Available online at www.ispacs.com/jnaa
Volume 2012, Year 2012 Article ID jnaa-00108, 11 Pages
doi:10.5899/2012/jnaa-00108
Research Article

NONLINEAR
Analysis and Application

A subclass of harmonic univalent functions with positive coefficients associated with fractional calculus operator

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Abstract
In the present paper, we are defining a subclass of harmonic univalent functions with positive coefficients in unit disc U by using fractional calculus operator. Further we obtain distortion bounds, extreme points and radii of convexity for functions belonging to this class and discuss a class preserving integral operator. Here we also determine that the class studied in this paper is closed under convolution and convex combinations.

Keywords: Harmonic univalent, starlike, extreme points, fractional derivative operator.

1 Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, z \in D$. See Clunie and Sheil-Small [1].

In [1], S_H denote the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ with normalization $f(0) = f_z(0) - 1 = 0$.

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Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

The class S_H reduces to class S of normalized analytic univalent functions if co-analytic part of f i.e. $g \equiv 0$. For this class $f(z)$ may be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.2}$$

Recently, S. Yalcin et al. [15] studied the class $HP(\alpha)$, $(0 \leq \alpha < 1)$ the subclass of S_H satisfying the condition

$$\operatorname{Re}\left\{h'(z) + g'(z)\right\} > \alpha. \tag{1.3}$$

Let V_H denote the subclass of S_H consisting of the form $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1. \tag{1.4}$$

This class V_H has been studied by Dixit and Porwal [4].

Several equivalent definitions of fractional derivatives and fractional integrals have been given in the literature (cf. e.g., Kumar et.al. [5], Nishimoto [7]. We find it to be convenient to restrict ourselves to the following definitions used recently by Srivastava and Owa [12].

Definition 1.1. *The fractional integral of order λ is defined for a function $f(z)$, by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta$$

where $\lambda > 0$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.2. *The fractional derivative of order λ is defined, for a function $f(z)$, by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta$$

where $0 \leq \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1.1 above.

Definition 1.3. *Under the hypothesis of Definition 1.2, the fractional derivative of order $n + \lambda$ is defined for a function $f(z)$, by*

$$D_z^{n+\lambda} f(z) = \frac{d^n D_z^{\lambda} f(z)}{dz^n}$$

where $0 \leq \lambda < 1$ and n is a non-negative integer.

Let $R_H(\lambda,\beta)$, $(0 \le \lambda < 1,\ 1 < \beta \le 2)$, denote the subclass of V_H satisfying

$$Re\{\Gamma(2-\lambda)z^{\lambda-1}D_z^{\lambda}h(z)+\Gamma(2-\lambda)z^{\lambda-1}D_z^{\lambda}g(z)\}<\beta,\ (z\in D) \tag{1.5}$$

for $0 \le \lambda < 1,\ 1 < \beta \le 2$; here $D_{\lambda}^{z}f(z)$ denotes the fractional derivative of function $f(z)$ of the form (1.2) of order λ is defined below

$$D_z^0 f(z) = f(z) \text{ and } D_z^1 f(z) = f'(z) \tag{1.6}$$

By putting specific values of parameters, we obtain various classes introduced by earlier researchers.

(i) For $\lambda = 1$, we obtain the class $R_H(1,\beta)$ studied by Dixit and Porwal [4].

(ii) For $g = 0$, the class $R_H(\lambda,\beta)$ reduces to $R_\lambda(\beta)$ studied by Dixit and Pathak [3].

(iii) For $\lambda = 1$ and $g = 0$, the class $R_H(\lambda,\beta)$ reduces to $R(\beta)$ studied by Uralegaddi, et al. [14].

Recently, Dixit and Porwal [3] (see also [6]) studied the subclasses of harmonic univalent function with negative coefficient by using fractional calculus operator. In this paper, An attempt has been made to study the subclass of harmonic univalent function with positive coefficient by using fractional calculus operator.

Here, we determine a number of sharp results coefficient estimates, distortion bounds, convex combinations, extreme points, convolution and other interesting properties.

2 Main Results

The coefficient estimates given by Theorem 2.1 lays the foundation of our systematic study of the class $R_H(\lambda,\beta)$ defined in preceding section.

Theorem 2.1. *$f \in R_H(\lambda,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_n|+\sum_{n=1}^{\infty}\frac{\Gamma(n+1))\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|b_n|\le\beta-1. \tag{2.7}$$

Proof. Let $\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_n|+\sum_{n=1}^{\infty}\frac{\Gamma(n+1))\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|b_n| \le \beta-1$. It suffices to prove that

$$\left|\frac{\Gamma(2-\lambda)z^{\lambda-1}D_z^{\lambda}h(z)+\Gamma(2-\lambda)z^{\lambda-1}D_z^{\lambda}g(z)-1}{\Gamma(2-\lambda)z^{\lambda-1}D_z^{\lambda}h(z)+\Gamma(2-\lambda)z^{\lambda-1}D_z^{\lambda}g(z)-(2\beta-1)}\right|<1, \quad z\in U.$$

We have

$$\left|\frac{\Gamma(2-\lambda)z^{\lambda-1}D_z^{\lambda}h(z)+\Gamma(2-\lambda)z^{\lambda-1}D_z^{\lambda}g(z)-1}{\Gamma(2-\lambda)z^{\lambda-1}D_z^{\lambda}h(z)+\Gamma(2-\lambda)z^{\lambda-1}D_z^{\lambda}g(z)-(2\beta-1)}\right|$$

$$=\left|\frac{\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|a_n|z^{n-1}+\sum_{n=1}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|b_n|z^{n-1}}{\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|a_n|\ z^{n-1}+\sum_{n=1}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|b_n|\ z^{n-1}-2(\beta-1)}\right|$$

$$\leq\frac{\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|a_n|\ |z|^{n-1}+\sum_{n=1}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|b_n|\ |z|^{n-1}}{2(\beta-1)-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|a_n|\ |z|^{n-1}-\sum_{n=1}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|b_n|\ |z|^{n-1}}$$

$$\leq\frac{\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|a_n|+\sum_{n=1}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|b_n|}{2(\beta-1)-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|a_n|-\sum_{n=1}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|b_n|}$$

which is bounded above by 1 since

$$\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|a_n|+\sum_{n=1}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|b_n|\leq 2(\beta-1)$$
$$-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|a_n|-\sum_{n=1}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|b_n|$$

or
$$\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|a_n|+\sum_{n=1}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}|b_n|\leq \beta-1$$

But (2.1) is true by hypothesis and the sufficient part is proved.
Conversely suppose that

$$\text{Re}\left\{\Gamma(2-\lambda)z^{\lambda-1}D_z^{\lambda}h(z)+\Gamma(2-\lambda)z^{\lambda-1}D_z^{\lambda}g(z)\right\}<\beta,(z\in U)\quad\text{i.e.}$$
$$\text{Re}\left\{1+\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_n|z^{n-1}+\sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|b_n|z^{n-1}\right\}<\beta,\quad z\in U.$$

The above condition must hold for all values of $z, |z|=r<1$.
Upon choosing the value of z to be real and let $z\to I^-$, we get

$$\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_n|+\sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|b_n|\leq\beta-1,$$

which gives the necessary part. The proof of the theorem is convenient.
Next, we determine bounds for the class $R_H(\lambda,\beta)$. □

Theorem 2.2. *If $f \in R_H(\lambda, \beta)$, then*

$$|f(z)| \le (1 + |b_1|)r + \frac{2-\lambda}{2}(\beta - 1 - |b_1|)r^2, |z| = r < 1 \tag{2.8}$$

and

$$|f(z)| \ge (1 - |b_1|)r - \frac{2-\lambda}{2}(\beta - 1 - |b_1|)r^2, |z| = r < 1 \tag{2.9}$$

Proof. Let $f \in R_H(\lambda, \beta)$. Taking the absolute value of f, we have

$$\begin{aligned}
|f(z)| &\le (1 + |b_1|)r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n \\
&\le (1 + |b_1|)r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^2 \\
&\le (1 + |b_1|)r + \frac{(2-\lambda)}{2}\sum_{n=2}^{\infty}\frac{2}{2-\lambda}(|a_n| + |b_n|)r^2 \\
&\le (1 + |b_1|)r + \frac{(2-\lambda)}{2}\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}(|a_n| + |b_n|)r^2 \\
&\le (1 + |b_1|)r + \frac{(2-\lambda)}{2}(\beta - 1 - |b_1|)r^2,
\end{aligned}$$

$$\begin{aligned}
\text{and} \qquad |f(z)| &\ge (1 - |b_1|)r - \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n \\
&\ge (1 - |b_1|)r - \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^2 \\
&\ge (1 - |b_1|)r - \frac{2-\lambda}{2}\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}(|a_n| + |b_n|)r^2 \\
&\ge (1 - |b_1|)r - \frac{2-\lambda}{2}(\beta - 1 - |b_1|)r^2.
\end{aligned}$$

Then functions $z + |b_1|\overline{z} + \frac{2-\lambda}{2}(\beta - 1 - |b_1|)\overline{z}^2$ and $z + |b_1|\overline{z} + \frac{2-\lambda}{2}(\beta - 1 - |b_1|)z^2$, for $|b_1| \le \beta - 1$ show that the bounds given in Theorem 2.2 are sharp.

The following result follows from the left hand inequality in Theorem 2.2. □

Corollary 2.1. *If $f \in R_H(\lambda, \beta)$, then*

$$\left\{w : |w| < \frac{1}{2}(4 - \lambda - (2-\lambda)\beta - \lambda|b_1|)\right\} \subset f(U). \tag{2.10}$$

Next, we determine the extreme points of closed convex hulls of $R_H(\lambda, \beta)$, denoted by clco $R_H(\beta)$.

Theorem 2.3. *$f \in$ clco $R_H(\lambda, \beta)$, if and only if*

$$f(z) = \sum_{n=1}^{\infty} X_n h_n + Y_n g_n \tag{2.11}$$

where

$$h_1(z) = z,\ h_n(z) = z + \frac{(\beta-1)\Gamma(n+1-\lambda)}{\Gamma(n+1)\Gamma(2-\lambda)} z^n, \quad (n = 2, 3, 4, ...)$$
$$g_n(z) = z + \frac{(\beta-1)\Gamma(n+1-\lambda)}{\Gamma(n+1)\Gamma(2-\lambda)} \overline{z}^n, \quad (n = 1, 2, 3, ...)$$

and $\sum_{n=1}^{\infty}(X_n + Y_n) = 1,\ X_n \geq 0$ *and* $Y_n \geq 0$. *In particular the extreme points of* $R_H(\lambda, \beta)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. For function f of the form (2.5), we write

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n) \\ &= z + \sum_{n=2}^{\infty} \left(\frac{(\beta-1)\Gamma(n+1-\lambda)}{\Gamma(n+1)\Gamma(2-\lambda)} \right) X_n z^n + \sum_{n=1}^{\infty} \left(\frac{(\beta-1)\Gamma(n+1-\lambda)}{\Gamma(n+1)\Gamma(2-\lambda)} \right) Y_n \overline{z}^n. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{(\beta-1)\Gamma(n+1-\lambda)} \left(\frac{(\beta-1)\Gamma(n+1-\lambda)}{\Gamma(n+1)\Gamma(2-\lambda)} X_n \right) \\ &+ \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{(\beta-1)\Gamma(n+1-\lambda)} \left(\frac{(\beta-1)\Gamma(n+1-\lambda)}{\Gamma(n+1)\Gamma(2-\lambda)} Y_n \right) \\ &= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n \\ &= 1 - X_1 \leq 1, \end{aligned}$$

and so $f \in \text{clco } R_H(\lambda, \beta)$.

Conversely suppose that $f \in \text{clco } R_H(\lambda, \beta)$. Set

$$X_n = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} |a_n|, \quad (n = 2, 3, ...)$$

and

$$Y_n = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} |b_n|, \quad (n = 1, 2, 3, ...).$$

Then note that by Theorem 2.1, $0 \leq X_n \leq 1$, $(n = 2, 3...)$ and $0 \leq Y_n \leq 1$, $(n = 1, 2, 3...)$. We define $X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n$ and note that by Theorem 2.1, $X_1 \geq 0$. Consequently, we obtain $f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n)$ as required. □

The study of starlikeness condition is an interesting problems in analytic as well as harmonic univalent functions e.g. see the work of ([4], [9] and [15]). In our next theorem we show that the class $R_H(\lambda, \beta)$ is a subclass of starlike harmonic univalent functions.

Theorem 2.4. *$R_H(\lambda,\beta) \subseteq S_H^*$, where $0 \le \lambda < 1, 1 < \beta \le 2$.*

Proof. Let $f \in R_H(\lambda,\beta)$. Then by Theorem 2.1

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)}|a_n| + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)}|b_n| \le 1 \tag{2.12}$$

$$\begin{aligned}\text{Now} \quad & \sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \\ \le & \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)}|a_n| + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)}|b_n| \\ \le & 1 \qquad \text{by using (2.6)}\end{aligned}$$

Thus $f \in S_H^*$.

This completes the proof of Theorem 2.4. □

Theorem 2.5. *Each function in the class $R_H(\lambda,\beta)$ maps a disks U_r where $r < \inf_n \left\{ \frac{1}{n(\beta-1-|b_1|)} \right\}^{1/n-1}$ onto convex domains for $\beta > 1 + |b_1|$.*

Proof. Let $f \in R_H(\lambda,\beta)$ and let r be fixed is that $0 < r < 1$, then $r^{-1}f(rz) \in R_H(\lambda,\beta)$ and we have

$$\begin{aligned}\sum_{n=2}^{\infty} n^2(|a_n|+|b_n|)r^{n-1} &= \sum_{n=2}^{\infty} n(|a_n|+|b_n|)(nr^{n-1}) \\ &\le \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}(|a_n|+|b_n|) \\ &\le \beta - 1 - |b_1| \le 1, \\ \text{provided} \qquad nr^{n-1} &\le \frac{1}{\beta-1-|b_1|} \\ \text{or} \qquad r &< \inf_n \left\{ \frac{1}{n(\beta-1-|b_1|)} \right\}^{1/n-1}.\end{aligned}$$

The proof of Theorem 2.5 is complete.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n \text{ and } F(z) = z + \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z}^n$$

we define their convolution as

$$(f * F)(z) = f(z) * F(z) = z + \sum_{n=2}^{\infty} |a_n A_n| z^n + \sum_{n=1}^{\infty} |b_n B_n| \overline{z}^n \tag{2.13}$$

Using this definition, we show that the class $R_H(\lambda,\beta)$ is closed under convolution. □

Theorem 2.6. *For $1 < \beta \leq \alpha \leq 2$ let $f \in R_H(\lambda, \beta)$, and $F \in R_H(\lambda, \alpha)$. Then $f * F \in R_H(\lambda, \beta) \subseteq R_H(\lambda, \alpha)$.*

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n$ be in $R_H(\lambda, \beta)$ and $F(z) = z + \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z}^n$ be in $R_H(\lambda, \alpha)$.

Then the convolution $f * F$ is given by (2.7), we wish to show that the coefficient of $f * F$ satisfy the required condition given in Theorem 2.1. For $F(z) \in R_H(\lambda, \alpha)$ we note that $|A_n| \leq 1$ and $|B_n| \leq 1$.

Now, for the convolution function $f * F$ we have

$$
\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} |a_n A_n| + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} |b_n B_n| \\
\leq &\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} |a_n| + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} |b_n| \\
\leq &1. \qquad \text{(Since } f \in R_H(\lambda, \beta))
\end{aligned}
$$

Therefore $f * F \in R_H(\lambda, \beta) \subseteq R_H(\lambda, \alpha)$.

Now, we show that $R_H(\lambda, \beta)$ is closed under convex combinations of its members. □

Theorem 2.7. *The class $R_H(\lambda, \beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3...$ let $f_i(z) \in R_H(\lambda, \beta)$, where $f_i(z)$ is given by

$$
f_i(z) = z + \sum_{n=2}^{\infty} |a_{n_i}| z^n + \sum_{n=1}^{\infty} |b_{n_i}| \overline{z}^n \tag{2.14}
$$

Then by Theorem 2.1, we have

$$
\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} |a_{n_i}| + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} |b_{n_i}| \leq 1.
$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1$, the convex combination of $f_i(z)$ may be written as

$$
\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n_i}| \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{n_i}| \right) \overline{z}^n
$$

Then by Theorem 2.1, we have

$$
\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} \left(\sum_{i=1}^{\infty} t_i |a_{n_i}| \right) + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} \left(\sum_{i=1}^{\infty} t_i |b_{n_i}| \right) \\
= &\sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} |a_{n_i}| + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} |b_{n_i}| \right) \\
\leq &\sum_{i=1}^{\infty} t_i = 1.
\end{aligned}
$$

Therefore $\sum_{i=1}^{\infty} t_i f_i(z) \in R_H(\lambda, \beta)$.

The δ-neighborhood of f is the set

$$N_\delta(f) = \Bigg\{ F : F(z) = z + \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z}^n \text{ and } \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} (|a_n - A_n| + |b_n - B_n|) \leq \delta \Bigg\}.$$

See[8]

□

Theorem 2.8. *Let $f \in R_H(\lambda, \beta)$ and $\delta \leq 2 - \beta$. If $F \in N_\delta(f)$, then F is harmonic starlike function.*

Proof. Let $F(z) = z + \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z}^n$ belongs to $N_\delta(f)$. We have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} |A_n| + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} |B_n| \\
\leq &\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} (|a_n - A_n| + |b_n - B_n|) \\
+ &\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} (|a_n| + |b_n| + |b_1 - B_1| + |b_1|) \\
\leq &\delta + \beta - 1 \\
\leq &1.
\end{aligned}$$

Hence, $F(z)$ is harmonic starlike function. □

3 A family of class preserving integral operator

Let $f(z) = h(z) + \overline{g(z)}$ be defined by (1.1) then $F(z)$ defined by the relation

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t) dt + \overline{\frac{c+1}{z^c} \int_o^z t^{c-1} g(t) dt}, \quad (c > -1). \tag{3.15}$$

Theorem 3.1. *Let $f(z) = h(z) + \overline{g(z)} \in S_H$ be given by (1.4) and $f(z) \in R_H(\lambda, \beta)$ then $F(z)$ be defined by (3.15) also belong to $R_H(\lambda, \beta)$.*

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n$ be in $R_H(\lambda, \beta)$ then by Theorem 2.1, we have

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} |a_n| + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} |b_n| \leq 1.$$

By definition of $F(z)$, we have

$$F(z) = z + \sum_{n=2}^{\infty} \frac{c+1}{c+n}|a_n|z^n + \sum_{n=1}^{\infty} \frac{c+1}{c+n}|b_n|\overline{z}^n.$$

Now

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} \left(\frac{c+1}{c+n}|a_n|\right) + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)} \left(\frac{c+1}{c+n}|b_n|\right)$$
$$\leq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)}|a_n| + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(\beta-1)}|b_n|$$
$$\leq 1.$$

Thus $F(z) \in R_H(\lambda, \beta)$. □

Acknowledgment

The authors are thankful to Prof. K.K. Dixit, Department of Mathematics, Gwalior Institute of Information Technology, Gwalior (M.P.) India for his fruitful comments and suggestions.

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