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Monotonicity properties of the Neumann heat kernel in the ball ☆

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Abstract

A well-known conjecture of R. Laugesen and C. Morpurgo asserts that the diagonal of the Neumann heat kernel of the unit ball $\mathbb{U} \subset \mathbb{R}^n$ is a strictly increasing radial function. In this paper we use probabilistic arguments to settle this conjecture and to prove some inequalities for the Neumann heat kernel in the ball.

Keywords: Neumann heat kernel; Reflecting Brownian motion; Hot Spots conjecture

1. Introduction

We learned from Rodrigo Bañuelos the following conjecture of Richard Laugesen and Carlo Morpurgo which arose in connection with their work on conformal extremals of zeta functions of eigenvalues under Neumann boundary conditions in [9]:

Conjecture 1 *(Laugesen–Morpurgo Conjecture). Let $p_{\mathbb{U}}(t,x,y)$ denote the heat kernel for the Laplacian with Neumann boundary conditions on the unit ball $\mathbb{U} = \{z \in \mathbb{R}^n\colon \ \|z\| < 1\}$ in $\mathbb{R}^n$ $(n \geqslant 1)$.*

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For any $t > 0$ we have

$$p_{\mathbb{U}}(t, x, x) < p_{\mathbb{U}}(t, y, y), \tag{1}$$

for all $x, y \in \overline{\mathbb{U}}$ with $\|x\| < \|y\|$.

Remark 2. The fact that the Laugesen–Morpurgo conjecture is true in the 1-dimensional case is known (see for example [4], Remark 5.4 for an analytic proof, or [12,14] for two different probabilistic proofs). Our main result in Theorem 9 (in the case $n = 1$) gives another probabilistic proof of the Laugesen–Morpurgo conjecture in the 1-dimensional case.

Surprisingly, despite the seemingly simple nature of this conjecture and the fact that it seems to have been well known since 1994, we do not know of any progress on it, aside from some partial related results (see [12–14]). A more recent result related to this conjecture is due to Bañuelos et al. [4], in which the authors show that if we replace $p_{\mathbb{U}}(t, x, y)$ by the transition density of the n-dimensional Bessel process on $(0, 1]$ reflected at 1, then the monotonicity (1) in the Laugesen–Morpurgo conjecture holds in the case $n > 2$, and that this is false for $n = 2$. Since the absolute value of an n-dimensional Brownian motion is a Bessel process of order n, this is equivalent to the monotonicity with respect to $r \in [0, 1]$ of the integral mean

$$r^{n-1} \int_{\partial\mathbb{U}} p_{\mathbb{U}}(t, re_1, ru)\, d\sigma(u),$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$ and σ is the normalized surface measure on $\partial\mathbb{U}$.

In this paper, we use probabilistic arguments (couplings of reflecting Brownian motions) to settle the Laugesen–Morpurgo conjecture. The paper is organized as follows: in Section 2 we present the *mirror coupling* introduced by Kendall [8] and developed by Burdzy et al. ([6], and more recently [1,2]), and we establish the notation.

In Section 3, we begin with a detailed analysis of mirror coupling of reflecting Brownian motions in the unit ball, which shows that the hyperplane of symmetry between the two reflecting Brownian motions (the *mirror* of the coupling) moves towards the origin (Lemma 5).

Using Lemma 5, in Theorem 6 we obtain a comparison result for the transition probabilities of reflecting Brownian motion in the unit ball, which is the key for our proof of the Laugesen–Morpurgo conjecture. Using this result, we obtain a double inequality for Neumann heat kernel of the unit ball (the double inequality in Theorem 7), and as a corollary we conclude with a short proof of Laugesen–Morpurgo conjecture (Theorem 9).

2. Preliminaries

Our proof of Laugesen–Morpurgo conjecture relies on a certain property of the mirror coupling of reflecting Brownian motions in the unit disk and a representation of the Neumann heat kernel as an occupation time density of reflecting Brownian motion. We begin with a presentation of these results.

We denote by $\mathbb{U} = \{z \in \mathbb{R}^n\colon\ \|z\| < 1\}$ the open unit ball in $\mathbb{R}^n$ $(n \geqslant 1)$ and by $\nu(z) = -z$, $z \in \partial\mathbb{U}$, the inward unit vector field on the boundary of $\mathbb{U}$.

Given a hyperplane $\mathcal{H} \subset \mathbb{R}^n$, we say that the points $x, y \in \mathbb{R}^n$ are separated by $\mathcal{H}$ if x and y lie in different components of $\mathbb{R}^n - \mathcal{H}$, and we say that they are not separated by $\mathcal{H}$ otherwise.

We define the reflecting Brownian motion in $\mathbb{U}$ as a solution of the stochastic differential equation:

$$X_t = X_0 + B_t + \frac{1}{2}\int_0^t \nu(X_s)\,dL_s, \quad t \geqslant 0. \tag{2}$$

Formally we have:

Definition 3. X_t is a reflecting Brownian motion in $\mathbb{U}$ starting at $x_0 \in \overline{\mathbb{U}}$ if it satisfies (2), where:

(a) B_t is an n-dimensional Brownian motion started at 0,
(b) L_t is a continuous non-decreasing process which increases only when $X_t \in \partial\mathbb{U}$,
(c) X_t is $(\mathcal{F}_t^B)$-adapted, and almost surely $X_0 = x_0$ and $X_t \in \overline{\mathbb{U}}$ for all $t \geqslant 0$.

Remark 4. For pathwise existence and uniqueness of reflecting Brownian motion in the sense of the above definition see for example [5].

The notion of *mirror coupling* was introduced by W.S. Kendall in [8] in the case of Brownian motions on a complete Riemannian manifold with nonnegative Ricci curvature, and was considered in [15] in the case of reflected processes.

In [6], and more recently in [1] and [2], K. Burdzy et al. gave a detailed construction of the mirror coupling of reflecting Brownian motions X_t and Y_t in a smooth planar domain $D \subset \mathbb{R}^n$, and used it in order to derive various properties related to Neumann eigenvalues and eigenfunctions of the Laplacian on D. The construction of mirror couplings was extended recently by the first author in [10] to the case when the two reflecting Brownian motions live in different smooth domains $D_1, D_2 \subset \mathbb{R}^n$ satisfying an additional assumption (this condition is satisfied in particular if D_1, D_2 have non-tangential boundaries and $D_1 \cap D_2$ is a convex domain). We will briefly present the construction of the mirror coupling in a smooth domain $D \subset \mathbb{R}^n$, and then we give the formal construction in the case of the unit ball $\mathbb{U}$ in $\mathbb{R}^n$, $n \geqslant 1$.

The idea of the mirror coupling is that the two processes behave like ordinary Brownian motions (symmetric with respect to a hyperplane, called the *mirror* of the coupling) when both of them are inside the domain D. When one of the processes hits the boundary, the mirror $\mathcal{M}_t$ gets a minimal push towards the inward unit normal at the corresponding point at the boundary, needed in order to keep both processes in $\overline{D}$.

Considering the coupling time $\tau = \inf\{t > 0\colon\ X_t = Y_t\}$, the mirror coupling evolves as described above for $t \leqslant \tau$, and we let $X_t = Y_t$ for $t \geqslant \tau$ (the two processes move together after the coupling time). For definiteness, for $t \geqslant \tau$ we define the mirror $\mathcal{M}_t$ as the hyperplane parallel to $\mathcal{M}_\tau$ passing through $X_t = Y_t$.

Formally, in the case $D = \mathbb{U}$, given two arbitrarily fixed points $x, y \in \overline{\mathbb{U}}$, we define the mirror coupling of reflecting Brownian motions in the unit ball $\mathbb{U} \subset \mathbb{R}^n$ as a pair $(X_t, Y_t)_{t\geqslant 0}$ of stochastic processes given by

$$\begin{cases} X_t = x + W_t + \int_0^t \nu(X_s)\, dL_s^X, \\ Y_t = y + Z_t + \int_0^t \nu(Y_s)\, dL_s^Y, \end{cases} \tag{3}$$

where W_t is an n-dimensional Brownian motion starting at $W_0 = 0$, Z_t is the mirror image of the Brownian motion W_t with respect to the hyperplane $\mathcal{M}_t$ of symmetry between X_t and Y_t, that is

$$Z_t = W_t - 2\int_0^t \frac{X_s - Y_s}{\|X_s - Y_s\|^2}(X_s - Y_s) \cdot dW_s, \tag{4}$$

and L_t^X and L_t^Y denote the boundary local times of the reflecting Brownian motions X_t and respectively Y_t.

The processes X_t and Y_t evolve according to (3) above for $t \leqslant \tau$, where τ is the coupling time

$$\tau = \inf\{t > 0\colon\ X_t = Y_t\} \in \mathbb{R} \cup \{\infty\},$$

and they evolve together after the coupling time (i.e. $X_t = Y_t$ for $t \geqslant \tau$).

3. Main results

The key for proving the Laugesen–Morpurgo conjecture (Conjecture 1) is the double inequality (14) in Theorem 7, which in turn relies on proving the following inequality:

$$p_{\mathbb{U}}(t, y, z) \leqslant p_{\mathbb{U}}(t, x, z), \quad t > 0, \tag{5}$$

for all $x, y, z \in \mathbb{U}$ satisfying $\|x - z\| \leqslant \|y - z\|$ and $\|y\| \leqslant \|x\|$.

Consider a mirror coupling X_t, Y_t of reflecting Brownian motions in $\mathbb{U}$ given by (3)–(4), with starting points $X_0 = x, Y_0 = y \in \overline{\mathbb{U}}$.

For $t < \tau = \inf\{t > 0\colon\ X_t = Y_t\}$, the mirror $\mathcal{M}_t$ of the coupling (the hyperplane of symmetry between X_t and Y_t) is given by

$$\mathcal{M}_t = \left\{z \in \mathbb{R}^n\colon\ \left(z - \frac{X_t + Y_t}{2}\right) \cdot (X_t - Y_t) = 0\right\}. \tag{6}$$

The idea for proving the inequality (5) is that the mirror $\mathcal{M}_t$ moves towards the origin, in the sense of Lemma 5 below. This property is a rigorous version of Example 4.5 in [6], used by the authors to prove the efficiency of the mirror coupling in the case of the unit disk.

Lemma 5. *Let X_t, Y_t be a mirror coupling of reflecting Brownian motions in $\mathbb{U}$ with starting points $X_0 = x$, $Y_0 = y \in \overline{\mathbb{U}}$, and let $\tau = \inf\{t > 0\colon\ X_t = Y_t\}$ be the coupling time and $\tau_1 = \inf\{t > 0\colon\ 0 \in \mathcal{M}_t\}$.*

For all times $t < \tau \wedge \tau_1$, the mirror $\mathcal{M}_t$ moves towards the origin, in such a way that if a point $P \in \mathbb{U}$ and the origin are separated by $\mathcal{M}_{t_1}$ for some $t_1 \in [0, \tau \wedge \tau_1)$, then the point P and the origin are separated by $\mathcal{M}_{t_2}$ for all $t_2 \in [t_1, \tau \wedge \tau_1)$ (see Fig. 1).

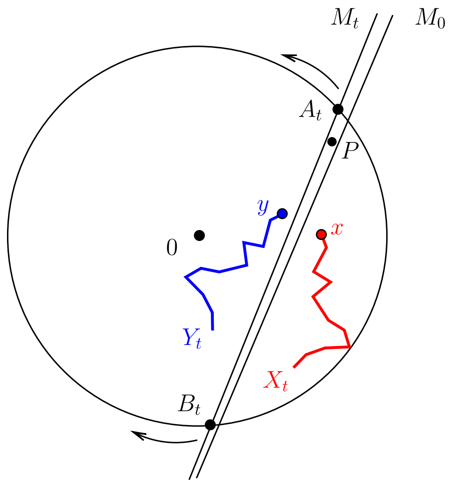

Fig. 1. Mirror coupling of reflecting Brownian motions in the unit disk (the case $n = 2$).

Proof. If $\|x\| = \|y\|$, then $\tau_1 = 0$ and there is nothing to prove in this case (the mirror $\mathcal{M}_0$ passes through the origin). Without loss of generality we may therefore assume that $\|x\| > \|y\|$.

Setting

$$\begin{cases} U_t = X_t - Y_t, \\ V_t = X_t + Y_t, \end{cases} \quad t \geqslant 0, \tag{7}$$

from the definition (3)–(4) of the mirror coupling we obtain

$$\begin{cases} U_t^i = x^i - y^i + W_t^i - Z_t^i - \int_0^t X_s^i \, dL_s^X + \int_0^t Y_s^i \, dL_s^Y, \\ V_t^i = x^i + y^i + W_t^i + Z_t^i - \int_0^t X_s^i \, dL_s^X - \int_0^t Y_s^i \, dL_s^Y, \end{cases} \quad i = 1, \ldots, n,$$

for all $t \leqslant \tau$, where the superscript i indicates the ith Cartesian coordinate of the given point.

Using the definition (4) of Z_t, we have

$$\begin{cases} U_t^i = x^i - y^i + 2\int_0^t \frac{U_s^i}{\|U_s\|^2} U_s \cdot dW_s - \int_0^t X_s^i \, dL_s^X + \int_0^t Y_s^i \, dL_s^Y, \\ V_t^i = x^i + y^i + 2W_t^i - 2\int_0^t \frac{U_s^i}{\|U_s\|^2} U_s \cdot dW_s - \int_0^t X_s^i \, dL_s^X - \int_0^t Y_s^i \, dL_s^Y, \end{cases} \tag{8}$$

for all $i = 1, \ldots, n$ and $t < \tau$, and therefore we obtain the following formulae for the quadratic variation of the processes U and V:

$$\begin{cases} \langle U^i, U^j \rangle_t = 4 \int_0^t \frac{U_s^i U_s^j}{\|U_s\|^2} \, ds, \\ \langle V^i, V^j \rangle_t = 4 \int_0^t \delta_{ij} - \frac{U_s^i U_s^j}{\|U_s\|^2} \, ds, \\ \langle U^i, V^j \rangle_t = 0, \end{cases} \quad i, j, = 1, \ldots, n. \tag{9}$$

Note that since $\|X_0\| = \|x\| > \|y\| = \|Y_0\|$, it follows that for all $t < \tau \wedge \tau_1$ we have

$$U_t \cdot V_t = (X_t - Y_t) \cdot (X_t + Y_t) = \|X_t\|^2 - \|Y_t\|^2 > 0, \tag{10}$$

and therefore for $t < \tau \wedge \tau_1$ we may define the process A_t by

$$A_t = \frac{2}{U_t \cdot V_t} U_t. \tag{11}$$

We will first show that for $t < \tau \wedge \tau_1$ the components of the process A_t are processes of bounded variation, satisfying

$$dA_t^i = \frac{2}{U_t \cdot V_t} \left(A_t^i - \frac{U_t^i + V_t^i}{2} \right) dL_t^X, \quad i = 1, \ldots, n. \tag{12}$$

Applying the Itô formula to the C^2 function $f(u, v) = \frac{u^i}{u \cdot v}$ and to the processes U_t and V_t, we have

$$\begin{aligned} \frac{1}{2} dA_t^i &= d\left(\frac{U_t^i}{U_t \cdot V_t} \right) \\ &= \frac{1}{(U_t \cdot V_t)^2} \sum_{j=1}^n \left(\left(\delta_{ij} U_t \cdot V_t - U_t^i V_t^j \right) dU_t^j - U_t^i U_t^j \, dV_t^j \right) \\ &\quad + \frac{1}{2(U_t \cdot V_t)^3} \sum_{j,k=1}^n \left(2U_t^i V_t^j V_t^k - \delta_{ij} V_t^k U_t \cdot V_t - \delta_{ik} V_t^j U_t \cdot V_t \right) d\langle U^j, U^k \rangle_t \\ &\quad + \frac{1}{2(U_t \cdot V_t)^3} \sum_{j,k=1}^n \left(2U_t^i U_t^j U_t^k \right) d\langle V^j, V^k \rangle_t \\ &\quad + \frac{1}{(U_t \cdot V_t)^3} \sum_{j,k=1}^n \left(2U_t^i U_t^k V_t^j - \delta_{ij} U_t^k U_t \cdot V_t - \delta_{jk} U_t^i U_t \cdot V_t \right) d\langle U^j, V^k \rangle_t. \end{aligned}$$

Using the relations in (8) it can be seen that the martingale part in the above expression reduces to zero, and combining with (9) we obtain

$$
\begin{aligned}
\frac{1}{2}\,dA_t^i &= \frac{1}{(U_t\cdot V_t)^2}\sum_{j=1}^{n}\bigl(\bigl(\delta_{ij}U_t\cdot V_t - U_t^iV_t^j\bigr)\bigl(-X_t^j\,dL_t^X + Y_t^j\,dL_t^Y\bigr)\bigr)\\
&\quad - \frac{1}{(U_t\cdot V_t)^2}\sum_{j=1}^{n}\bigl(U_t^iU_t^j\bigl(-X_t^j\,dL_t^X - Y_t^j\,dL_t^Y\bigr)\bigr)\\
&\quad + \frac{1}{2(U_t\cdot V_t)^3}\sum_{j,k=1}^{n}\bigl(2U_t^iV_t^jV_t^k - \delta_{ij}V_t^kU_t\cdot V_t - \delta_{ik}V_t^jU_t\cdot V_t\bigr)4\frac{U_t^jU_t^k}{\|U_t\|^2}\,dt\\
&\quad + \frac{1}{2(U_t\cdot V_t)^3}\sum_{j,k=1}^{n}\bigl(2U_t^iU_t^jU_t^k\bigr)4\left(\delta_{jk} - \frac{U_t^jU_t^k}{\|U_t\|^2}\right)dt\\
&= \frac{1}{(U_t\cdot V_t)^2}\sum_{j=1}^{n}\bigl(U_t^iU_t^j + U_t^iV_t^j - \delta_{ij}U_t\cdot V_t\bigr)X_t^j\,dL_t^X\\
&\quad + \frac{1}{(U_t\cdot V_t)^2}\sum_{j=1}^{n}\bigl(U_t^iU_t^j - U_t^iV_t^j + \delta_{ij}U_t\cdot V_t\bigr)Y_t^j\,dL_t^Y.
\end{aligned}
$$

Using the fact that $L_t^Y \equiv 0$ on the time interval $[0,\tau\wedge\tau_1)$ (the process Y_t cannot reach the boundary $\partial\mathbb{U}$ before either coupling first with X_t or before the first time when $\|X_t\| = \|Y_t\| = 1$, that is before $0\in\mathcal{M}_t$), and that L_t^X increases only when $X_t\in\partial\mathbb{U}$, that is only when $\|X_t\| = \|\frac{U_t+V_t}{2}\| = 1$, we obtain

$$
\begin{aligned}
\frac{1}{2}\,dA_t^i &= \frac{1}{(U_t\cdot V_t)^2}\sum_{j=1}^{n}\bigl(U_t^iU_t^j + U_t^iV_t^j - \delta_{ij}U_t\cdot V_t\bigr)X_t^j\,dL_t^X\\
&= \frac{U_t^i}{2(U_t\cdot V_t)^2}\sum_{j=1}^{n}\bigl(U_t^j + V_t^j\bigr)^2\,dL_t^X - \frac{U_t^i+V_t^i}{2U_t\cdot V_t}\,dL_t^X\\
&= \frac{U_t^i}{2(U_t\cdot V_t)^2}\|U_t+V_t\|^2\,dL_t^X - \frac{U_t^i+V_t^i}{2U_t\cdot V_t}\,dL_t^X\\
&= \frac{2U_t^i}{(U_t\cdot V_t)^2}\,dL_t^X - \frac{U_t^i+V_t^i}{2U_t\cdot V_t}\,dL_t^X\\
&= \frac{1}{U_t\cdot V_t}\left(A_t^i - \frac{U_t^i+V_t^i}{2}\right)dL_t^X,
\end{aligned}
$$

thus proving the claim (12).

To prove the claim of the lemma, assume by contradiction that there exists a point $P\in\mathbb{U}$ and times $0<t_1<t_2<\tau\wedge\tau_1$ such that the point P and the origin are separated by $\mathcal{M}_{t_1}$, but are not separated by $\mathcal{M}_{t_2}$. By eventually changing the point P, without loss of generality we may assume that $P\notin\mathcal{M}_{t_2}$, and using (6) and (7) we obtain

$$
P\cdot U_{t_2} - \frac{1}{2}U_{t_2}\cdot V_{t_2} < 0 < P\cdot U_{t_1} - \frac{1}{2}U_{t_1}\cdot V_{t_1},
$$

or equivalently (recall the definition (11) of the process A_t and that $U_t \cdot V_t > 0$ for $t \in [0, \tau \wedge \tau_1)$)

$$P \cdot A_{t_2} < 1 < P \cdot A_{t_1}.$$

Setting $t_0 = \inf\{t > t_1\colon\ P \cdot A_t < 1\} \in (t_1, t_2)$ and using (12), we obtain

$$\begin{aligned} P \cdot A_{t_0} &= P \cdot A_{t_1} + \int_{t_1}^{t_0} P \cdot dA_t \\ &= P \cdot A_{t_1} + \int_{t_1}^{t_0} \frac{2}{U_t \cdot V_t}\left(P \cdot A_t - \frac{1}{2} P \cdot (U_t + V_t)\right) dL_t^X \\ &\geqslant P \cdot A_{t_1} \\ &> 1, \end{aligned}$$

since $P \cdot A_t \geqslant 1$ for $t \in [t_1, t_0]$ and

$$\left| \frac{1}{2} P \cdot (U_t + V_t) \right| = |P \cdot X_t| \leqslant \|P\| \|X_t\| \leqslant 1.$$

By the continuity of the process A_t and the choice of t_0 we must also have $P \cdot A_{t_0} = 1$, contradiction which concludes the proof of the lemma. □

From the previous lemma we obtain the following:

Theorem 6. *For any points $x, y \in \overline{\mathbb{U}}$ with $\|y\| \leqslant \|x\|$ and any $z \in \overline{\mathbb{U}}$ such that $\|x - z\| \leqslant \|y - z\|$, we have*

$$P^y\big(\|Y_t - z\| < \varepsilon\big) \leqslant P^x\big(\|X_t - z\| < \varepsilon\big), \tag{13}$$

for any $t \geqslant 0$ and $\varepsilon \in (0, \min\{\|z\|, 1 - \|z\|\})$, where X_t and Y_t are reflecting Brownian motions in $\mathbb{U}$ starting at x, respectively y, and P^x, P^y denote the corresponding probability measures.

Proof. Without loss of generality we may assume that x and y are distinct points.

Let X_t, Y_t be a mirror coupling of reflecting Brownian motions in $\mathbb{U}$ with starting points $X_0 = x$ and $Y_0 = y$, and let τ be the coupling time and $\tau_1 = \inf\{t > 0\colon\ 0 \in \mathcal{M}_t\}$.

If $\mathcal{M}_t$ separates X_t and z for some $t < \tau \wedge \tau_1$, there exists a point $P \in \mathbb{U}$ such that the origin and the point P are separated by $\mathcal{M}_0$, but are not separated by $\mathcal{M}_t$, contradicting Lemma 5. It follows that the mirror $\mathcal{M}_t$ does not separate the points X_t and z for all $t < \tau \wedge \tau_1$, and therefore the distance from X_t to z is not greater than the distance from Y_t to z in this case.

Since for $t \geqslant \tau \wedge \tau_1$, either the processes X_t and Y_t are symmetric with respect to the (fixed) hyperplane $\mathcal{M}_{\tau \wedge \tau_1}$ passing through the origin (for $t \in (\tau \wedge \tau_1, \tau)$), or they have coupled (for $t \in (\tau, \infty)$), it follows that the distance from X_t to z is also not greater than the distance from Y_t to z.

In all cases we obtained that the distance from X_t to z is not greater than the distance from Y_t to z, and the claim follows. □

Denoting by $p_{\mathbb{U}}(t, x, y)$ the heat kernel for the Laplacian with Neumann boundary conditions on the unit ball $\mathbb{U} \subset \mathbb{R}^n$ (or equivalently, the transition density of reflecting Brownian motion in $\mathbb{U}$), we can now prove the following double inequality:

Theorem 7. *For any $x \in \mathbb{U} - \{0\}$, $r \in (0, \min\{\|x\|, 1 - \|x\|\})$ and $t > 0$ we have*

$$\int_{\partial \mathbb{U}} p_{\mathbb{U}}(t, x + ru, x)\, d\sigma(u) \leqslant p_{\mathbb{U}}\left(t, x + r\frac{x}{\|x\|}, x\right) \leqslant p_{\mathbb{U}}\left(t, x + r\frac{x}{\|x\|}, x + r\frac{x}{\|x\|}\right), \tag{14}$$

where σ is the normalized surface measure on $\partial \mathbb{U}$.

Proof. Using the continuity of the transition density $p_{\mathbb{U}}(t, x, y)$ of reflecting Brownian motion in the space variable, it follows $p_{\mathbb{U}}(t, x, y)$ can be written as

$$p_{\mathbb{U}}(t, x, y) = \lim_{\varepsilon \searrow 0} \frac{1}{c_n \varepsilon^n} \int_{\|y - z\| < \varepsilon} p_{\mathbb{U}}(t, x, z)\, dz = \lim_{\varepsilon \searrow 0} \frac{1}{c_n \varepsilon^n} P^x\big(\|W_t - y\| < \varepsilon\big), \tag{15}$$

where W_t is a reflecting Brownian motion in the unit ball $\mathbb{U} \subset \mathbb{R}^n$ starting at $W_0 = x$, P^x denotes the corresponding probability measure and $c_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the unit ball $\mathbb{U} \subset \mathbb{R}^n$.

For $u \in \mathbb{U}$ fixed, it is easy to see that the hypotheses of Theorem 6 are verified if we replace x, y and z respectively by $x + r\frac{x}{\|x\|}$, $x + ru$ and x. From this theorem, and combining with the above representation, we obtain

$$p_{\mathbb{U}}(t, x + ru, x) \leqslant p_{\mathbb{U}}\left(t, x + r\frac{x}{\|x\|}, x\right), \quad u \in \mathbb{U}, \tag{16}$$

and integrating with respect to $u \in \mathbb{U}$ we obtain the left inequality in (14).

The right inequality in (14) can be proved similarly, replacing x, y and z in Theorem 6 respectively by $x + r\frac{x}{\|x\|}$, x and $x + r\frac{x}{\|x\|}$, and using the symmetry of $p_{\mathbb{U}}(t, x, y)$ in $x, y \in \mathbb{U}$. □

Remark 8. The inequality (16) obtained in the previous proof might be of independent interest, and can be interpreted as an extremal property of reflecting Brownian motion in the unit ball $\mathbb{U}$, as follows:

$$\max_{y \in \mathbb{U}:\ \|y - x\| = r} p_{\mathbb{U}}(t, x, y) = p_{\mathbb{U}}\left(t, x, x + r\frac{x}{\|x\|}\right),$$

that is, among all reflecting Brownian motions in the unit ball $\mathbb{U}$ with starting points on the sphere $\{y \in \mathbb{R}^n\colon \|y - x\| = r\}$, the reflecting Brownian motion starting closest to the boundary of $\mathbb{U}$ (i.e. at the point $x + r\frac{x}{\|x\|}$) is most likely to return to (a neighborhood of) x. This extremal property of reflecting Brownian motion is the key of our proof of the Laugesen–Morpurgo conjecture.

As a corollary of the above theorem, we obtain the following resolution of the Laugesen–Morpurgo conjecture:

Theorem 9. *Let $p_{\mathbb{U}}(t,x,y)$ denote the heat kernel for the Laplacian with Neumann boundary conditions on the unit ball $\mathbb{U}=\{z\in\mathbb{R}^n\colon \|z\|<1\}$ in $\mathbb{R}^n$ $(n\geqslant 1)$.*

For any $t>0$ we have

$$p_{\mathbb{U}}(t,x,x)<p_{\mathbb{U}}(t,y,y), \tag{17}$$

for all $x,y\in\overline{\mathbb{U}}$ with $\|x\|<\|y\|$.

Proof. First note that for $t>0$ fixed, by the radial symmetry of the problem it follows that $p_{\mathbb{U}}(t,x,x)$ is a function of $\|x\|\in[0,1]$.

For an arbitrarily fixed $x\in\mathbb{U}-\{0\}$, from Theorem 7 we obtain

$$\begin{aligned} p_{\mathbb{U}}\left(t,x+r\frac{x}{\|x\|},x+r\frac{x}{\|x\|}\right)-p_U(t,x,x) &\geqslant \int_{\partial\mathbb{U}} p_{\mathbb{U}}(t,x+ru,x)\,d\sigma(u)-p_{\mathbb{U}}(t,x,x)\\ &= \int_{\partial\mathbb{U}} \big(p_{\mathbb{U}}(t,x+ru,x)-p_U(t,x,x)\big)\,d\sigma(u), \end{aligned}$$

for any $r\in(0,\min\{\|x\|,1-\|x\|\})$. Dividing by r and passing to the limit with $r\searrow 0$, we obtain

$$\begin{aligned} \frac{d}{d\|x\|}p_{\mathbb{U}}(t,x,x) &= \lim_{r\searrow 0}\frac{p_{\mathbb{U}}(t,x+r\frac{x}{\|x\|},x+r\frac{x}{\|x\|})-p_U(t,x,x)}{r}\\ &\geqslant \lim_{r\searrow 0}\int_{\partial\mathbb{U}}\frac{p_{\mathbb{U}}(t,x+ru,x)-p_{\mathbb{U}}(t,x,x)}{r}\,d\sigma(u). \end{aligned}$$

By bounded convergence theorem ($p_{\mathbb{U}}(t,\cdot,x)$ is a C^2 function in the second variable, hence $\nabla p_U(t,\cdot,x)$ is bounded in a neighborhood of x), we obtain

$$\frac{d}{d\|x\|}p_{\mathbb{U}}(t,x,x)\geqslant\int_{\partial\mathbb{U}}\nabla p_U(t,x,x)\cdot u\,d\sigma(u)=0,$$

where we denoted by $\nabla p_{\mathbb{U}}$ the gradient of $\nabla p_{\mathbb{U}}(t,\cdot,x)$ in the second variable.

Since $x\in\mathbb{U}-\{0\}$ was arbitrarily fixed, we have

$$\frac{d}{d\|x\|}p_{\mathbb{U}}(t,x,x)\geqslant 0,\quad x\in(0,1),$$

which shows that $p_{\mathbb{U}}(t,x,x)$ is a non-decreasing function of $\|x\|\in(0,1)$, and by continuity this also holds for $\|x\|\in[0,1]$.

Since $p_{\mathbb{U}}(t,x,x)$ is the diagonal of a heat kernel of an operator with real analytic coefficients, $p_{\mathbb{U}}(t,x,x)$ is a real analytic function. If $p_{\mathbb{U}}(t,x,x)$ were constant on a non-empty open subset of $\overline{\mathbb{U}}$, then it would be identically constant in $\overline{\mathbb{U}}$, which is impossible (it can be shown that $p_{\mathbb{U}}(t,0,0)<p_{\mathbb{U}}(t,1,1)$ for any $t>0$). This, together with the fact that $p_{\mathbb{U}}(t,x,x)$ is a non-decreasing radial function shows that $p_{\mathbb{U}}(t,x,x)$ is in fact a strictly increasing radial function for any $t>0$, concluding the proof. □

We conclude with the remark that the Laugesen–Morpurgo conjecture implies the famous Hot Spots conjecture of J. Rauch (see for example [3,7,11]) in the case of the unit ball $\mathbb{U} \subset \mathbb{R}^n$, and that extending the Laugesen–Morpurgo conjecture to more general domains would also give a resolution of the Hot Spots conjecture for the corresponding domains (the Hot Spots conjecture is only partially solved at the present moment).

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