stochastic integral representation on the set $\mathscr{F}$ of finite particle vectors,*
3. *h is of the form*

$$h = K + \mathscr{M}_\gamma,$$

where K is a Hilbert–Schmidt operator and $\mathscr{M}_\gamma$ is a multiplication by an essentially bounded function γ.

Surprisingly also, the exact same theorem holds for differential second quantizations.

Theorem 2. *Let h be a bounded operator on $L^2(\mathbb{R}_+)$. The following properties are equivalent*:

1. *$\lambda(h)$ and $\lambda(h^*)$ have a quantum stochastic integral representation on the set $\mathscr{E}(L^2(\mathbb{R}_+))$ of exponential vectors,*
2. *$\lambda(h)$ and $\lambda(h^*)$ have a quantum stochastic integral representation on the set $\mathscr{F}$ of finite particle vectors,*
3. *h is of the form*

$$h = K + \mathscr{M}\gamma$$

where K is a Hilbert–Schmidt operator and $\mathscr{M}\gamma$ is a multiplication by an essentially bounded function γ.

We furthermore derive fully explicit formulas for the integrands in the integral representation in both cases (differential and nondifferential); we give sufficient conditions for the existence of integral representations of such operators in the case of unbounded operators h. We also prove various results concerning the obtained representations.

The paper is organized as follows: in Section 1 we give all the necessary theoretical background and notations. Section 2 is the core of this article: the above quoted theorem for (non-differential) second quantizations is proved. Characterizations of the fact that $\Gamma(h)$ defines a regular semimartingale (see [At1]) are also given and the counterexample of Journé and Meyer is discussed. In Section 3 we prove the result for differential second quantization operators. Section 4 handles the extension of our characterizations and formulas to the case of Fock space of higher (possibly infinite) multiplicity. In Section 5 we treat the case of second quantization and differential second quantizations of *unbounded* operators, which often arise in physical applications of the theory. In that case we give simple sufficient conditions for the existence of a quantum stochastic integral representation.

1. Preliminaries

1.1. Quantum stochastic calculus

The usual framework of quantum stochastic calculus is the symmetric Fock space $\Gamma_{\mathrm{sym}}(L^2(\mathbb{R}_+))$ over the space $L^2(\mathbb{R}_+)$, that is, the completion of

$$\bigoplus_{n\geqslant 0} L^2(\mathbb{R}_+)^{\circ n},$$

where $L^2(\mathbb{R}_+)^{\circ n}$ denotes the n-fold symmetric tensor product of $L^2(\mathbb{R}_+)$ for $n\geqslant 1$ and $L^2(\mathbb{R}_+)^{\circ 0} = \mathbb{C}$ by convention.

We denote by Φ that space; according to the above definition, the elements of Φ are sequences $(f_0, f_1, \dots)$, where $f_0 \in \mathbb{C}$, f_n is a symmetric function on $\mathbb{R}_+^n$ for $n\geqslant 1$, and

$$|f_0|^2 + \sum_{n\geqslant 1} \int_{s_1<\dots<s_n} |f_n(s_1, \dots, s_n)|^2\, ds_1 \dots ds_n < +\infty.$$

We will use the more concise notation of Guichardet [Gui]; we identify sequences $(f_0, f_1, \dots)$ with functions f on $\mathscr{P}$, $\mathscr{P}$ being the set of finite subsets of $\mathbb{R}_+$. Such a function belongs to Φ if

$$|f(\emptyset)|^2 + \sum_{n\geqslant 1} \int_{s_1<\dots<s_n} |f(\{s_1, \dots, s_n\})|^2\, ds_1 \dots ds_n < +\infty.$$

On $\mathscr{P}$ we can define a measure simply by taking its restriction to the subset of $\mathscr{P}$ that has n-tuples for elements to be equal to the n-dimensional Lebesgue measure and taking the empty set to be an atom of mass one. Equipped with such a measure, $L^2(\mathscr{P})$ can be seen to be isomorphic to the space $\Gamma_{\mathrm{sym}}(L^2(\mathbb{R}_+))$. From now on we will always work with this realization of Fock space and write indifferently Φ or $L^2(\mathscr{P})$. The canonical element in $\mathscr{P}$ will be denoted by σ and the infinitesimal element by $d\sigma$. We will also follow the convention that subsets $\{s_1, \dots, s_n\}$ are always written in ordered form: the s_i's are always assumed to satisfy $s_1 < \cdots < s_n$ unless otherwise stated.

Among vectors of $L^2(\mathscr{P})$ we denote by Ω the vacuum vector, which is the indicator of the empty set. For any t in $\mathbb{R}_+$ we will denote by Φ_t or by $L^2(\mathscr{P}_t)$ the subspace of $L^2(\mathscr{P})$ made of functions with support in $[0, t]$: for a function f in $L^2(\mathscr{P})$,

$$f \text{ belongs to } L^2(\mathscr{P}_t) \;\Leftrightarrow\; f(\sigma) = 0 \quad \text{for a.a. } \sigma \text{ such that } \sigma \not\subset [0, t].$$

A vector f of Φ which belongs to $L^2(\mathscr{P}_t)$ is said to be *t-adapted*. For any n in $\mathbb{N}$ we also define the nth chaos as the subspace of Φ made of functions with support in the n-tuples: for a function f in $L^2(\mathscr{P})$,

$$f \text{ belongs to the } n\text{th chaos} \;\Leftrightarrow\; f(\sigma) = 0 \quad \text{if } card\, \sigma \neq n.$$

The nth chaos is therefore equal to the closed subspace generated by all vectors of the form $u_1 \circ \cdots \circ u_n$ with $u_1, \dots, u_n$ in $L^2(\mathbb{R}_+)$.

In $L^2(\mathscr{P})$ we will often consider three most important subspaces: the first is the exponential domain $\mathscr{E}(L^2(\mathbb{R}_+))$, the second is the subspace $\mathscr{J}(L^2(\mathbb{R}_+))$ introduced by Coquio [Coq], the last is the finite particle vector $\mathscr{F}$.

To any function u in $L^2(\mathbb{R}_+)$ we then associate a function $\mathscr{E}(u)$ on $\mathscr{P}$ by

$$\mathscr{E}(u)(\emptyset) = 1,$$

$$\mathscr{E}(u)(\{s_1, \dots, s_n\}) = u(s_1) \dots u(s_n).$$

This $\mathscr{E}(u)$ is an element of $L^2(\mathscr{P})$ as one can see from the equality

$$\int_{s_1 < \cdots < s_n} |u(s_1) \dots u(s_n)|^2 \, ds_1 \dots ds_n = \frac{1}{n!} \int_{\mathbb{R}_+} |u(s)|^2 \, ds$$

which implies

$$||\mathscr{E}(u)||^2 = e^{||u||^2}.$$

The set $\mathscr{E}(L^2(\mathbb{R}_+))$ is total in $L^2(\mathscr{P})$ (see [Mey]).

The set $\mathscr{J}(L^2(\mathbb{R}_+))$ is the subspace of $L^2(\mathscr{P})$ generated by the vacuum vector and by the vectors $j(v, u)$ defined for any u, v in $L^2(\mathbb{R}_+)$ by

$$j(v, u)(\emptyset) = 0,$$

$$j(v, u)(\{s_1, \dots, s_n\}) = v(s_n) u(s_1) \dots u(s_{n-1}).$$

From the relation $\mathscr{E}(u) = \Omega + j(u,u)$ one sees that $\mathscr{J}(L^2(\mathbb{R}_+))$ contains the linear span of $\mathscr{E}(L^2(\mathbb{R}_+))$ so that $\mathscr{J}(L^2(\mathbb{R}_+))$ is dense.

The finite particle domain $\mathscr{F}$ is defined as the algebraic sum of nth chaoses for n in $\mathbb{N}$. Thus for a vector f in $L^2(\mathscr{P})$,

$$f \in \mathscr{F} \quad \Leftrightarrow \quad \exists N \in \mathbb{N} \quad \text{s.t. } f(\sigma) = 0 \quad \text{if } \mathit{card}\,\sigma \geqslant N.$$

It is clear from the definition of Φ that $\mathscr{F}$ is a dense subset as well and that it is generated by Ω and all vectors of the form $u_1 \circ \cdots \circ u_n$.

1.1.1. Abstract Ito calculus

For details on all objects defined in this paragraph, see [A-M] or [At3]. Let us consider for all t the element χ_t of $L^2(\mathscr{P})$ defined by

$$\chi_t(\sigma) = \begin{cases} 1 & \text{if } \sigma = \{s\} \text{ and } s < t, \\ 0 & \text{otherwise.} \end{cases}$$

A family $(f_t)_{t \geqslant 0}$ of elements of $L^2(\mathscr{P})$ is called an *adapted process* if for almost all t, the function f_t is t-adapted, that is, f_t belongs to $L^2(\mathscr{P}_t)$. For an adapted process $(f_t)_{t \geqslant 0}$ satisfying $\int ||f_t||^2 \, dt < +\infty$, one can define the integral of $(f_t)_{t \geqslant 0}$ with respect to the curve $(\chi_t)_{t \geqslant 0}$. This defines an element of $L^2(\mathscr{P})$, denoted

$$\int_0^\infty f_t \, d\chi_t$$

which we call the *Ito integral* of $(f_t)_{t \geqslant 0}$. It has square norm

$$\left|\left| \int_0^\infty f_t \, d\chi_t \right|\right|^2 = \int_0^\infty ||f_t||^2 \, dt. \tag{1.1}$$

That integral can be constructed from the case of step processes, extending the definition by the isometry formula (1.1). One can check that, as a function on $\mathscr{P}$, the integral $\int_0^\infty f_t \, d\chi_t$ is given by

$$\int_0^\infty f_t \, d\chi_t(\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset, \\ f_{\vee\sigma}(\sigma \backslash \vee\sigma) & \text{otherwise,} \end{cases} \tag{1.2}$$

where $\vee\sigma$ represents the largest element in σ. We define integrals $\int_a^b f_t \, d\chi_t$ to be the integral of the process $(f_t \, \mathbb{1}_{[a,b]}(t))_{t \geqslant 0}$.

Apart from this integral, the two main tools of abstract Ito calculus are the following families of operators: the adapted projections $(P_t)_{t \geqslant 0}$ and adapted gradients $(D_t)_{t \geqslant 0}$, which we now define.

For any f in Φ, the image of f by these operators is defined by

$$P_t f(\sigma) = \begin{cases} f(\sigma) & \text{if } \sigma \subset [0,t], \\ 0 & \text{otherwise,} \end{cases}$$

$$D_t f(\sigma) = \begin{cases} f(\sigma \cup \{t\}) & \text{if } \sigma \subset [0,t], \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that for all f in $L^2(\mathscr{P})$, all t in $\mathbb{R}_+$, $P_t f$ is a well-defined element of $L^2(\mathscr{P})$; the question of definiteness of D_t will be addressed a little further. What is clear is that, as soon as $P_t f$, $D_t f$ are defined, they belong to $L^2(\mathscr{P}_t)$: in particular the operator P_t is simply the orthogonal projection on the subspace $L^2(\mathscr{P}_t)$. Thus the families $(P_t f)_{t\geqslant 0}$ and $(D_t f)_{t\geqslant 0}$ are adapted; this allows us to consider the integral $\int_0^\infty D_t f \, d\chi_t$. One deduces from the definition of integral (1.2) the following equality, valid for all f in Φ:

$$f = f(\emptyset)\Omega + \int_0^\infty D_t f \, d\chi_t \tag{1.3}$$

together with the isometry formula

$$||f||^2 = |f(\emptyset)|^2 + \int_0^\infty ||D_t f||^2 \, dt. \tag{1.4}$$

Equality (1.3) is known as the previsible representation of f.

In the definition of operators D_t we have omitted to discuss the question of definiteness; now formula (1.4) shows that for all f in Φ, almost all t in $\mathbb{R}_+$, the function $D_t f$ is a well-defined element of Φ. This cannot be improved, so that an individual D_t is not defined as an operator.

On vectors of $\mathscr{E}(L^2(\mathbb{R}_+))$ or $\mathscr{J}(L^2(\mathbb{R}_+))$ the operators P_t or D_t satisfy

$$\begin{aligned} P_t\mathscr{E}(u) &= \mathscr{E}(p_t u), & D_t\mathscr{E}(u) &= u(t)\mathscr{E}(p_t u), \\ P_t j(v,u) &= j(p_t v, p_t u), & D_t j(v,u) &= v(t)\mathscr{E}(p_t u), \end{aligned}$$

where p_t is the operator of multiplication by the indicator function of $[0,t]$ on $L^2(\mathbb{R}_+)$.

This gives, in particular, the previsible representations of exponential vectors and vectors $j(v,u)$:

$$\mathscr{E}(u) = \Omega + \int_0^\infty u(t)\mathscr{E}(p_t u) \, d\chi_t$$

and

$$j(v,u) = \int_0^\infty v(t)\mathscr{E}(p_t u)\, d\chi_t.$$

1.1.2. Second quantization operators

For any bounded operator h on $L^2(\mathbb{R}_+)$ we define operators $\Gamma(h)$, $\lambda(h)$ on vectors of Φ of the form $u_1 \circ \cdots \circ u_n$ with $u_1, \ldots, u_n$ in $L^2(\mathbb{R}_+)$ by

$$\Gamma(h)(u_1 \circ \cdots \circ u_n) = hu_1 \circ \cdots \circ hu_n, \tag{1.5}$$

$$\lambda(h)(u_1 \circ \cdots \circ u_n) = hu_1 \circ u_2 \circ \cdots \circ u_n + \cdots + u_1 \circ \cdots \circ u_{n-1} \circ hu_n. \tag{1.6}$$

These operators $\Gamma(h)$ and $\lambda(h)$ are then extended by linearity and closure; one can check that their domain is the set of all f in $L^2(\mathscr{P})$ such that, denoting f_n the restriction of f to the nth chaos, we have

$$\sum_n ||\Gamma(h)f_n||^2 < +\infty$$

or, respectively,

$$\sum_n ||\lambda(h)f_n||^2 < +\infty.$$

The action of these operators on exponential vectors is easily expressed:

$$\Gamma(h)\mathscr{E}(u) = \mathscr{E}(hu),$$

$$\lambda(h)\mathscr{E}(u) = a^+_{hu}\mathscr{E}(u).$$

These two types of operators are linked by the following formula, which can be taken as the definition of differential second quantization operators, and explains the term differential: for any bounded h on $L^2(\mathbb{R}_+)$, any t in $\mathbb{R}$, one has

$$\Gamma(e^{ith}) = e^{it\lambda(h)}.$$

In the case of a self-adjoint operator h this means that $\lambda(h)$ is the (uniquely defined) generator of the unitary semigroup obtained by second quantization of the unitary semigroup with generator h.

We will mention in Section 5 second quantizations of unbounded operators h. The definition for such operators is easily adapted from the above formulas (1.5) and (1.6).

1.1.3. Quantum stochastic integration

We will now define integrals of operator processes with respect to the three quantum noises da_t^+, da_t^-, da_t°.

We consider quantum stochastic integrals as defined by Attal and Meyer [A-M]. To describe it we first need to define adapted processes of operators:

Definition 1.1. An adapted process of operators on Φ is a family $(H_s)_{s\geqslant 0}$ of operators on Φ such that, for almost all s, the operator H_s is s-adapted, that is

- *Dom* H_s is stable by the operators P_s and D_u for a.a. $u\geqslant s$;
- for any f in *Dom* H_s the following equalities hold:

$$H_s P_s f = P_s H_s f,$$
$$H_s D_u f = D_u H_s f \quad \text{for a.a. } u\geqslant s.$$

Now, for three adapted families $(H_s^+)_{s\geqslant 0}$, $(H_s^-)_{s\geqslant 0}$, $(H_s^\circ)_{s\geqslant 0}$ of operators on Φ and a scalar λ, the integral

$$\lambda\, Id + \int_0^\infty H_s^+\, da_s^+ + \int_0^\infty H_s^-\, da_s^- + \int_0^\infty H_s^\circ\, da_s^\circ$$

is defined as the only operator H that satisfies the following equality:

$$Hf = \lambda f(\emptyset) + \int_0^\infty H_s D_s f\, d\chi_s + \int_0^\infty H_s^+ P_s f\, d\chi_s + \int_0^\infty H_s^- D_s f\, ds + \int_0^\infty H_s^\circ D_s f\, d\chi_s, \tag{1.7}$$

where H_s is $P_s H P_s$. The domain of H is then the set of all vectors f in Φ such that Eq. (1.7) is meaningful, that is:

- for almost all s in $\mathbb{R}_+$, $D_s f$ belongs to *Dom* H_s, *Dom* H_s^-, *Dom* H_s° and $P_s f$ belongs to *Dom* H_s^+.
- equality (1.7) holds (remark that H appears on both sides of the equation).

We define integrals $\int_a^b H_s^\varepsilon\, da_s^\varepsilon$ to be the integral of the process $(H_s^\varepsilon\, \mathbb{1}_{[a,b]}(s))_{s\geqslant 0}$. Then it can be seen that an operator H_t is recovered as the restriction to Φ_t of

$$\lambda\, Id + \int_0^t H_s^+\, da_s^+ + \int_0^t H_s^-\, da_s^- + \int_0^t H_s^\circ\, da_s^\circ.$$

In the theory of quantum stochastic integration a fourth type of integrals is usually considered: integrals $\int_0^\infty H_s^\times\, da_s^\times$ with respect to the noise $da_s^\times$. The noise $da_s^\times$ is actually equal to ds and that new integral is simply the strong integral $\int_0^\infty H_s^\times da_s^\times$.

Therefore, there is no interest in representing a *single* operator H as

$$\lambda\, Id + \int_0^\infty H_s^+ \, da_s^+ + \int_0^\infty H_s^- \, da_s^- + \int_0^\infty H_s^\circ \, da_s^\circ + \int_0^\infty H_s^\times \, da_s^\times.$$

Note that the picture is different if one looks for representation of processes of operators, in which case the introduction of the time integral is in general necessary.

In this paper we are interested in representing operators individually, so that we will consider representations as integrals

$$\lambda\, Id + \int_0^\infty H_s^+ \, da_s^+ + \int_0^\infty H_s^- \, da_s^- + \int_0^\infty H_s^\circ \, da_s^\circ$$

only. What's more, we will always look for representations in which, for a.a. s, the operators H_s^+, H_s^-, H_s° are closable; otherwise the unicity of the representation does not hold (see [At2]).

1.1.4. The Journé–Meyer counterexample

The question of whether all operators on Fock space are representable as quantum stochastic integrals assuming simple domain assumptions was given a negative answer by Journé and Meyer [J-M]: their counterexample consists of a bounded operator on $L^2(\mathscr{P})$ which is not representable on the whole of the exponential domain. The reason why we include this counterexample here is that the considered operator is a second quantization: Journé and Meyer consider the second quantization operator $\Gamma(h)$ where h is the Hilbert transform on $L^2(\mathbb{R}_+)$ (more precisely, to a function f in $L^2(\mathbb{R}_+)$ it associates the restriction to $\mathbb{R}_+$ of the Hilbert transform of f seen as an element of $L^2(\mathbb{R})$). Since the Hilbert transform is a unitary operator, the operator h is a contraction of $L^2(\mathbb{R}_+)$ so that the associated $\Gamma(h)$ is a bounded operator; yet this operator $\Gamma(h)$ is not representable on the whole of $\mathscr{E}(L^2(\mathbb{R}_+))$—not even on the subset $\mathscr{E}(L^2 \cap L^\infty(\mathbb{R}_+))$. Indeed Journé and Meyer prove that, if some u in $L^2 \cap L^\infty(\mathbb{R}_+)$ is such that $\mathscr{E}(u)$ is in the domain of some integral operator H, then $t \mapsto P_t H \mathscr{E}(u_t)$ has finite quadratic variation. Then they construct explicitly some u in $L^2 \cap L^\infty(\mathbb{R}_+)$ for which $t \mapsto P_t H \mathscr{E}(u_t)$ does not have finite quadratic variation.

1.1.5. The case of Fock space of higher multiplicity

We present here briefly the counterpart of the above definitions in the case of Fock space of higher multiplicity.

For that consider some separable Hilbert space $\mathscr{K}$ and fix a hilbertian basis $(e_i)_{i \in \mathscr{I}}$, where we assume for notational convenience that $\mathscr{I}$ does not contain the index zero. The Fock space with multiplicity space $\mathscr{K}$, which we denote by $\Phi_{\mathscr{K}}$, is defined as the completion of

$$\bigoplus_{n \geqslant 0} L^2(\mathbb{R}_+, \mathscr{K})^{\circ n}$$

as before, but again Guichardet's shorthand notation gives a more concise form and we describe it now without details. The Fock space $\Phi_{\mathscr{K}}$ is identified with $L^2(\mathscr{P}_{\mathscr{I}})$ where elements of $\mathscr{P}_{\mathscr{I}}$ are finite subsets of $\mathscr{P} \times \mathscr{I}$. An element of $\Phi_{\mathscr{K}}$ is now a function with variable σ of the form

$$\{(s_1, i_1), \dots, (s_n, i_n)\},$$

where $n \in \mathbb{N}$, the s_k's belong to $\mathbb{R}_+$ and the i_k's belong to $\mathscr{I}$. As before we always write such a σ so that $s_1 < \cdots < s_n$. The integrability condition for a function f on $\mathscr{P}_{\mathscr{I}}$ to belong to $\Phi_{\mathscr{K}}$ is now

$$\sum_n \sum_{i_1, \dots, i_n \in \mathscr{I}} \int_{s_1 < \cdots < s_n} |f(\{(s_1, i_1), \dots, (s_n, i_n)\})|^2 \, ds_1 \dots ds_n < +\infty.$$

The exponential vectors $\mathscr{E}(u)$, vectors $j(v,u)$ and finite particle vectors are defined in ways analogous to the case of simple Fock space Φ: exponential vectors are now constructed with respect to functions u in $L^2(\mathbb{R}_+, \mathscr{K})$ by

$$\mathscr{E}(u)(\{(s_1, i_1), \dots, (s_n, i_n)\}) = u(s_1, i_1) \cdots u(s_n, i_n),$$

the set $\mathscr{J}(L^2(\mathbb{R}_+, \mathbb{K}))$ of vectors $j(v,u)$ for v, u in $L^2(\mathbb{R}_+, \mathscr{K})$ is defined by

$$j(v,u) = \Omega + \sum_{i \in \mathscr{I}} \int_0^\infty v(s,i) \mathscr{E}(u_s) \, d\chi_s^i,$$

and the finite particle domain is generated by Ω and vectors of the form

$$u_1 \circ \cdots \circ u_n$$

for $u_1, \dots, u_n$ in $L^2(\mathbb{R}_+, \mathscr{K})$.

The abstract Ito calculus uses now a set of curves χ_t^i and operators D_t^i for $i \in \mathscr{I}$. For any f in $\Phi_{\mathscr{K}}$, we define $D_t^i f$ and $P_t f$ by

$$P_t f(\sigma) = \begin{cases} f(\sigma) & \text{if } \sigma = \{(s_1, i_1), \dots, (s_n, i_n)\} \text{ with all } s_i \text{ in } [0,t], \\ 0 & \text{otherwise,} \end{cases}$$

$$D_t^i f(\sigma) = \begin{cases} f(\sigma \cup (t,i)) & \text{if } \sigma = \{(s_1, i_1), \dots, (s_n, i_n)\} \text{ with all } s_i \text{ in } [0,t], \\ 0 & \text{otherwise.} \end{cases}$$

To define an abstract Ito integral we consider again a family of elements $(\chi_t^i)_{t \geqslant 0}$ of $\Phi_{\mathscr{K}}$. For an adapted process $(f_t)_{t \geqslant 0}$ in $\Phi_{\mathscr{K}}$ an integral $\int_0^\infty f_t \, d\chi_t^i$ is defined for each i in $\mathscr{I}$ by

$$\int_0^\infty f_t \, d\chi_t^i(\sigma) = \begin{cases} f_{s_n}(\sigma \backslash (s_n, i_n)) & \text{if } \sigma = \{(s_1, i_1), \dots, (s_n, i_n)\} \text{ with } i_n = i, \\ 0 & \text{otherwise} \end{cases}$$

and all integrals are defined as zero on the null set. To unify our notations we will denote by $\int f_t\, d\chi_t^0$ the time integral $\int f_t\, dt$, and by D_t^0 the projection P_t.

Now any vector f of $\Phi_{\mathscr{K}}$ has a previsible representation of the form

$$f = f(\emptyset) + \sum_{i \in \mathscr{I}} \int_0^\infty D_t^i f\, d\chi_t^i$$

with associated isometry formula

$$||f||^2 = |f(\emptyset)|^2 + \sum_{i \in \mathscr{I}} \int_0^\infty ||D_t^i f||^2\, dt;$$

second quantization and differential second quantization operators are defined as before, but now for operators h on $L^2(\mathbb{R}_+, \mathscr{K})$.

Stochastic integrals of operators are now to be considered with respect to a set of quantum noises $da_t^{i,j}$ for i,j both in $\mathscr{I} \cup \{0\}$. The noise $da_t^{0,0}$ will represent the time differential $dt\, Id$; $da_t^{0,i}$, $da_t^{i,0}$ and $da_t^{i,i}$ represent, respectively, the creation, annihilation and conservation operators at site $i \in \mathscr{I}$, the remaining ones $da_t^{i,j}$ for $i \neq j$ representing *exchange* operators.

Now an integral

$$\lambda\, Id + \sum_{i,j \in \mathscr{I} \cup \{0\}} \int_0^\infty H_s^{i,j}\, da_s^{i,j}$$

is defined as the operator H that satisfies

$$Hf = \lambda f(\emptyset) + \sum_{i \in \mathscr{I}} \int_0^\infty H_s D_s^i f\, d\chi_s^i + \sum_{i,j \in \mathscr{I} \cup \{0\}} \int_0^\infty H_s^{i,j} D_s^j f\, d\chi_s^i.$$

This definition meets the one for Fock space of multiplicity one if $\mathscr{I}$ is made of a single element.

Note that, as before, we will be interested in representing operators as sums of integrals that exclude integration with respect to time: only the noises $da_s^{i,j}$ for $(i,j) \neq (0,0)$ will be considered in quantum stochastic integral representations.

2. Second quantization operators

2.1. A representation theorem

The main theorem to be proved in this section is the characterization of operators h on $L^2(\mathbb{R}_+)$ such that the operators $\Gamma(h)$, $\Gamma(h^*)$ are representable on the exponential domain or on the finite particle domain.

The explicit formulas for the integrands to appear in the representation are given along the proof, in (2.12) and (2.13).

Theorem 2.1. *Let h be a bounded operator on* $L^2(\mathbb{R}_+)$. *The following properties are equivalent*:

1. $\Gamma(h)$ *and* $\Gamma(h^*)$ *have a stochastic integral representation on the set* $\mathscr{E}(L^2(\mathbb{R}_+))$ *of exponential vectors*,
2. $\Gamma(h)$ *and* $\Gamma(h^*)$ *have a stochastic integral representation on the set* $\mathscr{F}$ *of finite particle vectors*,
3. *h is of the form*

$$h = K + \mathscr{M}_\gamma,$$

where K is a Hilbert–Schmidt operator and $\mathscr{M}_\gamma$ *is a multiplication by an essentially bounded function* γ.

Besides, *if one of these holds*, *then the representation holds on the set* $\mathscr{J}(L^2(\mathbb{R}_+))$.

First we will prove the equivalence of 1 and 3; the proof of the equivalence of 2 and 3 will then appear as a simplication.

The proof will be decomposed as follows:

- we prove in Proposition 2.2 below that 1 implies a seemingly weaker set of conditions (C),
- we prove the equivalence of (C) and 3 in Lemma 2.6,
- we prove implication $3 \Rightarrow 1$,
- we extend a representation to the domain $\mathscr{J}(L^2(\mathbb{R}_+))$.

We recall that we denote by p_t the operator on $L^2(\mathbb{R}_+)$ of multiplication by the indicator function of $[0,t]$. We will also denote by $\mathbb{1}_{[a,b]}$ the indicator function of $[a,b] \subset \mathbb{R}_+$ and by $\mathbb{1}_{t]}$ the indicator of $[0,t]$.

Proposition 2.2. *If* $\Gamma(h)$ *and* $\Gamma(h^*)$ *have a stochastic integral representation on the set* $\mathscr{E}(L^2(\mathbb{R}_+))$ *then h satisfies the following set of conditions* (C):

$$(\mathrm{C}) \quad \begin{cases} \bullet \ \textit{for a.a. } t, p_t h \mathbb{1}_{[t,t+\varepsilon]}/\varepsilon \textit{ converges in } L^2(\mathbb{R}_+) \textit{ to a function } \alpha_t \textit{ as } \varepsilon \to 0, \\ \bullet \ \textit{for a.a. } t, p_t h^* \mathbb{1}_{[t,t+\varepsilon]}/\varepsilon \textit{ converges in } L^2(\mathbb{R}_+) \textit{ to a function } \beta_t \textit{ as } \varepsilon \to 0, \\ \bullet \ \textit{the functions } t \mapsto ||\alpha_t||_{L^2(\mathbb{R}_+)}, \ t \mapsto ||\beta_t||_{L^2(\mathbb{R}_+)} \textit{ are square-integrable}, \\ \bullet \ \textit{for a.a. } t, \frac{1}{\varepsilon}\int_t^{t+\varepsilon} h \mathbb{1}_{[t,t+\varepsilon]}(s)\, ds \textit{ converges to a scalar } \gamma(t) \textit{ as } \varepsilon \to 0 \textit{ and} \\ \bullet \ \textit{the function } t \mapsto \gamma(t) \textit{ is essentially bounded}. \end{cases}$$

Proof. We suppose here that the equality

$$\Gamma(h) = \lambda\, Id + \int_0^\infty H_s^+\, da_s^+ + \int_0^\infty H_s^+\, da_s^- + \int_0^\infty H_s^\circ\, da_s^\circ, \tag{2.1}$$

holds on $\mathscr{E}(L^2(\mathbb{R}_+))$. We also assume that $\Gamma(h^*)$ has such a representation.

First let us remark that $\Gamma(h)\Omega = \Omega$; then equality (2.1) implies

$$\Omega = \lambda\Omega + \int_0^\infty H_s^+\Omega\, d\chi_s$$

so that $\lambda = 1$ (and $H_s^+\Omega = 0$ for almost all s, but we do not need this).

Now we compute $P_t\Gamma(h)\mathscr{E}(p_{t+\varepsilon}u)$ and $P_t\Gamma(h)\mathscr{E}(p_t u)$ for some function u in $L^2(\mathbb{R}_+)$:

$$\begin{aligned}\Gamma(h)\mathscr{E}(p_{t+\varepsilon}u) = \Omega &+ \int_0^{t+\varepsilon} u(s)H_s\mathscr{E}(p_s u)\, d\chi_s + \int_0^{+\infty} H_s^+\mathscr{E}(p_{s\wedge(t+\varepsilon)}u)\, d\chi_s \\ &+ \int_0^{t+\varepsilon} u(s)H_s^-\mathscr{E}(p_s u)\, ds + \int_0^{t+\varepsilon} u(s)H_s^\circ\mathscr{E}(p_s u)\, d\chi_s,\end{aligned}$$

so that

$$\begin{aligned}P_t\Gamma(h)\mathscr{E}(p_{t+\varepsilon}u) = \Omega &+ \int_0^t u(s)H_s\mathscr{E}(p_s u)\, d\chi_s + \int_0^t H_s^+\mathscr{E}(p_s u)\, d\chi_s \\ &+ \int_0^{t+\varepsilon} u(s)P_tH_s^-\mathscr{E}(p_s u)\, ds + \int_0^t u(s)H_s^\circ\mathscr{E}(p_s u)\, d\chi_s,\end{aligned}$$

whereas

$$\begin{aligned}P_t\Gamma(h)\mathscr{E}(p_t u) = \Omega &+ \int_0^t u(s)H_s\mathscr{E}(p_s u)\, d\chi_s + \int_0^t H_s^+\mathscr{E}(p_s u)\, d\chi_s \\ &+ \int_0^t u(s)H_s^-\mathscr{E}(p_s u)\, ds + \int_0^t u(s)H_s^\circ\mathscr{E}(p_s u)\, d\chi_s.\end{aligned}$$

Hence

$$\frac{1}{\varepsilon} P_t(\Gamma(h)\mathscr{E}(p_{t+\varepsilon}u) - \Gamma(h)\mathscr{E}(p_t u)) = P_t \frac{1}{\varepsilon}\int_t^{t+\varepsilon} u(s)H_s^-\mathscr{E}(p_s u)\, ds.$$

But $s \mapsto |u(s)|\, ||H_s^-\mathscr{E}(p_s u)||$ is integrable by definiteness of the da^- integral on $\mathscr{E}(L^2(\mathbb{R}_+))$, so that, for almost all t, the integral on the right-hand side above tends in norm to $u(t)H_t^-\mathscr{E}(p_t u)$. The operator P_t is a projection and the limit $H_t^-\mathscr{E}(u_t)$ belongs to its image $\Phi_{t]}$, so that the right-hand side also converges to $u(t)H_t^-\mathscr{E}(p_t u)$.

For simplicity, we denote by $p_{[t,t+\varepsilon]}$ the operator $p_{t+\varepsilon} - p_t$. When restricted to the first chaos, the left-hand side, with this notation, is simply $(p_t h p_{[t,t+\varepsilon]}u)/\varepsilon$ so that we have proved that

$$\text{for a.a. } t, \quad p_t h p_{[t,t+\varepsilon]} u/\varepsilon \xrightarrow{\varepsilon\to 0} u(t) P^1 H_t^- \mathscr{E}(p_t u) \quad \text{in } L^2 \text{ norm,} \tag{2.2}$$

where P^1 is the projection on the first chaos.

Applying that result in the case where u is taken to be $\mathbb{1}_{N]}$ for some N in $\mathbb{N}$ yields the fact that

$$\text{for a.a. } t, \quad p_t h \mathbb{1}_{[t,t+\varepsilon]}/\varepsilon \xrightarrow{\varepsilon\to 0} P^1 H_t^- \mathscr{E}(\mathbb{1}_{t]}) \quad \text{in } L^2 \text{ norm,} \tag{2.3}$$

We denote from now on $P^1 H_t^- \mathscr{E}(\mathbb{1}_t)$, by α_t and remark that $t \mapsto ||\alpha_t||$ is a locally integrable function since $t \mapsto |\mathbb{1}_{N]}(t)|\, ||H_t^- \mathscr{E}(p_t \mathbb{1}_{N]})||$ is integrable for any N in $\mathbb{N}$.

By the assumption that $\Gamma(h^*)$ has an integral representation we obtain the dual assumptions, so that we have proved the two first conditions of (C). We have not proved the third, that is, the square-integrability of $||\alpha_t||$ and $||\beta_t||$; we will address that issue after the last proof of convergence.

We prove the fourth property of (C). Let us consider $\Gamma(h)\mathscr{E}(p_{t+\varepsilon}u)$ on the first chaos, for some u in $L^2(\mathbb{R}_+)$. For almost any $s \leqslant t+\varepsilon$,

$$\begin{aligned}\Gamma(h)\mathscr{E}(p_{t+\varepsilon}u)(s) &= u(s) H_s \mathscr{E}(p_s u)(\emptyset) + H_s^+ \mathscr{E}(p_s u)(\emptyset)\\ &\quad + \int_s^{t+\varepsilon} u(r) H_r^- \mathscr{E}(p_r u)(s)\, dr + u(s) H_s^\circ \mathscr{E}(p_s u)(\emptyset)\end{aligned}$$

where the lower bound s for the a^- integral arises because $H_r^- \mathscr{E}(p_r u)(s)$ is zero for $r < s$ by adaptedness.

Thus one has, using the fact that $\Gamma(h)\mathscr{E}(p_{t+\varepsilon}u)(s) = hp_{t+\varepsilon}u(s)$,

$$\begin{aligned}\frac{1}{\varepsilon}\int_t^{t+\varepsilon} hp_{t+\varepsilon}u(s)\, ds &= \frac{1}{\varepsilon}\int_t^{t+\varepsilon} u(s)\, ds + \frac{1}{\varepsilon}\int_t^{t+\varepsilon} H_s^+ \mathscr{E}(p_s u)(\emptyset)\, ds\\ &\quad + \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_s^{t+\varepsilon} u(r) H_r^- \mathscr{E}(p_r u)(s)\, dr\, ds\\ &\quad + \frac{1}{\varepsilon}\int_t^{t+\varepsilon} u(r) H_r^\circ \mathscr{E}(p_r u)(\emptyset)\, dr.\end{aligned} \tag{2.4}$$

For almost all t:

- the first term converges to $u(t)$,
- the second and fourth terms converge, respectively, to $H_t^+ \mathscr{E}(p_t u)(\emptyset)$ and $u(t) H_t^\circ \mathscr{E}(p_t u)(\emptyset)$ because of the integrability properties of $s \mapsto H_s^+ \mathscr{E}(p_s u)$, $s \mapsto u(s) H_s^\circ \mathscr{E}(p_s u)$,

- the third term vanishes. Indeed, an application of Fubini's theorem shows that

$$\left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_s^{t+\varepsilon} u(r)H_r^-\mathscr{E}(p_r u)(s)\,dr\,ds\right| \leqslant \frac{1}{\varepsilon}\int_t^{t+\varepsilon}|u(r)|\int_t^r |H_r^-\mathscr{E}(p_r u)(s)|\,ds\,dr$$

  and the integral $\int_t^r |H_r^-\mathscr{E}(p_r u)(s)|\,ds$ is smaller than $\sqrt{\varepsilon}||H_r^-\mathscr{E}(p_r u)||$ for any r in $[t, t+\varepsilon]$ by the Cauchy–Schwarz inequality. That third term is therefore equal to $\sqrt{\varepsilon}$ times a convergent term.

Thus we have shown that

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} hp_{t+\varepsilon}u(s)\,ds \text{ converges as } \varepsilon \to 0. \tag{2.5}$$

But then

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} hp_t u(s)\,ds = \langle (h^*\mathbb{1}_{[t,t+\varepsilon]})/\varepsilon, p_t u\rangle,$$

and $p_t H^*\mathbb{1}_{[t,t+\varepsilon]}/\varepsilon$ converges in $L^2(\mathbb{R}_+)$, so that

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} hp_{t+\varepsilon}u(s)\,ds \text{ converges as } \varepsilon \to 0. \tag{2.6}$$

Subtracting (2.6) from (2.5) one finally obtains that for all u in $L^2(\mathbb{R}_+)$,

$$\text{for almost all } t,\ \frac{1}{\varepsilon}\int_t^{t+\varepsilon} hp_{[t,t+\varepsilon]}u(s)\,ds \text{ has a limit as } \varepsilon \text{ tends to zero.}$$

In the case where u is equal to some indicator function $\mathbb{1}_{N]}$ with $t < N$, this implies that

$$\text{for almost all } t,\ \frac{1}{\varepsilon}\int_t^{t+\varepsilon} h\mathbb{1}_{[t,t+\varepsilon]}(s)\,ds \text{ has a limit as } \varepsilon \text{ tends to zero,}$$

and in that particular case we denote the limit by $\gamma(t)$. For every ε,

$$\begin{aligned}\left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon} h\mathbb{1}_{[t,t+\varepsilon]}(s)\,ds\right| &\leqslant \frac{1}{\sqrt{\varepsilon}}\left(\int |h\mathbb{1}_{[t,t+\varepsilon]}|^2\right)^{1/2}\\ &\leqslant ||h||\end{aligned}$$

so that the function γ is necessarily essentially bounded.

We now prove the square-integrability of α_t and β_t. The argument is the following: we prove in Lemma 2.4 below that

$$u(t)\alpha_t = u(t)P^1 H_t^-\mathscr{E}(p_t u) \quad \text{almost everywhere in } \mathscr{P}$$

therefore showing that $t \mapsto u(t)||\alpha_t||_{L^2([0,t])}$ is integrable for any u in $L^2(\mathbb{R}_+)$, and conclude thanks to the following lemma:

Lemma 2.3. *Let $p \in [1, +\infty[$. Let v be a measurable function such that uv is integrable for every u is in $L^p(\mathbb{R}_+)$. Then v is in $L^q(\mathbb{R}_+)$, where $q \in]1, +\infty]$ is the conjugate index of p.*

Proof. If $u \mapsto \int uv$ is bounded, then the classical duality theorem imply that $v \in L^q(\mathbb{R}_+)$. It is therefore enough to prove that $u \mapsto uv$ is continuous between the two Banachs $L^p(\mathbb{R}_+)$ and $L^1(\mathbb{R}_+)$. By the closed graph theorem it is enough to prove that the graph of that application is closed: for that let us suppose that $u_n \to u$ in $L^p(\mathbb{R}_+)$ with $u_n v$ converging in $L^1(\mathbb{R}_+)$. Almost-everywhere convergence of subsequences holds in both cases, and this shows that the limit of $u_n v$ is uv. □

The following is therefore a crucial step of our demonstration:

Lemma 2.4. *For all $u \in L^2(\mathbb{R}_+)$, almost all t,*

$$P^1 H_t^- \mathscr{E}(p_t u) = P^1 H_t^- \mathscr{E}(\mathbb{1}_{t]}). \tag{2.7}$$

Proof. First notice two simple consequences of (2.3) and (2.2):

- for all u, v, one has $(u+v)(t)P^1 H_t^- \mathscr{E}(p_t(u+v)) = u(t)P^1 H_t^- \mathscr{E}(p_t u) + v(t)P^1 H_t^- \mathscr{E}(p_t v)$,
- for any almost everywhere differentiable function u, the desired equality (2.7) holds for almost all t.

To prove the second point, consider an almost everywhere differentiable function u. For almost all t, the following three conditions hold:

$$p_t h p_{[t,t+\varepsilon]} u/\varepsilon \to u(t) P^1 H_t^- \mathscr{E}(p_t u), \tag{2.8}$$

$$p_t h(u(t)\mathbb{1}_{[t,t+\varepsilon]})/\varepsilon \to u(t)\alpha_t \tag{2.9}$$

and u is differentiable at t. The differentiability of u at t implies that the function $|u - u(t)|/\varepsilon$ is bounded on $[t, t+\varepsilon]$ for small enough ε. Therefore the L^2 norm of $p_{[t,t+\varepsilon]}\frac{u-u(t)}{\varepsilon}$ converges to zero as ε goes to zero and so does the L^2 norm of $hp_{[t,t+\varepsilon]}\frac{u-u(t)}{\varepsilon}$. We deduce equality (2.7) from (2.8) and (2.9).

Thanks to the first point, one can restrict the proof of Lemma 2.4 to the case of a positive function u. Besides, it is enough to prove equality (2.7) for almost all t in a compact interval $[0, T]$.

Consider therefore an almost everywhere positive function u in $L^2([0,T])$. The sequence $(v_n)_{n\geqslant 0}$ with $v_n = \sup(u, \frac{1}{n})$ converges to u in $L^2([0,T])$. Besides, classical convolution techniques allow us to approximate any v_n by a sequence of smooth functions which are almost everywhere greater than $\frac{1}{n}$. Combining these two remarks provides us with a sequence of almost everywhere strictly positive smooth functions $(u_n)_{n\geqslant 0}$ which approximate u in $L^2([0,T])$.

For every n we have, since u_n is smooth,

$$\text{for almost all } t \text{ in } [0,T], \quad u_n(t)P^1 H_t^- \mathscr{E}(p_t u_n) = u_n(t) P^1 H_t^- \mathscr{E}(\mathbb{1}_{t]}).$$

The following lemma strengthens that equality:

Lemma 2.5. *For any function u such that $u(t)P^1 H_t^- \mathscr{E}(p_t u) = u(t)P^1 H_t^- \mathscr{E}(\mathbb{1}_{t]})$ for almost all t, we have*

$$u(t)H_t^- \mathscr{E}(p_t u) = u(t) \int_0^t \alpha_t(s)\, d\chi_s \circ \mathscr{E}(p_t h p_t u) \tag{2.10}$$

for almost all t also.

Proof. One of the first steps of the proof of Proposition 2.2 was to show that

$$\frac{1}{\varepsilon} P_t(\Gamma(h)\mathscr{E}(p_{t+\varepsilon}u) - \Gamma(h)\mathscr{E}(p_t u))$$

converges to $u(t)H_t^- \mathscr{E}(p_t u)$. Now, on the nth chaos, the left-hand side is equal to

$$\frac{1}{\varepsilon} P_t((hp_{t+\varepsilon}u)^{\circ n} - (hp_t u)^{\circ n})$$

and writing $hp_{t+\varepsilon}u = hp_{[t,t+\varepsilon]u} + hp_t u$, and expanding that expression, we obtain that the above is

$$\frac{1}{\varepsilon} P_t(hp_{[t,t+\varepsilon]}u \circ (hp_t u)^{\circ(n-1)} + (hp_{[t,t+\varepsilon]}u)^{\circ 2} \circ (hp_t u)^{\circ(n-2)} + \cdots + (hp^{\circ n}_{[t,t+\varepsilon]u}))$$

and since $\frac{1}{\varepsilon} p_t hp_{[t,t+\varepsilon]}u$ converges to $u(t)\alpha_t$, the normalisations imply that the above converges almost everywhere to $u(t)\alpha_t \circ (p_t h p_t u)^{\circ(n-1)}$, on the nth chaos, so that we obtain equality (2.10). □

We now apply Lemma 2.5 to each of the functions u_n. In each relation we can simplify by $u_n(t)$ and still have an equality which holds almost everywhere. Besides, we have a countable set of almost-everywhere equalities so that:

For almost all t in $[0, T]$:

$$\text{for all } n, \quad H_t^- \mathscr{E}(p_t u_n) = \int_0^t \alpha_t(s)\, d\chi_s \circ \mathscr{E}(p_t h p_t u_n);$$

the right-hand side converges in $L^2(\mathscr{P})$, hence the left-hand side also does.

Since $\mathscr{E}(p_t u_n)$ converges to $\mathscr{E}(p_t u)$ which is in the domain of the closable operator H_t^-, the closability of H_t^- implies that for almost all t,

$$H_t^- \mathscr{E}(p_t u) = \int_0^t \alpha_t(s)\, d\chi_s \circ \mathscr{E}(p_t h p_t u),$$

a relation which, projected on the first chaos, yields

$$\text{for almost all } t, \quad P^1 H_t^- \mathscr{E}(p_t u) = \alpha_t. \qquad \square$$

This concludes the proof of Proposition 2.2. □

We now take on the second step of our proof, that is, the equivalence of conditions (C) and 2.

Lemma 2.6. *For a bounded operator h, conditions* (C) *in Proposition* 2.2 *are equivalent to the existence of a Hilbert–Schmidt operator K such that*

$$h = K + \mathscr{M}_\gamma,$$

where $\mathscr{M}_\gamma$ is the multiplication operator by γ.

Proof. That h being of the mentioned form implies conditions (C) is straightforward (and unnecessary for our purpose). Now, to prove the converse, let us define a kernel κ by

$$\begin{cases} \kappa(s,t) = \overline{\beta_t(s)} & \text{if } s < t, \\ \kappa(s,t) = \alpha_s(t) & \text{if } s > t. \end{cases} \tag{2.11}$$

Our assumptions on α and β show that

$$\int_{0<s<t} |\kappa(s,t)|^2\, ds\, dt < +\infty$$

and

$$\int_{0<t<s} |\kappa(s,t)|^2 ds\, dt < +\infty$$

so that the kernel κ defines a Hilbert–Schmidt operator K which is in particular a bounded operator on $L^2(\mathbb{R}_+)$. The operator of multiplication by γ is also bounded.

Therefore we can consider $h - K - \mathscr{M}_\gamma$ instead of h and show that, if h satisfies (C) with α, β, γ all null then h is the null operator.

To prove that claim, observe that if $u \in L^2(\mathbb{R}_+)$, and $a < b$, then for almost every $t > b$:

$$\begin{aligned} h(u\mathbb{1}_{[a,b]})(t) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} h(u\mathbb{1}_{[a,b]})(s)\, ds \\ &= \lim_{\varepsilon \to 0} \langle (h^* \mathbb{1}_{[t,t+\varepsilon]})/\varepsilon, u\mathbb{1}_{[a,b]} \rangle \end{aligned}$$

and the limit is zero since the restriction of $(h^*\mathbb{1}_{[t,t+\varepsilon]})/\varepsilon$ to $[0, t]$ converges to zero in L^2. Therefore $h(u\mathbb{1}_{[a,b]})$ is a.e. null on $[b, +\infty[$. The same property holds by symmetry for h^*.

For $c < d < a$,

$$\int_c^d h(u\mathbb{1}_{[a,b]})(s)\, ds = \langle h^* \mathbb{1}_{[c,d]}, u\mathbb{1}_{[a,b]} \rangle$$

which is zero by the previous step.

Therefore $h(u\mathbb{1}_{[a,b]})$ has support in $[a, b]$, so that for any $u \in L^2(\mathbb{R}_+)$, any interval I of $\mathbb{R}_+$, one has

$$h(u\mathbb{1}_I) = \mathbb{1}_I(hu).$$

From this one deduces that h is a multiplication operator. It is straightforward from the last condition in (C) with $\gamma = 0$ that this multiplication is null. $\square$

Proof of $3 \Rightarrow 1$ **of Theorem 2.1.** Suppose that h is of the form

$$h = K + \mathscr{M}_\gamma,$$

where K is a Hilbert–Schmidt operator with kernel κ. We then define for any $u \in L^2(\mathbb{R}_+)$

$$H_t^+ \mathscr{E}(p_t u) = \int_0^t \kappa(s,t) u(s)\, ds\, \mathscr{E}(p_t h p_t u),$$

$$H_t^- \mathscr{E}(p_t u) = \mathscr{E}(p_t h p_t u) \circ \int_0^t \kappa(t,s)\, d\chi_s,$$

$$H_t^\circ \mathscr{E}(p_t u) = (\gamma(t) - 1) \mathscr{E}(p_t h p_t u), \tag{2.12}$$

and extend all three operators by adaptedness. Remark that this means that on the set $\mathscr{E}(L^2(\mathbb{R}_+))$,

$$\begin{aligned} H_t^+ P_t &= P_t \Gamma(h)\, a^-_{\overline{\kappa(.,t)}}\, P_t, \\ H_t^- P_t &= P_t a^+_{\kappa(t,.)} \Gamma(h) P_t, \\ H_t^\circ P_t &= (\gamma(t) - 1) P_t \Gamma(h) P_t. \end{aligned} \tag{2.13}$$

From the above equations one can check that, if one denotes by $\tilde{H}_t^+$, $\tilde{H}_t^-$, $\tilde{H}_t^\circ$ the operators defined in an analogous way from h^* instead of h, then the equalities

$$\begin{aligned} \tilde{H}_t^+ &= (H_t^-)^*, \\ \tilde{H}_t^- &= (H_t^+)^*, \\ \tilde{H}_t^\circ &= (H_t^\circ)^* \end{aligned}$$

hold on $\mathscr{E}(L^2(\mathbb{R}_+))$; this implies that the integrands we have defined are closable.

From formulas (2.12) one also derives the estimates

$$\begin{aligned} \|H_t^+ \mathscr{E}(p_t u)\|^2 &\leqslant \int_0^t |\kappa(s,t)|^2\, ds\, \|p_t u\|^2 \exp(\|h\|^2 \|p_t u\|^2), \\ \|H_t^- \mathscr{E}(p_t u)\|^2 &\leqslant \int_0^t |\kappa(t,s)|^2\, ds \exp(\|h\|^2 \|p_t u\|^2), \\ \|H_t^\circ \mathscr{E}(p_t u)\|^2 &\leqslant (\|\gamma\|_\infty + 1)^2 \exp(\|h\|^2 \|p_t u\|^2), \end{aligned} \tag{2.14}$$

which imply that for all $u \in L^2(\mathbb{R}_+)$, $|u(t)|\, \|H_t^- \mathscr{E}(p_t u)\|$ is integrable and $\|H_t^+ \mathscr{E}(p_t u)\|$, $|u(t)|\, \|H_t^\circ \mathscr{E}(p_t u)\|$ are square-integrable.

The operator

$$H = Id + \int_0^\infty H_s^+ da_s^+ + \int_0^\infty H_s^- da_s^- + \int_0^\infty H_s^\circ da_s^\circ$$

is therefore well defined on the set $\mathscr{E}(L^2(\mathbb{R}_+))$. One can check that the operator

$$Id + \int_0^\infty \tilde{H}_s^+\, da_s^+ + \int_0^\infty \tilde{H}_s^-\, da_s^- + \int_0^\infty \tilde{H}_s^\circ\, da_s^\circ$$

is adjoint to H on the exponential domain, so that H has a densely defined adjoint, and therefore is closable.

We now prove that H actually equals $\Gamma(h)$ on $\mathcal{E}(L^2(\mathbb{R}_+))$. Let u belong to $L^2(\mathbb{R}_+)$. Then

$$H\mathcal{E}(u) = \Omega + \int_0^\infty u(s)H_s\mathcal{E}(p_s u)\, d\chi_s + \int_0^\infty H_s^+\mathcal{E}(p_s u)\, d\chi_s \\ + \int_0^\infty u(s)H_s^-\mathcal{E}(p_s u)\, ds + \int_0^\infty u(s)H_s^\circ\mathcal{E}(p_s u)\, d\chi_s$$

and for almost all $t \in \mathbb{R}_+$,

$$D_t H\mathcal{E}(u) = u(t)H_t\mathcal{E}(p_t u) + H_t^+\mathcal{E}(p_t u) + D_t\int_0^\infty u(s)H_s^-\mathcal{E}(p_s u)\, ds \\ + u(t)H_t^\circ\mathcal{E}(p_t u)$$

so that, for $\sigma < t$ one has

$$D_t H\mathcal{E}(u)(\sigma) = u(t)H_t\mathcal{E}(p_t u)(\sigma) + \int_0^t \kappa(s,t)u(s)\, ds\, (hp_t u)(\sigma) \\ + \int_t^{+\infty} u(s)H_s^-\mathcal{E}(p_s u)(\sigma\cup t)\, ds + u(t)(\gamma(t)-1)(hp_t u)(\sigma).$$

Where the a^- integral is restricted to $[t, +\infty[$ for reasons of adaptedness.

We will write $(hp_t u)(\sigma)$ for $\prod_{a\in\sigma}(hp_t u)(a)$ or the equivalent short notation for other functions, as is customary. We now use the fact that for a in σ,

$$hp_t u(a) = \int_0^t \kappa(s,a)u(s)\, ds + u(a)\gamma(a)$$

and develop such expressions as $(hp_t u)(\sigma)$:

$$D_t H\mathcal{E}(u)(\sigma) = u(t)(H_t\mathcal{E}(p_t u) - \mathcal{E}(hp_t u))(\sigma) \\ + \sum_{\tau\subset\sigma}\prod_{a\in\tau\cup t}\left(\int_0^t \kappa(s,a)u(a)\, ds\right)(u\gamma)(\sigma\backslash\tau) \\ + \int_t^{+\infty} u(s)\sum_{b\in\sigma\cup t}\kappa(s,b)(hp_s u)(\sigma\cup t\backslash b)\, ds \\ + \sum_{\tau\subset\sigma}\prod_{a\in\tau}\left(\int_0^t \kappa(s,a)u(a)\, ds\right)(u\gamma)(\sigma\cup t\backslash\tau).$$

The term with the $\int_t^{+\infty}$ integral is equal to

$$\int_t^{+\infty} \kappa(s,t)u(s) \sum_{\tau\subset\sigma} \prod_{a\in\tau} \int_0^s \kappa(r,a)u(r)\, dr\, ds\, (u\gamma)(\sigma\backslash\tau)$$
$$+ \sum_{\tau\cup v\cup\{b\}=\sigma} \int_t^{+\infty} \kappa(s,b)u(s) \prod_{a\in\tau} \int_0^s \kappa(r,a)u(r)\, dr\, ds\, (u\gamma)(v+t)$$
$$+ \sum_{\tau\cup v\cup\{b\}=\sigma} \int_t^{+\infty} \kappa(s,b)u(s) \prod_{a\in\tau\cup t} \int_0^s \kappa(r,a)u(r)\, dr\, ds\, (u\gamma)(\tau).$$

From this formulation one can see that $(D_t H\mathscr{E}(u) - u(t)H_t\mathscr{E}(p_t u) - \mathscr{E}(hp_t u))(\sigma)$ is

$$((u\gamma)(t) + \int_0^\infty \kappa(s,t)u(s)ds) \sum_{\tau\cup v=\sigma} \int_0^\infty \prod_{a\in\tau} \kappa(r,a)u(r)dr(u\gamma)(v)$$

which equals $\mathscr{E}(hu)(\sigma\cup\{t\})$, and we have proved that

$$D_t H\mathscr{E}(u) - D_t\mathscr{E}(hu) = u(t)P_t(\mathscr{E}(hu) - \mathscr{E}(hp_t u)).$$

Since $H\mathscr{E}(u)(\emptyset) = \mathscr{E}(hu)(\emptyset)$, the previsible representation isometry (1.4) yields

$$\begin{aligned} ||H\mathscr{E}(u) - \mathscr{E}(hu)||^2 &= \int_0^\infty |u(t)|^2 ||P_t(H\mathscr{E}(p_t u) - \mathscr{E}(hp_t u))||^2\, dt \\ &\leqslant \int_0^\infty |u(t)|^2 ||(H\mathscr{E}(p_t u) - \mathscr{E}(hp_t u))||^2\, dt \end{aligned}$$

and this allows us to conclude that $H\mathscr{E}(p_T u) = \mathscr{E}(hp_T u)$ for any T in $\mathbb{R}_+$ thanks to Gronwall's lemma. The closability of H, which we have proved before, entails $H\mathscr{E}(u) = \mathscr{E}(hu)$.

This ends the proof of the equivalence of 1 and 3.

2.1.1. *Extension of the representation to $\mathscr{J}$*

We consider the stochastic integral representation defined on $\mathscr{E}(L^2(\mathbb{R}_+))$ above and prove now that it holds on all of the domain $\mathscr{J}$. First of all, since $D_t j(g,f) = g(t)\mathscr{E}(f_t)$, estimates (2.14) are enough to prove that $t\mapsto ||H_t^- D_t j(g,f)||$ and $t\mapsto ||H_t^\circ D_t j(g,f)||^2$ are integrable. Now one needs to show that H_t^+ can be extended to $j(g,f)$: one defines it through (2.13). To prove that it is well defined it is enough to show, since the restriction of $\Gamma(h)$ to the nth chaos has norm $||h||^n$, that, denoting by P^n the projection on the nth chaos,

$$\sum_{n\geqslant 0} ||h||^{2n} ||P^n(a^-_{\overline{\kappa(.,t)}} P_t j(g,f))||^2 < +\infty$$

and this series would be an upper bound for $||H_t^+ P_t j(g,f)||^2$. One can see that, for almost all $s_1 < \cdots < s_n$,

$$a^-_{\overline{\kappa(.,t)}} P_t j(g,f)(s_1,\dots,s_n) = \int_0^{s_n} \kappa(r,t) f(r)\, dr\, j(g,f)(s_1,\dots,s_n) + \int_{s_n}^t \kappa(r,t) g(r)\, dr\, \mathscr{E}(f_t)(s_1,\dots,s_n),$$

so that

$$||P^n a^-_{\overline{\kappa(.,t)}} P_t j(g,f)(s_1,\dots,s_n)||^2 \leqslant 2\int_0^t |\kappa(r,t)|^2\, dr\, (||f||^2 ||P^n j(g,f)||^2 + ||g||^2 ||P^n \mathscr{E}(f)||^2).$$

Since $\sum_{n\geqslant 0} ||h||^{2n} ||P^n j(g,f)||^2$ and $\sum_{n\leqslant 0} ||h||^{2n} ||P^n \mathscr{E}(f)||^2$ are finite, this proves both that H_t^+ is well defined on $\mathscr{J}$ and that $t \mapsto ||H_t^+ P_t j(g,f)||$ is square-integrable. Therefore, the considered quantum stochastic integral is defined on $\mathscr{J}$ and a proof that it actually equals $\Gamma(h)$ on that set would be similar to the proof of equality on $\mathscr{E}(L^2(\mathbb{R}_+))$.

2.1.2. Proof in the case of the finite particle domain

To obtain the equivalence of 2 and 3 in Theorem 2.1 we need to obtain the following implications:

- condition 2 implies conditions (C),
- condition 3 on the form of h implies condition 2,

because the other parts of the proof are unchanged.

To prove conditions (C) from 2 one first computes for u in $L^2(\mathbb{R}_+)$

$$P_t \Gamma(h) \int_0^{t+\varepsilon} u(s)\, d\chi_s - P_t \Gamma(h) \int_0^t u(s)\, d\chi_s$$

and shows that this implies

$$(p_t h p_{[t,t+\varepsilon]} u)/\varepsilon = \frac{1}{\varepsilon} P_t \int_t^{t+\varepsilon} u(s) H_s^- \, \Omega ds$$

so that $p_t h p_{[t,t+\varepsilon]} u/\varepsilon$ converges to $u(t) H_t^- \Omega$. Since

$$u(t) H_t^- \Omega = H_t^- D_t\left(\int_0^\infty u(s)\, d\chi_s\right),$$

this implies as before that $P^1 H_t^- \Omega$ belongs to $L^2(\mathbb{R}_+)$ and that $t \mapsto ||\alpha_t||$ is square-integrable. By duality this proves the first three conditions in (C).

The proof of the fourth and fifth conditions in (C) is similar to the one we have given in the case of the exponential domain, the assumed representation being applied to $\int u(s)\,d\chi_s$ instead of $\mathscr{E}(u)$.

Assuming that $h = K + \mathscr{M}_\gamma$ with Hilbert–Schmidt K and essentially bounded γ we know from the previous case that the quantum stochastic integral with integrands given by (2.12) is well defined and equal to $\Gamma(h)$ on $\mathscr{E}(L^2(\mathbb{R}_+))$. It is immediate to see that the integral is also defined on $\mathscr{F}$; besides, it is clear from the formulas for the integrands that any chaos space is stable by the so defined integral operator. Therefore the equality of $\Gamma(h)$ and the integral operator on an exponential $\mathscr{E}(u)$ imply, for any n in $\mathbb{N}$, the equality on $u^{\circ n}$, hence by polarization the equality on any $u_1 \circ \cdots \circ u_n$.

This ends the proof of Theorem 2.1.

2.1.3. Possible extensions of Theorem 2.1

It is clear that the hypothesis in 1 that the integral representation of the considered operator is valid on the whole of the exponential domain is most important to our proof: it is what enables us to prove that the functions $t \mapsto ||\alpha_t||$, $t \mapsto ||\beta_t||$ are square-integrable, implying that the kernel operator we construct is of Hilbert–Schmidt type. It would be interesting for applications to consider weaker assumptions such as the representability on $\mathscr{E}(L^2 \cap L^p)$, the set of exponentials of functions both in $L^2(\mathbb{R}_+)$ and $L^p(\mathbb{R}_+)$. In that case we obtain the property that $t \mapsto ||\alpha_t||$ and $t \mapsto ||\beta_t||$ belong to $L^2(\mathbb{R}_+) + L^q(\mathbb{R}_+)$ where $q = p/p - 1$, but this has no such simple consequences on the form of h as in the case we have treated. See [Pau] for detailed results and proofs.

2.2. The Journé–Meyer counterexample

First of all one should remark that the counterexample of Journé and Meyer is a counterexample to the representability of $\Gamma(h)$; nothing is said of the representability of $\Gamma(h^*)$. Yet the same phenomena appear as in our case and the representability of $\Gamma(h)$ on $\mathscr{E}(L^2(\mathbb{R}_+))$ would be equivalent to the fact that h is of the form $K + \mathscr{M}_\gamma$ as before.

Indeed, here $h^* = -h$; this has no *a priori* consequence on the representability of $\Gamma(h^*)$ but it nevertheless allows us to obtain all conditions (C): the representability of $\Gamma(h)$ alone gives the first condition in (C), the relation $h = h^*$ gives the second one and the rest is proved as before. Therefore h is of the form $h = K + \mathscr{M}_\gamma$.

In the considered case the operator h satisfies $h\mathbb{1}_{[a,b]}(s) = \frac{1}{\pi}\log\frac{s-a}{s-b}$ for a.a. s (see [J-M]). This implies that $\frac{1}{\varepsilon}h\mathbb{1}_{[t,t+\varepsilon]}$ converges almost everywhere to $s \mapsto \frac{1}{\pi}\frac{1}{t-s}$, and this last function is not square-integrable, so that the above convergence cannot hold in the L^2 sense. This operator h is therefore far from fulfilling conditions (C): the associated α_t is such that $\int_0^t \alpha_t(s)\,d\chi_s$ does not define an element of Φ, so that the equality

$$H_t^- \mathscr{E}(p_t u) = \int_0^t \alpha_t(s)\, d\chi_s \circ \mathscr{E}(p_t h p_t u),$$

cannot be given a definite meaning for any u.

This is why a stochastic integral representation can be defined only in a weaker way: indeed the representation defined in Parthasarathy's response [Par] is probably the best possible one: $H_t^+ \mathscr{E}(p_t u)$ is defined only for u's with some regularity (enough to make the integral $\int_0^t \frac{1}{t-s} u(s)\, ds$ meaningful) and H_t^- is only defined as a distribution, through the action of its adjoint.

Let us make an additional remark: the function u pointed out by Journé and Meyer is actually bounded with compact support, so that it belongs to all sets $L^p(\mathbb{R}_+)$; this proves that $\Gamma(h)$ is not representable even on the smaller subspaces $L^2 \cap L^p(\mathbb{R}_+)$. As we noted before, a more general characterization of representability on such a subspace is given in [Pau]; in this example the representability defect is actually so strong (the kernel is not even square integrable in one variable) that characterization does not teach us anything new about this particular case.

2.3. *Second quantization operators as regular martingales*

It is clear from formulas (2.12) or (2.13) that the boundedness of $\Gamma(h)$ is strongly linked to that of the operators H_t^+, H_t^-, H_t°. This allows us to obtain a pleasant characterization of the operators of $\Gamma(h)$ that belong to one of Attal's semimartingale algebras (see [At1]).

We recall the definition of the two semimartingale algebras:

Definition 2.7.

- An operator H is an element of $\mathscr{S}'$ if it has a quantum stochastic integral representation with bounded integrands H_t^+, H_t^- and H_t° that moreover are such that $t \mapsto ||H_t^+||, ||H_t^-||$ are square integrable functions and $t \mapsto ||H_t^\circ||$ is an essentially bounded function.
- An operator H is an element of $\mathscr{S}$ if it is an element of $\mathscr{S}'$ and is a bounded operator.

Proposition 2.8. *For a second quantization operator $\Gamma(h)$ the following are equivalent*:

1. *$\Gamma(h)$ belongs to $\mathscr{S}'$*,
2. *$\Gamma(h)$ belongs to $\mathscr{S}$*,
3. *h is of the form $K + \mathscr{M}_\gamma$ as in Theorem* 2.1 *and h is a contraction.*

Proof. We first prove that 3 implies that $\Gamma(h)$ is an element of $\mathscr{S}$. If h is a contraction then (2.14) shows that H_t^-, H_t° are bounded on $\mathscr{E}(L^2(\mathbb{R}_+))$, with norms smaller than

$\sqrt{\int_0^t |\kappa(t,s)|^2\, ds}$, $(||f||_\infty + 1)$, respectively. That is, both can be extended as bounded operators on Φ, with $t \mapsto ||H_t^-||$ and $t \mapsto ||H_t^\circ||$ respectively square-integrable and essentially bounded. Now to prove that H_t^+ is bounded we recall the following earlier remark: if one denotes by $\widetilde{H}_t^-$ the analogue of operator H_t^- associated to h^*, then the equality $\widetilde{H}_t^- = (H_t^+)^*$ holds on $\mathscr{E}(L^2(\mathbb{R}_+))$. Therefore the adjoint of H_t^+ is contained in a bounded operator with norm $\sqrt{\int_0^t |\kappa(s,t)|^2\, dr}$, the closure of H_t^+ is a bounded operator and $t \mapsto ||H_t^+||^2$. The operator $\Gamma(h)$ itself is bounded so that finally $\Gamma(h) \in \mathscr{S}$.

Now let us prove that 1 implies 3: we suppose that $\Gamma(h)$ has a representation as a quantum stochastic integral with the relevant boundedness assumptions on H_t^+, H_t^-, H_t°. Then the same property holds for $\Gamma(h^*)$ (see [At1]) and our Theorem 2.1 applies, and shows that h is of the form $K + \mathscr{M}_\gamma$ as before. Besides, since the integrands H_t^+, H_t^-, H_t° are bounded and therefore closable, Attal's uniqueness theorem applies (see [At2]) and shows that the integrands satisfy formulas (2.12). Therefore

$$||H_t^- \mathscr{E}(p_t u)||^2 = ||\alpha_t||^2 \exp ||(hp_t u)_t||^2$$

and is to be smaller than a constant times $\exp ||p_t u||^2$ for every u. Denote by p_t the projection in $L^2(\mathbb{R}_+)$ of restriction to $[0,t]$; it is necessary that $u \mapsto p_t h p_t$ be a contraction. This being true for every t, one has for any u in $L^2(\mathbb{R}_+)$, any s, any t larger than s that

$$||p_t h p_s u|| \leqslant ||u||,$$

so that hp_s is a contraction for any s. The boundedness of h implies that h is itself a contraction.

That 2 implies 1 is always true, so that the proof is complete. □

3. Differential second quantization operators

We consider now the case of differential second quantization operators and obtain a characterization which is exactly the same as in the previous case of non differential second quantizations. The explicit formulas for the integrands in the obtained representations are given below in (3.1) and (3.2).

Theorem 3.1. *Let h be a bounded operator on $L^2(\mathbb{R}_+)$. The following properties are equivalent*:

1. *$\lambda(h)$ and $\lambda(h^*)$ have a stochastic integral representation on the set $\mathscr{E}(L^2(\mathbb{R}_+))$ of exponential vectors,*
2. *$\lambda(h)$ and $\lambda(h^*)$ have a stochastic integral representation on the set $\mathscr{F}$ of finite particle vectors,*

3. *h is of the form*

$$h = K + \mathscr{M}_\gamma$$

where K is a Hilbert–Schmidt operator and $\mathscr{M}_\gamma$ *is a multiplication by a* L^∞ *function* γ.

Besides, if one of these holds, then the stochastic integral representation holds on the set $\mathscr{J}(L^2(\mathbb{R}_+))$.

Note that this theorem is an improvement of one proved by Coquio [Coq], where the representation of $\lambda(h)$ and $\lambda(h^*)$ in 1 was assumed to hold on the set $\mathscr{J}$.

Proof. Observe simply that our proof of Proposition 2.2 uses only equalities involving the stochastic integral operator and $\Gamma(h)\mathscr{E}(u)$ or $\Gamma(h^*)\mathscr{E}(u)$ on the chaoses of order zero and one, for functions u in $L^2(\mathbb{R}_+)$. But then $\lambda(h)\mathscr{E}(u)$ (respectively $\lambda(h^*)\mathscr{E}(u)$) coincides with $\Gamma(h)\mathscr{E}(u)$ (respectively $\Gamma(h^*)\mathscr{E}(u)$) on the chaos of order one, and is zero independently of u on the chaos of order zero whereas $\Gamma(h)\mathscr{E}(u)$ (respectively $\Gamma(h)\mathscr{E}(u)$) is one independently of u on that same chaos. Therefore it is easy to see that our proof of Proposition 2.2 would hold if we considered *differential* second quantization operators instead of the non-differential ones. That 1 implies 2 is then proved by Lemma 2.6. The proofs for the converse and the extension to $\mathscr{J}$ are also similar to the previous: if 2 is assumed then we define H_t^+, H_t^-, H_t° by the following formulas:

$$\begin{aligned} H_t^+\mathscr{E}(p_tu) &= \int_0^t \kappa(s,t)u(s)\,ds\,\mathscr{E}(p_tu), \\ H_t^-\mathscr{E}(p_tu) &= \mathscr{E}(p_tu)\circ\int_0^t \kappa(t,s)\,d\chi_s, \\ H_t^\circ\mathscr{E}(p_tu) &= (\gamma(t)-1)\mathscr{E}(p_tu), \end{aligned} \tag{3.1}$$

then the definiteness of the integral

$$H = \int H_s^+\,da_s^+ + \int H_s^-\,da_s^- + \int H_s^\circ\,da_s^\circ$$

on $\mathscr{J}$ and the equality $D_tHj(v,u) = D_t\lambda(h)j(v,u)$ are obtained as before. $\square$

Note that above defined integrals have a more general expression:

$$\begin{aligned} H_t^+P_t &= a^-_{\overline{\kappa(.,t)}}P_t, \\ H_t^-P_t &= a^+_{\kappa(t,.)}P_t, \\ H_t^\circ P_t &= (\gamma(t)-1)P_t. \end{aligned} \tag{3.2}$$

4. The case of Fock space of higher multiplicity

Once again the proofs in this section would be simple rewritings of the proof of Theorem 2.1; our task will be therefore to point out the similarities and to define a correct way of writing our conditions in a concise form.

Let us consider as in the preliminaries a fixed hilbertian basis $(e_i)_{i\in\mathscr{I}}$ of our multiplicity space $\mathscr{K}$. Let us define, or recall, the terms to appear below:

Definition 4.1.

- A Hilbert–Schmidt operator in $L^2(\mathbb{R}_+,\mathscr{K})$ is an operator K for which there exists a family $(\kappa_{i,j})_{i,j\in\mathscr{I}}$ of functions in $L^2(\mathbb{R}_+\times\mathbb{R}_+)$ such that

$$\sum_{i,j\in\mathscr{I}}\int_{\mathbb{R}_+\times\mathbb{R}_+}|\kappa_{i,j}(s,t)|^2\,ds\,dt<+\infty$$

  and for all i in $\mathscr{I}$, almost all s in $\mathbb{R}_+$,

$$Kf(s,i)=\sum_{j\in\mathscr{I}}\int_0^\infty \kappa_{i,j}(r,s)f(r,j)\,dr.$$

- A matrix multiplication operator is an operator M_γ for which there exists a set of functions $(\gamma_{i,j})_{i,j\in\mathscr{I}}$ on $\mathbb{R}_+$ such that for almost all s in $\mathbb{R}_+$, the quantity

$$||\gamma||(s)=\left(\sum_{i,j\in\mathscr{I}}|\gamma_{i,j}(s)|^2\right)^{1/2}$$

  is finite and

$$M_\gamma f(s,i)=\sum_{j\in\mathscr{I}}\gamma_{i,j}(s)f(s,j).$$

Here are Theorems 2.1 and 3.1 rolled into one in the case of Fock space of infinite multiplicity. The formulas for the integrands are given below, in (4.2), (4.3) for the integrands of $\Gamma(h)$ and (4.4), (4.5) for the integrands of $\lambda(h)$.

Theorem 4.2. *Let h be a bounded operator on $L^2(\mathbb{R}_+,\mathscr{K})$. The following properties are equivalent*:

1. *$\Gamma(h)$ and $\Gamma(h^*)$ have a stochastic integral representation on the set $\mathscr{E}(L^2(\mathbb{R}_+,\mathbb{K}))$ of exponential vectors.*
2. *$\Gamma(h)$ and $\Gamma(h^*)$ have a stochastic integral representation on the set $\mathscr{F}_{\mathscr{K}}$ of finite particle vectors.*
3. *$\lambda(h)$ and $\lambda(h^*)$ have a stochastic integral representation on the set $\mathscr{E}(L^2(\mathbb{R}_+,\mathbb{K}))$ of exponential vectors.*

4. *$\lambda(h)$ and $\lambda(h^*)$ have a stochastic integral representation on the set $\mathscr{F}_{\mathscr{K}}$ of finite particle vectors.*
5. *h is of the form*

$$h = K + \mathscr{M}_\gamma,$$

where K is a Hilbert–Schmidt operator and $\mathscr{M}_\gamma$ is a multiplication by a matrix $(\gamma_{i,j})_{i,j \in \mathscr{I}}$ such that the function $\|\gamma\|$ is essentially bounded on $\mathbb{R}_+$.

Besides, if one of these holds, then the representations hold on the set $\mathscr{J}(L^2(\mathbb{R}_+, \mathscr{K}))$.

Proof. The proof is essentially the same as before: consider for instance the equivalence of 1 and 3. By the same arguments as in the case of multiplicity one, the "i,j matrix element" $\Gamma(h)_{i,j}$ of $\Gamma(h)$ is of the form

$$\Gamma(h)_{i,j} = K_{i,j} + \mathscr{M}_{\gamma_{i,j}}, \tag{4.1}$$

where $\gamma_{i,j}$ is an essentially bounded function and $K_{i,j}$ is a Hilbert–Schmidt operator with kernel $\kappa_{i,j}$. Besides, the operator $\Gamma(h)$ is bounded and equality (4.1) for all i,j imply that the operator K with kernel $(\kappa_{i,j})_{i,j \in \mathscr{I}}$ is bounded. Indeed, for any t in $\mathbb{R}_+$, $P_t\Gamma(h)_{i,j}P_t = P_tK_{i,j}P_t$ and from this and the dual relation one deduces that $P_t\Gamma(h)P_t$ is bounded uniformly in t. The operator of multiplication by the matrix $(\gamma_{i,j})_{i,j \in \mathscr{I}}$ is thus also bounded; the theory of Hilbert–Schmidt operators then implies that these operators satisfy integrability assumptions as in Definition 4.1.

The proof that 3 allows the construction of a well-defined integral which coincides with the desired operator is exactly the same as before. The formulas for integrands are the following:

Integrands for the representation of $\Gamma(h)$: The integrands in the representation of $\Gamma(h)$ are given by

$$\begin{aligned} H_t^{0,i}\mathscr{E}(p_tu) &= \sum_{j \in \mathscr{I}} \int_0^t \kappa_{i,j}(s,t)u(s,j)\,ds\,\mathscr{E}((hp_tu)_t), \\ H_t^{j,0}\mathscr{E}(p_tu) &= \mathscr{E}((hp_tu)_t) \circ \sum_{i \in \mathscr{I}} \int_0^t \kappa_{i,j}(t,s)\,d\chi_s^i, \\ H_t^{j,i}\mathscr{E}(p_tu) &= (\gamma_{i,j}(t) - 1)\mathscr{E}((hp_tu)_t). \end{aligned} \tag{4.2}$$

for all i, j in $\mathscr{I}$. Otherwise stated, they satisfy the following equalities on $\mathscr{E}(L^2(\mathbb{R}_+))$ and $\mathscr{F}$:

$$\begin{aligned} H_t^{0,i}P_t &= P_t\Gamma(h)P_t \sum_{j \in \mathscr{I}} \int_0^t \overline{\kappa_{i,j}(s,t)}\,da_s^{j,0}, \\ H_t^{j,0}P_t &= \sum_{i \in \mathscr{I}} \int_0^t \kappa_{i,j}(t,s)\,da_s^{0,i}\,P_t\Gamma(h)P_t, \end{aligned}$$

$$H_t^{j,i}P_t = (\gamma_{i,j}(t) - 1)P_t\Gamma(h)P_t, \tag{4.3}$$

and these equalities can be extended to $\mathscr{J}(L^2(\mathbb{R}_+))$.

Integrands for the representation of $\lambda(h)$: The integrands in the representation of $\lambda(h)$ are given by

$$\begin{aligned} H_t^{0,i}\mathscr{E}(p_t u) &= \sum_{j\in\mathscr{I}} \int_0^t \kappa_{i,j}(s,t)u(s,j)\,ds\,\mathscr{E}(p_t u),\\ H_t^{j,0}\mathscr{E}(p_t u) &= \mathscr{E}(p_t u)\circ \sum_{i\in\mathscr{I}} \int_0^t \kappa_{i,j}(t,s)\,d\chi_s^i,\\ H_t^{j,i}\mathscr{E}(p_t u) &= (\gamma_{i,j}(t) - 1)\mathscr{E}(p_t u). \end{aligned} \tag{4.4}$$

Otherwise stated, they satisfy the following equalities on $\mathscr{E}(L^2(\mathbb{R}_+))$ and $\mathscr{F}$:

$$\begin{aligned} H_t^{i,0}P_t &= \sum_{j\in\mathscr{I}} \int_0^t \overline{\kappa_{i,j}(s,t)}\,da_s^{0,i}P_t,\\ H_t^{0,j}P_t &= \sum_{i\in\mathscr{I}} \int_0^t \kappa_{i,j}(t,s)\,da_s^{j,0}\,P_t,\\ H_t^{i,j}P_t &= (\gamma_{i,j}(t) - 1)P_t \end{aligned} \tag{4.5}$$

and these equalities can be extended to $\mathscr{J}(L^2(\mathbb{R}_+))$.

5. The case of unbounded operators

This section is meant to handle the case where h is an unbounded operator. In this particular case, the lack of information about the domain makes it impossible to obtain simple conditions on the form of h. One can still obtain necessary conditions for the representability of $\Gamma(h)$: for example, for any u in the domain of h, one has for almost all t

- $(hu_{t,t+\varepsilon})/\varepsilon$ converges in $L^2([0,t])$,
- $\frac{1}{\varepsilon}\int_0^\varepsilon hu_{[t,t+\varepsilon]}(s)\,ds$ converges to some complex number.

But these conditions do not translate to a more satisfactory condition on h.

Nevertheless, sufficient conditions are very easy to obtain: the proof of Theorem 2.1 makes it clear that it is enough for $\Gamma(h)$ to be representable on $\mathscr{E}(L^2(\mathbb{R}_+))$ that the expressions in 2.12 and 3.1 be well defined on the set $\mathscr{E}(Dom\,h)$ with estimates that make the quantum stochastic integrals also well defined. This is summarized in the next proposition:

Proposition 5.1. *Let h be an operator on* $L^2(\mathbb{R}_+)$ *with domain* Dom h. *Assume that there exist a function* $\kappa : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ *and a function* $\gamma : \mathbb{R}_+ \to \mathbb{C}$ *such that, for any u in* Dom h, *almost all s in* $\mathbb{R}_+$,

$$hu(s) = \int_0^\infty \kappa(r,s)u(r)\,dr + u(s)f(s).$$

Consider the following conditions:

1. $t \mapsto \int_0^t |\kappa(s,t)|^2\,ds$ *is integrable*,
2. *for any u in* Dom h, $t \mapsto |u(t)|\sqrt{\int_0^t |\kappa(t,s)|^2\,ds}$ *and* $t \mapsto |f(t)-1||u(t)|$ *are integrable*,
3. *for fixed u in* Dom h, $||(hp_t u)_t||$ *is uniformly bounded in t*.

Then:

- *If conditions* 1 *and* 2 *hold, then* $\lambda(h)$ *is representable as a quantum stochastic integral on* $\mathscr{J}(\mathrm{Dom}\,h)$, *that is, the set of vectors of the form* $j(g,f)$ *with* g, f *in* Dom h.
- *If conditions* 1–3 *hold, then* $\Gamma(h)$ *is representable as a quantum stochastic integral on* $\mathscr{J}(\mathrm{Dom}\,h)$.

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