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Quasiconformality of harmonic extensions

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Dedicated to Professor Haakon Waadeland on the occasion of his 70th birthday

Abstract

Continued from Partyka and Sakan (Bull. Soc. Sci. Letters Łódź 47 (1997) 51–63) this paper aims at giving necessary and sufficient conditions on sense-preserving homeomorphisms of the unit circle for the quasiconformality of their harmonic extensions to the unit disk. In particular, all such homeomorphisms with a bounded derivative are well characterized. In consequence, a generalization of Martio's result is obtained. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $\mathrm{Hom}^+(\boldsymbol{T})$ stand for the class of all sense-preserving homeomorphic self-mappings γ of the unit circle $\boldsymbol{T}:=\{z\in\mathbb{C}\colon |z|=1\}$ and let $L^1(\boldsymbol{T})$ be the space of all complex-valued functions Lebesgue integrable on $\boldsymbol{T}$. According to the famous Radó–Kneser–Choquet theorem (cf. e.g. [1, p. 22]) the Poisson extension $\mathscr{P}[\gamma]$ of $\gamma\in\mathrm{Hom}^+(\boldsymbol{T})$ to the unit disk $\Delta:=\{z\in\mathbb{C}\colon |z|<1\}$ is a harmonic and homeomorphic self-mapping of Δ, where for every $f\in L^1(\boldsymbol{T})$,

$$\mathscr{P}[f](z):=\frac{1}{2\pi}\int_{\boldsymbol{T}} f(u)\operatorname{Re}\frac{u+z}{u-z}|\mathrm{d}u|,\quad z\in\Delta. \tag{1.1}$$

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Moreover, $\mathscr{P}[\gamma]$ is sense-preserving, i.e. the Jacobian $\mathscr{J}[\mathscr{P}[\gamma]] = |\partial\mathscr{P}[\gamma]|^2 - |\bar{\partial}\mathscr{P}[\gamma]|^2$ is positive on $\varDelta$, where $\partial := \frac{1}{2}((\partial/\partial x) - i(\partial/\partial y))$ and $\bar{\partial} := \frac{1}{2}((\partial/\partial x) + i(\partial/\partial y))$ are the so-called formal derivatives operators; cf. e.g. [6, pp. 42–43]. For $K \geqslant 1$ let $\mathbb{Q}_{\boldsymbol{T}}(K)$ be the class of all $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ which admit a K-quasiconformal (K-qc. for short) extension to $\varDelta$, and let $\mathbb{Q}_{\boldsymbol{T}} := \bigcup_{K \geqslant 1} \mathbb{Q}_{\boldsymbol{T}}(K)$. Obviously, by definition any $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ belongs to $\mathbb{Q}_{\boldsymbol{T}}$ provided $\mathscr{P}[\gamma]$ is a qc. mapping. Thus the natural question is whether the Poisson extension $\mathscr{P}[\gamma]$ is qc. provided $\gamma \in \mathbb{Q}_{\boldsymbol{T}}$. The answer is negative. Let $\mathbb{Q}^*_{\boldsymbol{T}}$ denote the class of all $\gamma \in \mathbb{Q}_{\boldsymbol{T}}$ such that $\mathscr{P}[\gamma]$ is a qc. mapping. Yang has shown in [10] that $\mathbb{Q}^*_{\boldsymbol{T}} \neq \mathbb{Q}_{\boldsymbol{T}}$. Next, Laugesen improved this by showing (cf. [4, Corollary 3]) that for each $K > 1$ there exists $\gamma \in \mathbb{Q}_{\boldsymbol{T}}(K) \backslash \mathbb{Q}^*_{\boldsymbol{T}}$. Several very simple and explicit examples of $\gamma \in \mathbb{Q}_{\boldsymbol{T}}(K) \backslash \mathbb{Q}^*_{\boldsymbol{T}}$ for each $K > 1$ were presented in [7, Examples 4.1–4.3]. The reader can find in [7] general simple techniques of constructing for each $K > 1$ an explicit homeomorphism $\gamma \in \mathbb{Q}_{\boldsymbol{T}}(K) \backslash \mathbb{Q}^*_{\boldsymbol{T}}$. This shows that the class $\mathbb{Q}^*_{\boldsymbol{T}}$ is substantially smaller than $\mathbb{Q}_{\boldsymbol{T}}$. However, the question how big is the class $\mathbb{Q}^*_{\boldsymbol{T}}$ is still open. In other words, the problem of characterizing homeomorphisms $\gamma \in \mathbb{Q}^*_{\boldsymbol{T}}$ arises. So far as the authors know, Martio was the first who studied this problem provided $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ is sufficiently smooth; cf. [5]. In what follows, we proceed with the study of the problem for irregular γ. In fact, our paper is a continuation of [7]. In Section 2 we derive some lower estimate of the radial limiting values of the Jacobian $\mathscr{J}[\mathscr{P}[\gamma]]$ on the boundary $\boldsymbol{T}$. Then as an application we present several necessary and sufficient conditions for $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ to belong to $\mathbb{Q}^*_{\boldsymbol{T}}$. Section 3 deals with the general case of arbitrary $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$. In Section 4 we restrict our attention to $\gamma \in \mathrm{Hom}^+(\mathbf{T})$ with a bounded derivative, in particular to $\gamma \in \mathrm{Hom}^+(\mathbf{T})$ being a Lipschitz function. We show that a homeomorphism with bounded derivative has a qc. harmonic extension if and only if its derivative is bounded away from zero and the Cauchy–Stieltjes transform of its differential is bounded; see Theorem 4.1. In Section 5 we give a few examples and compare our results with Martio's theorem; cf. [5, Theorem 1]. These results were presented by the first named author on the conference "Continued Fractions and Geometric Function Theory", Trondheim (Norway), June 24–28, 1997. In general, we adopt all notations from our paper [7]. Thus for a function $F : \varDelta \to \mathbb{C}$ (resp. $F : \mathbb{C} \backslash \overline{\varDelta} \to \mathbb{C}$) and $z \in \boldsymbol{T}$ we define

$$\hat{\partial}_r^- F(z) := \lim_{t \to 1^-} F(tz) \quad \text{(resp. } \hat{\partial}_r^+ F(z) := \lim_{t \to 1^+} F(tz)\text{)},$$

whenever the limit exists, while $\hat{\partial}_r^- F(z) := 0$ (resp. $\hat{\partial}_r^+ F(z) := 0$) otherwise.

2. A lower estimate of $\hat{\partial}_r^- \mathscr{J}[\mathscr{P}[\gamma]]$

Given a function $f : \boldsymbol{T} \to \mathbb{C}$ and $z \in \boldsymbol{T}$ we define

$$f'(z) := \lim_{\boldsymbol{T} \ni u \to z} \frac{f(u) - f(z)}{u - z}$$

provided the limit exists, while $f'(z) := 0$ otherwise. If the limit exists we say that f has the derivative $f'(z)$ at z. As in [7], for any $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ we can consider the Riemann–Stieltjes integral

$$\mathscr{C}_\gamma(z) := \frac{1}{2\pi\mathrm{i}} \int_{\boldsymbol{T}} \frac{\mathrm{d}\gamma(u)}{u - z}, \quad z \in \varDelta \cup (\mathbb{C} \backslash \bar{\varDelta}) \quad \text{and} \quad \mathscr{C}_\gamma(\infty) := 0.$$

Lemma 2.1. *If $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ then for a.e. $z \in \boldsymbol{T}$, both the functions $\partial\mathscr{P}[\gamma]$ and $\bar{\partial}\mathscr{P}[\gamma]$ have nontangential limiting values at z and*

$$\lim_{r\to 1^-} \partial\mathscr{P}[\gamma](rz) = \frac{\bar{z}}{2}\lim_{r\to 1^-}\left[\frac{\mathrm{d}}{\mathrm{d}r}\mathscr{P}[\gamma](rz) + z\gamma'(z)\right] = \frac{\bar{z}}{2}\lim_{r\to 1^-}\left[\frac{\gamma(z)-\mathscr{P}[\gamma](rz)}{1-r} + z\gamma'(z)\right] \tag{2.1}$$

and

$$\lim_{r\to 1^-} \bar{\partial}\mathscr{P}[\gamma](rz) = \frac{z}{2}\lim_{r\to 1^-}\left[\frac{\mathrm{d}}{\mathrm{d}r}\mathscr{P}[\gamma](rz) - z\gamma'(z)\right] = \frac{z}{2}\lim_{r\to 1^-}\left[\frac{\gamma(z)-\mathscr{P}[\gamma](rz)}{1-r} - z\gamma'(z)\right]. \tag{2.2}$$

Proof. As is remarked just below (3.9), $\mathscr{C}_\gamma$ has nontangential limiting values a.e. on $\boldsymbol{T}$. Hence by (3.1) both the functions $\partial\mathscr{P}[\gamma]$ and $\bar{\partial}\mathscr{P}[\gamma]$ have nontangential limiting values a.e. on $\boldsymbol{T}$, as well. From (1.1) it follows that for every $r \in (0,1)$ and for every $z \in \boldsymbol{T}$,

$$\begin{aligned}
\partial\mathscr{P}[\gamma](rz) &= \frac{1}{2\pi}\int_{\boldsymbol{T}} \gamma(u)\frac{u}{(u-rz)^2}|\mathrm{d}u| = -\frac{1}{2\pi}\frac{1}{2\mathrm{i}rz}\int_0^{2\pi}\gamma(\mathrm{e}^{\mathrm{i}t})\frac{-2\mathrm{i}rz\mathrm{e}^{\mathrm{i}t}}{(\mathrm{e}^{\mathrm{i}t}-rz)^2}\,\mathrm{d}t \\
&= -\frac{1}{2\pi}\frac{1}{2\mathrm{i}rz}\int_0^{2\pi}\gamma(\mathrm{e}^{\mathrm{i}t})\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{e}^{\mathrm{i}t}+rz}{\mathrm{e}^{\mathrm{i}t}-rz}\right)\mathrm{d}t \\
&= -\frac{1}{2\pi}\frac{1}{2\mathrm{i}rz}\left[\int_0^{2\pi}\gamma(\mathrm{e}^{\mathrm{i}t})\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\,\frac{\mathrm{e}^{\mathrm{i}t}+rz}{\mathrm{e}^{\mathrm{i}t}-rz}\,\mathrm{d}t + i\int_0^{2\pi}\gamma(\mathrm{e}^{\mathrm{i}t})\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}\,\frac{\mathrm{e}^{\mathrm{i}t}+rz}{\mathrm{e}^{\mathrm{i}t}-rz}\,\mathrm{d}t\right] \\
&= \frac{1}{2}\frac{1}{2\pi\mathrm{i}rz}\left[\int_0^{2\pi}\mathrm{Re}\,\frac{\mathrm{e}^{\mathrm{i}t}+rz}{\mathrm{e}^{\mathrm{i}t}-rz}\mathrm{d}\gamma(\mathrm{e}^{\mathrm{i}t}) + \mathrm{i}r\int_0^{2\pi}\gamma(\mathrm{e}^{\mathrm{i}t})\mathrm{Re}\,\frac{2z\mathrm{e}^{\mathrm{i}t}}{(\mathrm{e}^{\mathrm{i}t}-rz)^2}\,\mathrm{d}t\right] \\
&= \frac{1}{2}\frac{1}{2\pi\mathrm{i}rz}\int_0^{2\pi}\mathrm{Re}\,\frac{\mathrm{e}^{\mathrm{i}t}+rz}{\mathrm{e}^{\mathrm{i}t}-rz}\mathrm{d}\gamma(\mathrm{e}^{\mathrm{i}t}) + \frac{1}{2}\frac{1}{2\pi z}\int_0^{2\pi}\gamma(\mathrm{e}^{\mathrm{i}t})\frac{\partial}{\partial r}\mathrm{Re}\,\frac{\mathrm{e}^{\mathrm{i}t}+rz}{\mathrm{e}^{\mathrm{i}t}-rz}\,\mathrm{d}t \\
&= \frac{1}{2}\frac{1}{2\pi\mathrm{i}rz}\int_0^{2\pi}\mathrm{Re}\,\frac{\mathrm{e}^{\mathrm{i}t}+rz}{\mathrm{e}^{\mathrm{i}t}-rz}\mathrm{d}\gamma(\mathrm{e}^{\mathrm{i}t}) + \frac{1}{2z}\frac{\mathrm{d}}{\mathrm{d}r}\mathscr{P}[\gamma](rz).
\end{aligned} \tag{2.3}$$

Assume γ has the derivative $\gamma'(z)$ at $z = \mathrm{e}^{\mathrm{i}\theta} \in \boldsymbol{T}$. Then $(\mathrm{d}/\mathrm{d}t)\gamma(\mathrm{e}^{\mathrm{i}t})_{|t=\theta} = \mathrm{i}z\gamma'(z)$ and by the Fatou theorem on Poisson integrals (cf. [9, Theorem 11.12]) we have

$$\lim_{r\to 1^-}\frac{1}{2}\frac{1}{2\pi\mathrm{i}rz}\int_0^{2\pi}\mathrm{Re}\,\frac{\mathrm{e}^{\mathrm{i}t}+rz}{\mathrm{e}^{\mathrm{i}t}-rz}\mathrm{d}\gamma(\mathrm{e}^{\mathrm{i}t}) = \frac{1}{2}\gamma'(z). \tag{2.4}$$

By (2.3) and (2.4), the limit $\lim_{r\to 1^-}\partial\mathscr{P}[\gamma](rz)$ exists iff the limit $\lim_{r\to 1^-}(1/2z)(\mathrm{d}/\mathrm{d}r)\mathscr{P}[\gamma](rz)$ exists. As shown in the proof of [7, Theorem 1.1] both the radial limits

$$\lim_{r\to 1^-}\partial\mathscr{P}[\gamma](rz) = \hat{\partial}_r^-\mathscr{C}_\gamma(z) \quad \text{and} \quad \lim_{r\to 1^-}\bar{\partial}\mathscr{P}[\gamma](rz) = z^2\hat{\partial}_r^+\mathscr{C}_\gamma(z) \tag{2.5}$$

exist for a.e. $z \in \boldsymbol{T}$. Since γ has the derivative $\gamma'(z)$ for a.e. $z \in \boldsymbol{T}$, we conclude from (2.3), (2.4) and (2.5) that the first equality in (2.1) holds. If for $z \in \boldsymbol{T}$ the limit $\lim_{r\to 1^-}(\mathrm{d}/\mathrm{d}r)\mathscr{P}[\gamma](rz)$ exists, then

$$\lim_{r\to 1^-}\frac{\gamma(z)-\mathscr{P}[\gamma](rz)}{1-r} = \lim_{r\to 1^-}\frac{1}{1-r}\int_r^1\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{P}[\gamma](tz)\,\mathrm{d}t = \lim_{r\to 1^-}\frac{\mathrm{d}}{\mathrm{d}r}\mathscr{P}[\gamma](rz).$$

This yields the second equality in (2.1). By (2.5), we can rewrite [7, (1.8)] as

$$\lim_{r\to 1^-}(\partial\mathscr{P}[\gamma](rz) - \bar{z}^2\bar{\partial}\mathscr{P}[\gamma](rz)) = \gamma'(z) \quad \text{for a.e. } z\in\boldsymbol{T}.$$

Combining this with (2.1) we obtain (2.2). □

Given $\gamma\in\mathrm{Hom}^+(\boldsymbol{T})$ define $d_\gamma := \operatorname{ess\,inf}_{z\in\boldsymbol{T}}|\gamma'(z)|$ and $j_\gamma := \operatorname{ess\,inf}_{z\in\boldsymbol{T}}\hat{\partial}_r^-\mathscr{J}[\mathscr{P}[\gamma]](z)$. We use Lemma 2.1 to find a lower estimate of the functional j_γ. The following theorem is in fact a generalization of Martio's [5, Lemma 3].

Theorem 2.2. *If $\gamma\in\mathrm{Hom}^+(\boldsymbol{T})$ then for a.e. $z\in\boldsymbol{T}$, the Jacobian $\mathscr{J}[\mathscr{P}[\gamma]]$ has a nontangential limiting value at z and*

$$\lim_{r\to 1^-}\mathscr{J}[\mathscr{P}[\gamma]](rz) = |\hat{\partial}_r^-\mathscr{C}_\gamma(z)|^2 - |\hat{\partial}_r^+\mathscr{C}_\gamma(z)|^2 \geqslant \frac{1}{2}|\gamma'(z)|\frac{1-|F_\gamma(0)|}{1+|F_\gamma(0)|}, \tag{2.6}$$

where $F_\gamma(0)$ is the point in $\varDelta$ which is uniquely determined by $\mathscr{P}[\gamma](F_\gamma(0)) = 0$. In particular,

$$j_\gamma \geqslant \frac{1}{2}d_\gamma\frac{1-|F_\gamma(0)|}{1+|F_\gamma(0)|}. \tag{2.7}$$

Proof. It is easy to check that for each $r\in[0,1)$ and $u,z\in\boldsymbol{T}$,

$$\frac{1}{1-r}\frac{1}{2\pi}\mathrm{Re}\,\frac{u+rz}{u-rz} = \frac{1}{2\pi}\frac{1+r}{|u-rz|^2} \geqslant \frac{1}{2\pi}\frac{1+r}{(1+r)^2} \geqslant \frac{1}{4\pi}. \tag{2.8}$$

Given $a\in\varDelta$ write $h_a(u) := (u-a)/(1-\bar{a}u)$ for $u\in\mathbb{C}\setminus\{1/\bar{a}\}$ and $h_a(1/\bar{a}) := \infty$. Set $a := -F_\gamma(0)$ and $\eta := \gamma\circ h_a$. Then $\mathscr{P}[\gamma\circ h_a](0) = \mathscr{P}[\gamma](h_a(0)) = \mathscr{P}[\gamma](F_\gamma(0)) = 0$, and so $0 = \int_{\boldsymbol{T}}\eta(u)|\mathrm{d}u| = \int_0^{2\pi}\mathrm{e}^{\mathrm{i}\hat{\eta}(t)}\,\mathrm{d}t = \int_0^{2\pi}\cos\hat{\eta}(t)\,\mathrm{d}t + \mathrm{i}\int_0^{2\pi}\sin\hat{\eta}(t)\,\mathrm{d}t$, where $\hat{\eta}$ is the angular parametrization of η; cf. (3.3). Thus for every $\theta\in\mathbb{R}$, $\int_0^{2\pi}\cos(\hat{\eta}(t)-\hat{\eta}(\theta))\,\mathrm{d}t = 0$. From this, (2.8) and Lemma 2.1 we see, following the proof of [5, Lemma 3], that for a.e. $z = \mathrm{e}^{\mathrm{i}\theta}\in\boldsymbol{T}$,

$$\begin{aligned}
\lim_{r\to 1^-}\mathscr{J}[\mathscr{P}[\eta]](rz) &= \lim_{r\to 1^-}(|\partial\mathscr{P}[\eta](rz)|^2 - |\bar{\partial}\mathscr{P}[\eta](rz)|^2)\\
&= \lim_{r\to 1^-}\mathrm{Re}\left(\frac{\eta(z)-\mathscr{P}[\eta](rz)}{1-r}\overline{z\eta'(z)}\right)\\
&= \lim_{r\to 1^-}|\eta'(z)|\mathrm{Re}\left(\overline{\eta(z)}\,\frac{\eta(z)-\mathscr{P}[\eta](rz)}{1-r}\right)\\
&= \lim_{r\to 1^-}|\eta'(z)|\frac{1}{1-r}\frac{1}{2\pi}\int_0^{2\pi}[1-\cos(\hat{\eta}(t)-\hat{\eta}(\theta))]\mathrm{Re}\,\frac{\mathrm{e}^{\mathrm{i}t}+rz}{\mathrm{e}^{\mathrm{i}t}-rz}\mathrm{d}t\\
&\geqslant \tfrac{1}{2}|\eta'(z)| = \tfrac{1}{2}|(\gamma\circ h_a)'(z)| = \tfrac{1}{2}|\gamma'(h_a(z))||h_a'(z)|.
\end{aligned} \tag{2.9}$$

Since $\mathscr{P}[\eta] = \mathscr{P}[\gamma\circ h_a] = \mathscr{P}[\gamma]\circ h_a$, (2.9) yields

$$\lim_{r\to 1^-}\mathscr{J}[\mathscr{P}[\gamma]](h_a(rz))|h_a'(rz)|^2 = \lim_{r\to 1^-}\mathscr{J}[\mathscr{P}[\gamma]\circ h_a](rz) = \tfrac{1}{2}|\gamma'(h_a(z))||h_a'(z)|. \tag{2.10}$$

From Lemma 2.1 it follows that for a.e. $z\in \boldsymbol{T}$, the Jacobian $\mathscr{J}[\mathscr{P}[\gamma]]$ has a nontangential limiting value at z, and (2.10) implies

$$\lim_{r\to 1^-} \mathscr{J}[\mathscr{P}[\gamma]](rz) \geqslant \frac{1}{2}|\gamma'(z)| \inf_{u\in \boldsymbol{T}} |h_a'(u)|^{-1} = \frac{1}{2}|\gamma'(z)|\frac{1-|F_\gamma(0)|}{1+|F_\gamma(0)|}.$$

Combining this with (2.5) we obtain (2.6). Moreover, by definition

$$j_\gamma = \operatorname*{ess\,inf}_{z\in \boldsymbol{T}} \hat{\partial}_r^- \mathscr{J}[\mathscr{P}[\gamma]](z) \geqslant \operatorname*{ess\,inf}_{z\in \boldsymbol{T}} \left(\frac{1}{2}|\gamma'(z)|\frac{1-|F_\gamma(0)|}{1+|F_\gamma(0)|}\right) = \frac{1}{2}d_\gamma\frac{1-|F_\gamma(0)|}{1+|F_\gamma(0)|},$$

which proves (2.7). □

Remark 2.3. It can be shown that $j_\gamma \geqslant d_\gamma^3$. However, the proof exceeds the scope of this paper and will be published elsewhere.

3. The main results

Let $\gamma\in \mathrm{Hom}^+(\boldsymbol{T})$. For the convenience of readers, we recall that for each $z\in \varDelta$,

$$\partial\mathscr{P}[\gamma](z) = \frac{1}{2\pi i}\int_{\boldsymbol{T}} \frac{d\gamma(u)}{u-z} = \mathscr{C}_\gamma(z) \quad \text{and} \quad \bar{\partial}\mathscr{P}[\gamma](z) = \frac{1}{\bar{z}^2}\mathscr{C}_\gamma\left(\frac{1}{\bar{z}}\right), \tag{3.1}$$

cf. [7, (1.3) and (1.4)]. Moreover, for $z\in \varDelta\setminus\{0\}$

$$\mathscr{C}_\gamma(z) - \mathscr{C}_\gamma\left(\frac{1}{\bar{z}}\right) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) d\tilde{\gamma}(\theta) \tag{3.2}$$

(cf. [7, (1.7)], where

$$\tilde{\gamma}(\theta) := \int_0^\theta \frac{d\gamma(e^{it})}{ie^{it}}, \quad \theta\in[0,2\pi]. \tag{3.3}$$

Following the proof of [7, Theorem 3.1] we first show

Theorem 3.1. *Suppose that $\gamma\in \mathrm{Hom}^+(\boldsymbol{T})$ and that there exists a sequence $\zeta_n\in \boldsymbol{T}$, $n\in\mathbb{N}$, such that the derivative $\gamma'(\zeta_n) = \lim_{u\to\zeta_n}(\gamma(u)-\gamma(\zeta_n))/(u-\zeta_n)$ exists for each $n\in\mathbb{N}$ and*

$$\lim_{n\to\infty} \gamma'(\zeta_n) = 0. \tag{3.4}$$

Then $\mathscr{P}[\gamma]$ is not a qc. mapping. In particular, if $d_\gamma = 0$ then $\mathscr{P}[\gamma]$ is not a qc. mapping.

Proof. Fix $n\in\mathbb{N}$. By the Fatou theorem on Poisson integrals (cf. [9, Theorem 11.12]), the Poisson–Stieltjes integral in (3.2) has the nontangential limit $\tilde{\gamma}'(\varphi) = \gamma'(e^{i\varphi})$ for each $\varphi\in[0,2\pi]$ such that the derivative $\tilde{\gamma}'(\varphi)$ exists; cf. [2, pp. 4–5]. Therefore, by (3.2) we obtain

$$\mathscr{C}_\gamma(t\zeta_n) - \mathscr{C}_\gamma(1/\overline{t\zeta_n}) \to \gamma'(\zeta_n) \quad \text{as } (0,1)\ni t\to 1. \tag{3.5}$$

As shown in [7, (1.10)], $2|\partial\mathscr{P}[\gamma](\zeta)|^2 \geqslant c(\gamma) > 0$ on Δ, where $c(\gamma)$ is a positive constant from [7, Theorem 1.1]. Hence by (3.1) we see that for $0 < t < 1$

$$0 \leqslant 1 - \left|\frac{\bar\partial\mathscr{P}[\gamma](t\zeta_n)}{\partial\mathscr{P}[\gamma](t\zeta_n)}\right| \leqslant 1 - \left|\frac{\overline{t\zeta_n}^2\,\bar\partial\mathscr{P}[\gamma](t\zeta_n)}{\partial\mathscr{P}[\gamma](t\zeta_n)}\right|$$

$$\leqslant \frac{|\mathscr{C}_\gamma(t\zeta_n) - \mathscr{C}_\gamma(1/\overline{t\zeta_n})|}{|\partial\mathscr{P}[\gamma](t\zeta_n)|} \leqslant \frac{\sqrt{2}|\mathscr{C}_\gamma(t\zeta_n) - \mathscr{C}_\gamma(1/\overline{t\zeta_n})|}{\sqrt{c(\gamma)}}.$$

Combining this with (3.5) and (3.4) we obtain

$$\sup_{z\in\Delta}\left|\frac{\bar\partial\mathscr{P}[\gamma](z)}{\partial\mathscr{P}[\gamma](z)}\right| \geqslant \liminf_{t\to 1^-}\left|\frac{\bar\partial\mathscr{P}[\gamma](t\zeta_n)}{\partial\mathscr{P}[\gamma](t\zeta_n)}\right| \geqslant 1 - \sqrt{\frac{2}{c(\gamma)}}|\gamma'(\zeta_n)| \to 1 \quad \text{as } n\to\infty,$$

and consequently $\mathscr{P}[\gamma]$ is not a qc. mapping.

If $d_\gamma = 0$, then obviously there exists a sequence $\zeta_n \in \boldsymbol{T}$, $n \in \mathbb{N}$, such that γ has the derivative at each ζ_n and such that (3.4) is satisfied. Thus $\mathscr{P}[\gamma]$ is not a qc. mapping. □

We write $L^\infty(\boldsymbol{T})$ for the class of all measurable functions $f: \boldsymbol{T} \to \mathbb{C}$ essentially bounded on $\boldsymbol{T}$, i.e. $||f||_\infty := \operatorname{ess\,sup}_{z\in\boldsymbol{T}}|f(z)| < \infty$. As a conclusion from Theorems 2.2 and 3.1 we derive a useful sufficient condition of quasiconformality of $\mathscr{P}[\gamma]$.

Theorem 3.2. *Suppose that $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$. If $d_\gamma > 0$ and $\hat\partial_r^-\mathscr{C}_\gamma \in L^\infty(\boldsymbol{T})$, then $\gamma \in \mathbb{Q}^*_{\boldsymbol{T}}$ and*

$$\sup_{z\in\Delta}\left|\frac{\bar\partial\mathscr{P}[\gamma](z)}{\partial\mathscr{P}[\gamma](z)}\right| \leqslant \sqrt{1 - \frac{j_\gamma}{||\hat\partial_r^-\mathscr{C}_\gamma||_\infty^2}}. \tag{3.6}$$

Proof. From [7, Theorem 1.1] it follows that

$$\sup_{z\in\Delta}\left|\frac{\bar\partial\mathscr{P}[\gamma](z)}{\partial\mathscr{P}[\gamma](z)}\right| = \operatorname*{ess\,sup}_{z\in\boldsymbol{T}}\left|\frac{\hat\partial_r^+\mathscr{C}_\gamma(z)}{\hat\partial_r^-\mathscr{C}_\gamma(z)}\right| \tag{3.7}$$

and

$$|\hat\partial_r^-\mathscr{C}_\gamma(u)|^2 \geqslant c(\gamma)/2 > 0, \tag{3.8}$$

for a.e. $u \in \boldsymbol{T}$, where $c(\gamma)$ is a constant. As shown in the proof of [7, Theorem 1.1] both the radial limits $\hat\partial_r^+\mathscr{C}_\gamma(u)$ and $\hat\partial_r^-\mathscr{C}_\gamma(u)$ exist for a.e. $u \in \boldsymbol{T}$. Hence Theorem 2.2 shows, by (3.1), that for a.e. $u \in \boldsymbol{T}$

$$|\hat\partial_r^-\mathscr{C}_\gamma(u)|^2 - |\hat\partial_r^+\mathscr{C}_\gamma(u)|^2 = \lim_{t\to 1^-}(|\partial\mathscr{P}[\gamma](tu)|^2 - |\bar\partial\mathscr{P}[\gamma](tu)|^2) = \lim_{t\to 1^-}\mathscr{J}[\mathscr{P}[\gamma]](tu) \geqslant j_\gamma.$$

Applying now (3.8) we have

$$1 - \left|\frac{\hat\partial_r^+\mathscr{C}_\gamma(u)}{\hat\partial_r^-\mathscr{C}_\gamma(z)}\right|^2 \geqslant \frac{j_\gamma}{|\hat\partial_r^-\mathscr{C}_\gamma(u)|^2} \quad \text{for a.e. } u \in \boldsymbol{T}.$$

By the assumptions and (2.7),

$$\operatorname*{ess\,sup}_{u\in \boldsymbol{T}}\left|\frac{\hat{\partial}_r^+\mathscr{C}_\gamma(u)}{\hat{\partial}_r^-\mathscr{C}_\gamma(z)}\right|^2 \leqslant 1-\frac{j_\gamma}{\|\hat{\partial}_r^-\mathscr{C}_\gamma\|_\infty^2}<1.$$

From this and (3.7) it follows that $\gamma\in\mathbb{Q}_{\boldsymbol{T}}^*$ and (3.6) holds. □

From [7, (1.6)] it follows that for $z\in\mathbb{C}\setminus\boldsymbol{T}$, $\mathscr{C}_\gamma(z)$ is represented by the Cauchy–Stieltjes type integral of $\tilde{\gamma}$ defined by (3.3),

$$\mathscr{C}_\gamma(z)=\frac{1}{2\pi}\int_{\boldsymbol{T}}\frac{u}{u-z}\frac{\mathrm{d}\gamma(u)}{\mathrm{i}u}=\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}-z}\,\mathrm{d}\tilde{\gamma}(\theta). \tag{3.9}$$

For every function $f:[0,2\pi]\to\mathbb{C}$ of bounded variation write $\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}f]$ for the Cauchy–Stieltjes type singular integral, i.e. for every $z=\mathrm{e}^{\mathrm{i}x}\in\boldsymbol{T}$

$$\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}f](z):=PV\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}-z}\mathrm{d}f(\theta):=\lim_{\varepsilon\to0^+}\frac{1}{2\pi}\int_{\varepsilon<|\theta-x|\leqslant\pi}\frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}-z}\mathrm{d}f(\theta)$$

whenever the limit exists and $\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}f](z):=0$ otherwise.

By Smirnov theorem [8, p. 65] and (3.9) the function $\mathscr{C}_\gamma$ belongs to the Hardy class $H^p(\varDelta)$ for arbitrary $p\in(0,1)$. Hence $\mathscr{C}_\gamma$ has nontangential limiting values a.e. on $\boldsymbol{T}$; cf. [8, p. 56]. Then the classical Privalov's Theorem (cf. [8, p. 135]) says that the singular integral exists a.e. on $\boldsymbol{T}$ and the formulas

$$\hat{\partial}_r^-\mathscr{C}_\gamma(u)=\tfrac{1}{2}\tilde{\gamma}'(x)+\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}](u)\quad\text{and}\quad\hat{\partial}_r^+\mathscr{C}_\gamma(u)=-\tfrac{1}{2}\tilde{\gamma}'(x)+\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}](u)$$

hold for a.e. $u=\mathrm{e}^{\mathrm{i}x}\in\boldsymbol{T}$. Since $\tilde{\gamma}'(x)=\gamma'(\mathrm{e}^{\mathrm{i}x})$ for a.e. $x\in[0,2\pi]$ we have

$$\hat{\partial}_r^-\mathscr{C}_\gamma=\tfrac{1}{2}\gamma'+\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}]\quad\text{and}\quad\hat{\partial}_r^+\mathscr{C}_\gamma=-\tfrac{1}{2}\gamma'+\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}]\quad\text{a.e. on }\boldsymbol{T}. \tag{3.10}$$

Applying the equalities (3.10) we can extend [7, Theorem 1.2 and Corollary 2.1] to an arbitrary homeomorphism $\gamma\in\mathrm{Hom}^+(\boldsymbol{T})$ with $\mathscr{C}_{\boldsymbol{T}}(\gamma')$ replaced by $\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}]$ as follows.

Theorem 3.3. *If $\gamma\in\mathrm{Hom}^+(\boldsymbol{T})$, then for a.e. $z\in\boldsymbol{T}$*

$$|\gamma'(z)+2\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}](z)|^2\geqslant 2c(\gamma)>0 \tag{3.11}$$

and

$$\sup_{z\in\varDelta}\left|\frac{\bar{\partial}\mathscr{P}[\gamma](z)}{\partial\mathscr{P}[\gamma](z)}\right|=\operatorname*{ess\,sup}_{z\in\boldsymbol{T}}\left|\frac{\gamma'(z)-2\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}](z)}{\gamma'(z)+2\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}](z)}\right|=\sqrt{\frac{1-m_\gamma}{1+m_\gamma}}, \tag{3.12}$$

where

$$m_\gamma:=\operatorname*{ess\,inf}_{z\in\boldsymbol{T}}\frac{4\mathrm{Re}[\overline{\gamma'(z)}\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}](z)]}{|\gamma'(z)|^2+4|\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}](z)|^2} \tag{3.13}$$

and $c(\gamma)$ is a constant from [7, *Theorem* 1.1]. *In particular, $m_\gamma\geqslant 0$, and the mapping $\mathscr{P}[\gamma]$ is qc. iff $m_\gamma>0$.*

Proof. The theorem follows from [7, Theorem 1.1] and the equalities (3.10). The second equality in (3.12) is obtained by simple computations. □

Corollary 3.4. *Suppose that* $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$. *Then* $\gamma \in \mathbb{Q}^*_{\boldsymbol{T}}$ *iff* $d_\gamma > 0$ *and*

$$0 < \operatorname*{ess\,inf}_{z\in \boldsymbol{T}} \mathrm{Re} \frac{\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}](z)}{\gamma'(z)} \leqslant \operatorname*{ess\,sup}_{z\in \boldsymbol{T}} \left| \frac{\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}](z)}{\gamma'(z)} \right| < \infty. \tag{3.14}$$

Proof. Assume $\mathscr{P}[\gamma]$ is a qc. mapping. Theorem 3.1 then implies that $d_\gamma > 0$. From Theorem 3.3 it follows that $m_\gamma > 0$. Hence (3.14) follows from (3.13). Conversely, if $d_\gamma > 0$ and (3.14) holds then from (3.13) it follows that $m_\gamma > 0$. Applying Theorem 3.3 we see that $\gamma \in \mathbb{Q}^*_{\boldsymbol{T}}$ as claimed. □

Remark 3.5. If $\gamma'(z) \neq 0$ for a.e. $z \in \boldsymbol{T}$ and (3.14) holds, then from (3.11) it follows that $d_\gamma > 0$. Thus in Corollary 3.4 the condition $d_\gamma > 0$ may be replaced by the condition that $\gamma'(z) \neq 0$ for a.e. $z \in \boldsymbol{T}$.

4. The case of homeomorphisms with a bounded derivative

In this section, we discuss the case where $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ has a bounded derivative. Our most essential result is

Theorem 4.1. *Suppose that* $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ *satisfies* $\gamma' \in L^\infty(\boldsymbol{T})$. *Then* $\gamma \in \mathbb{Q}^*_{\boldsymbol{T}}$ *iff* $d_\gamma > 0$ *and* $\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}] \in L^\infty(\boldsymbol{T})$.

Proof. Assume $\gamma \in \mathbb{Q}^*_{\boldsymbol{T}}$. Corollary 3.4 shows that $d_\gamma > 0$ and by (3.14),

$$\|\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}]\|_\infty \leqslant \|\gamma'\|_\infty \|\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}]/\gamma'\|_\infty < \infty,$$

so that $\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}] \in L^\infty(\boldsymbol{T})$. Conversely, if $\gamma' \in L^\infty(\boldsymbol{T})$ and $\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}] \in L^\infty(\boldsymbol{T})$, then by (3.10) we obtain $\hat{\partial}_r^- \mathscr{C}_\gamma = (1/2)\gamma' + \mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}] \in L^\infty(\boldsymbol{T})$. Since $d_\gamma > 0$ we conclude from Theorem 3.2 that $\gamma \in \mathbb{Q}^*_{\boldsymbol{T}}$ as claimed. □

We now turn to the case where $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ is absolutely continuous. For every $f \in L^1(\boldsymbol{T})$ define

$$f_{\boldsymbol{T}} := \frac{1}{2\pi} \int_{\boldsymbol{T}} f(u)|\mathrm{d}u|$$

and write $\mathscr{C}_{\boldsymbol{T}}(f)$ for the Cauchy singular integral, i.e. for every $z \in \boldsymbol{T}$

$$\mathscr{C}_{\boldsymbol{T}}(f)(z) := \mathrm{PV}\frac{1}{2\pi\mathrm{i}} \int_{\boldsymbol{T}} \frac{f(u)}{u-z}\mathrm{d}u := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi\mathrm{i}} \int_{\boldsymbol{T}\setminus \boldsymbol{T}(z,\varepsilon)} \frac{f(u)}{u-z}\mathrm{d}u,$$

whenever the limit exists and $\mathscr{C}_{\boldsymbol{T}}(f)(z) := 0$ otherwise, where $\boldsymbol{T}(\mathrm{e}^{\mathrm{i}x}, \varepsilon) := \{\mathrm{e}^{\mathrm{i}t} \in \boldsymbol{T} : |t-x| < \varepsilon\}$. We recall that the harmonic conjugation operator $\mathscr{A}$ is defined by the singular integral

$$\mathscr{A}(f)(z) := \frac{1}{2\pi} \lim_{\varepsilon\to 0^+} \int_{\varepsilon < |t-x| \leqslant \pi} f(\mathrm{e}^{\mathrm{i}t}) \cot\frac{x-t}{2}\mathrm{d}t, \quad z = \mathrm{e}^{\mathrm{i}x} \in \boldsymbol{T},$$

whenever the limit exists and $\mathscr{A}(f)(z) := 0$ otherwise; cf. e.g. [3, 11]. If $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ is absolutely continuous then $\gamma' \in L^1(\boldsymbol{T})$ and $\int_{\boldsymbol{T}} |\gamma'(u)||\mathrm{d}u| = 2\pi$. Moreover, by [7, (1.14)] we have for a.e. $z \in \boldsymbol{T}$

$$\mathscr{C}_{\boldsymbol{T}}[\mathrm{d}\tilde{\gamma}](z) = \mathscr{C}_{\boldsymbol{T}}(\gamma')(z) = \tfrac{1}{2}\gamma'_{\boldsymbol{T}} + \tfrac{\mathrm{i}}{2}\mathscr{A}(\gamma')(z). \tag{4.1}$$

Applying now Theorem 4.1 to Lipschitz functions we obtain

Corollary 4.2. *Suppose that $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ is a Lipschitz function, i.e. there exists a constant $L \geqslant 0$ such that*

$$|\gamma(u) - \gamma(w)| \leqslant L|u - w|, \quad u, w \in \boldsymbol{T}. \tag{4.2}$$

Then $\gamma \in \mathbb{Q}^*_{\boldsymbol{T}}$ iff $d_\gamma > 0$ and $\mathscr{C}_{\boldsymbol{T}}(\gamma') \in L^\infty(\boldsymbol{T})$.

The same is true with $\mathscr{C}_{\boldsymbol{T}}(\gamma')$ replaced by $\mathscr{A}(\gamma')$.

Proof. Since $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ satisfies (4.2), it follows that γ is absolutely continuous and $\gamma' \in L^\infty(\boldsymbol{T})$ with $||\gamma'||_\infty \leqslant L$. Therefore, the corollary follows from Theorem 4.1 and (4.1). □

Each $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ defines a unique continuous function $\hat{\gamma}$ satisfying $0 \leqslant \hat{\gamma}(0) < 2\pi$ and

$$\gamma(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}\hat{\gamma}(t)}, \quad t \in \mathbb{R}. \tag{4.3}$$

Actually, $\hat{\gamma}$ is an increasing homeomorphism of $\mathbb{R}$ onto itself satisfying $\hat{\gamma}(t + 2\pi) - \hat{\gamma}(t) = 2\pi$ for $t \in \mathbb{R}$, called the angular parametrization of γ. Sometimes it is more convenient to use the real-valued function $|\gamma'|$ instead of γ'. To this end we prove. □

Corollary 4.3. *Suppose that $\gamma \in \mathrm{Hom}^+(\boldsymbol{T})$ is a Lipschitz function. Then $\gamma \in \mathbb{Q}^*_{\boldsymbol{T}}$ iff $d_\gamma > 0$ and $\mathscr{A}(|\gamma'|) \in L^\infty(\boldsymbol{T})$.*

Proof. From (4.3) it follows that for a.e. $t \in \mathbb{R}$, $\gamma'(\mathrm{e}^{\mathrm{i}t}) = \hat{\gamma}'(t)\mathrm{e}^{\mathrm{i}(\hat{\gamma}(t)-t)}$. Hence for a.e. $x \in \mathbb{R}$,

$$\begin{aligned}
&|\mathscr{A}(\gamma')(\mathrm{e}^{\mathrm{i}x}) - \mathrm{e}^{\mathrm{i}(\hat{\gamma}(x)-x)}\mathscr{A}(|\gamma'|)(\mathrm{e}^{\mathrm{i}x})| \\
&= \frac{1}{2\pi}\left|\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|t-x|\leqslant\pi}(\gamma'(\mathrm{e}^{\mathrm{i}t}) - \mathrm{e}^{\mathrm{i}(\hat{\gamma}(x)-x)}|\gamma'(\mathrm{e}^{\mathrm{i}t})|)\cot\frac{x-t}{2}\,\mathrm{d}t\right| \\
&= \frac{1}{2\pi}\left|\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|t-x|\leqslant\pi}|\gamma'(\mathrm{e}^{\mathrm{i}t})|(\mathrm{e}^{\mathrm{i}(\hat{\gamma}(t)-t)} - \mathrm{e}^{\mathrm{i}(\hat{\gamma}(x)-x)})\cot\frac{x-t}{2}\,\mathrm{d}t\right| \\
&\leqslant \frac{||\gamma'||_\infty}{2\pi}\int_{x-\pi}^{x+\pi}\left|\frac{\mathrm{e}^{\mathrm{i}(\hat{\gamma}(t)-t)} - \mathrm{e}^{\mathrm{i}(\hat{\gamma}(x)-x)}}{x-t}\right|\frac{|x-t|}{|\sin\frac{x-t}{2}|}\,\mathrm{d}t \\
&\leqslant \frac{||\gamma'||_\infty}{2}\int_{x-\pi}^{x+\pi}\left(\left|\frac{\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{\mathrm{i}x}}{t-x}\right| + \left|\frac{\mathrm{e}^{\mathrm{i}\hat{\gamma}(t)} - \mathrm{e}^{\mathrm{i}\hat{\gamma}(x)}}{t-x}\right|\right)\mathrm{d}t \\
&\leqslant \frac{||\gamma'||_\infty}{2}\int_{x-\pi}^{x+\pi}\left(1 + \left|\frac{\hat{\gamma}(t) - \hat{\gamma}(x)}{x-t}\right|\right)\mathrm{d}t \\
&\leqslant \frac{||\gamma'||_\infty}{2}\int_{x-\pi}^{x+\pi}\left(1 + \frac{1}{|t-x|}\left|\int_x^t \hat{\gamma}'(s)\,\mathrm{d}s\right|\right)\mathrm{d}t \\
&\leqslant \pi||\gamma'||_\infty(1 + ||\gamma'||_\infty) < \infty.
\end{aligned}$$

Therefore, $\mathscr{A}(\gamma') \in L^\infty(\boldsymbol{T})$ iff $\mathscr{A}(|\gamma'|) \in L^\infty(\boldsymbol{T})$, and the corollary follows from Corollary 4.2. □

Following the proof of [7, Corollary 3.5] we easily obtain a version of Corollary 4.2 in terms of a variant $\mathscr{H}$ of the Hilbert transform defined for every locally integrable function $f:\mathbb{R}\to\mathbb{C}$ and every $x\in\mathbb{R}$ by the singular integral

$$\mathscr{H}(f)(x):=\frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|t-x|\leqslant\pi}\frac{f(t)}{x-t}\,\mathrm{d}t,$$

whenever the limit exists and $\mathscr{H}(f)(x):=0$ otherwise.

Corollary 4.4. *Suppose that* $\gamma\in\mathrm{Hom}^+(\boldsymbol{T})$ *and* $\hat{\gamma}$ *is absolutely continuous on* $\mathbb{R}$ *satisfying* $\operatorname{ess\,sup}_{t\in\mathbb{R}}\hat{\gamma}'(t)<\infty$. *Then* $\gamma\in\mathbb{Q}^*_{\boldsymbol{T}}$ *iff* $\operatorname{ess\,inf}_{t\in\mathbb{R}}\hat{\gamma}'(t)>0$ *and* $\operatorname{ess\,sup}_{t\in\mathbb{R}}|\mathscr{H}(\hat{\gamma}')(t)|<\infty$.

We end this section with the following corollary, which points out a relation with the Hardy class $H^\infty(\varDelta)$ of bounded analytic functions on $\varDelta$.

Corollary 4.5. *Suppose* $\gamma\in\mathrm{Hom}^+(\boldsymbol{T})$ *is a Lipschitz function. Then* $\mathscr{P}[\gamma]$ *is a qc. mapping iff there exists* $F\in H^\infty(\varDelta)$ *such that*

$$\hat{\partial}_r^-\operatorname{Re}F(z)=|\gamma'(z)|\geqslant d_\gamma>0 \quad \text{for a.e. } z\in\boldsymbol{T}. \tag{4.4}$$

Proof. Assume $\mathscr{P}[\gamma]$ is a qc. mapping. Corollary 4.3 then shows that $d_\gamma>0$ and $\operatorname{ess\,sup}_{z\in\boldsymbol{T}}|\mathscr{A}(|\gamma'|)(z)|<\infty$. By [3, p. 103], $\mathscr{P}[\mathscr{A}(|\gamma'|)]$ is identical with the harmonic conjugate function v of $\mathscr{P}[|\gamma'|]$ satisfying $v(0)=0$. Thus $F:=\mathscr{P}[|\gamma'|+\mathrm{i}\mathscr{A}(|\gamma'|)]$ is an analytic function on $\varDelta$. Since $|\gamma'|+\mathrm{i}\mathscr{A}(|\gamma'|)\in L^\infty(\boldsymbol{T})$, we conclude from [2, Corollary 2 to Theorem 3.1] that $F\in H^\infty(\varDelta)$. By Fatou's theorem on Poisson integrals (cf. [9, Theorem 11.12]), the radial limit $\hat{\partial}_r^-\operatorname{Re}F=|\gamma'|$ a.e. on $\boldsymbol{T}$, which ends the proof in the first direction. Assume now that (4.4) holds for some $F\in H^\infty(\varDelta)$. A classical result (cf. e.g. [3, p. 103]) states that $\hat{\partial}_r^-\operatorname{Im}F=\operatorname{Im}F(0)+\mathscr{A}(\hat{\partial}_r^-\operatorname{Re}F)$ a.e. on $\boldsymbol{T}$. Hence $\operatorname{ess\,sup}_{z\in\boldsymbol{T}}|\mathscr{A}(|\gamma'|)(z)|\leqslant|F(0)|+\sup_{z\in\varDelta}|F(z)|<\infty$, and $\mathscr{P}[\gamma]$ is qc. by Corollary 4.3. □

5. Comparison between our results and Martio's theorem

In this section, we compare our results with Martio's theorem; cf. [5, Theorem 1]. We show that Theorem 4.1 and Corollaries 4.2–4.5 essentially extend Martio's theorem.

If $\gamma\in\mathrm{Hom}^+(\boldsymbol{T})\cap C^1(\boldsymbol{T})$ and $|\gamma'|$ is Dini continuous on $\boldsymbol{T}$, then a classical result (cf. e.g. [3, p. 106]) shows that the function $\mathscr{A}(|\gamma'|)$ is continuous on $\boldsymbol{T}$, and hence bounded. Thus Corollary 4.3 leads to

Corollary 5.1 (Martio [5]). *Suppose* $\gamma\in\mathrm{Hom}^+(\boldsymbol{T})\cap C^1(\boldsymbol{T})$. *If* $|\gamma'|$ *is Dini continuous on* $\boldsymbol{T}$, *then* $\mathscr{P}[\gamma]$ *is qc. iff* $d_\gamma>0$.

Let $\mathrm{Diff}^+(T)$ denote the class of all sense-preserving diffeomorphic self-mappings of $\boldsymbol{T}$. For $0<\alpha\leqslant 1$ write $C^{1+\alpha}(\boldsymbol{T})$ for the class of all complex-valued functions continuously differentiable on $\boldsymbol{T}$, whose derivatives are α-Hölder continuous functions on $\boldsymbol{T}$.

Corollary 5.2. *For each* $\alpha \in (0, 1]$,

$$\mathrm{Diff}^+(\boldsymbol{T}) \cap C^{1+\alpha}(\boldsymbol{T}) \subset \mathbb{Q}_{\boldsymbol{T}}^*. \tag{5.1}$$

Moreover,

$$\mathrm{Diff}^+(\boldsymbol{T}) \backslash \mathbb{Q}_{\boldsymbol{T}}^* \neq \emptyset \quad \text{and} \quad \mathbb{Q}_{\boldsymbol{T}}^* \backslash \mathrm{Diff}^+(\boldsymbol{T}) \neq \emptyset. \tag{5.2}$$

Proof. Fix $\gamma \in \mathrm{Diff}^+(\boldsymbol{T})$. By the definition of a diffeomorphism, $d_\gamma > 0$. For any $\alpha \in (0, 1]$, if $\gamma \in C^{1+\alpha}(\boldsymbol{T})$ then $|\gamma'|$ is Dini continuous on $\boldsymbol{T}$, and the inclusion (5.1) follows from Corollary 5.1. Let $\mathscr{F}$ denote the class of all real-valued functions $f \in L^\infty(\boldsymbol{T})$ such that $\operatorname{ess\,inf}_{z \in \boldsymbol{T}} f(z) > 0$ and $\int_{\boldsymbol{T}} f(u)|\mathrm{d}u| = 2\pi$. Each function $f \in \mathscr{F}$ defines a homeomorphism $\gamma_f \in \mathrm{Hom}^+(\boldsymbol{T})$ whose angular parametrization $\hat{\gamma}_f$ is determined by the equality $\hat{\gamma}_f(x) = \int_0^x f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t$ for $x \in \mathbb{R}$. Obviously, $\hat{\gamma}_f$ is a Lipschitz function on $\mathbb{R}$, and so is γ_f on $\boldsymbol{T}$. Moreover, $d_{\gamma_f} > 0$. It is a simple matter to construct a continuous function $f \in \mathscr{F}$ such that $\mathscr{A}(f)$ is unbounded; see Example 5.3. Then $\gamma_f \in \mathrm{Diff}^+(\boldsymbol{T})$ and $\mathscr{A}(|\gamma_f'|) = \mathscr{A}(f)$ is unbounded. From Corollary 4.3 it follows that $\mathscr{P}[\gamma_f]$ is not a qc. mapping, i.e. $\gamma_f \notin \mathbb{Q}_{\boldsymbol{T}}^*$, which proves the first assertion in (5.2).

It is easy to find $F \in H^\infty(\varDelta)$ such that $f := \hat{\partial}_r^- \operatorname{Re} F \in \mathscr{F}$ but $|\gamma_f'|$ is not continuous on $\boldsymbol{T}$; see Example 5.4. Then $\gamma_f \notin \mathrm{Diff}^+(\boldsymbol{T})$ and Corollary 4.5 shows that $\gamma_f \in \mathbb{Q}_{\boldsymbol{T}}^*$, which proves the second assertion in (5.2). □

Example 5.3. For every $x \in \mathbb{R}$ define

$$p(x) := \begin{cases} 1 & \text{if } x \geqslant 1/e, \\ -1/\log x & \text{if } 0 < x < 1/e, \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

Since $p \in C^1(\mathbb{R} \backslash \{0, 1/e\})$, $\mathscr{H}(p)(x)$ exists for every $x \in \mathbb{R} \backslash \{0, 1/e\}$. Moreover, $p(t) = 0$ if $t \leqslant 0$ and $p(t) > 0$ if $t > 0$. Then for each $x \in (-1/e, 0)$ we have

$$\begin{aligned} \pi|\mathscr{H}(p)(x)| &= \left| \lim_{\varepsilon \to 0^+} \int_{\varepsilon < |t-x| \leqslant \pi} \frac{p(t)}{x-t} \mathrm{d}t \right| = \int_{x-\pi}^{x+\pi} \frac{p(t)}{t-x} \mathrm{d}t \geqslant \int_{-x/2}^{1/e} \frac{p(t)}{t-x} \mathrm{d}t \\ &= \int_{-x/2}^{1/e} \frac{t}{t-x} \left(-\frac{1}{t \log t} \right) \mathrm{d}t \geqslant \frac{1}{3} \int_{-x/2}^{1/e} -\frac{\mathrm{d}t}{t \log t} \quad (u = \log t) \\ &= -\frac{1}{3} \int_{\log|x/2|}^{-1} \frac{\mathrm{d}u}{u} = \frac{1}{3} \log \left| \log \left| \frac{x}{2} \right| \right| \to \infty, \quad \text{as } x \to 0^-. \end{aligned} \tag{5.3}$$

It is easily seen that there exists a function $q \in C^1(\mathbb{R})$ such that $(p+q)(-\pi) = (p+q)(\pi) > 0$ and $\min_{-\pi \leqslant t \leqslant \pi} (p+q)(t) > 0$. Set $c := \int_{-\pi}^{\pi} (p+q)(t)\,\mathrm{d}t > 0$, and define $f(\mathrm{e}^{\mathrm{i}t}) = (2\pi/c)(p(t) + q(t))$, $-\pi \leqslant t \leqslant \pi$. Obviously, $f \in \mathscr{F} \cap C(\boldsymbol{T})$, and thus γ_f is a Lipschitz function on $\boldsymbol{T}$. Since $q \in C^1(\mathbb{R})$, $\mathscr{H}(q)$ is continuous on $\mathbb{R}$, and thus bounded on $[-\pi, \pi]$. Hence by (5.3) we see that

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}} |\mathscr{H}(\hat{\gamma}_f')(x)| = \infty. \tag{5.4}$$

As in the proof of [7, Corollary 3.5], we can check that (5.4) implies

$$||\mathscr{A}(\gamma_f')||_\infty = \infty. \tag{5.5}$$

As shown in the proof of Corollary 4.3, we see that (5.5) implies $||\mathscr{A}(f)||_\infty = ||\mathscr{A}(|\gamma'_f|)||_\infty = \infty$.

Example 5.4. For $z \in \varDelta$ define $G(z) := \exp(-(1+z)/(1-z))$. Clearly

$$|G(z)| = \exp\left(-\mathrm{Re}\frac{1+z}{1-z}\right) \leqslant \mathrm{e}^0 = 1, \quad z \in \varDelta, \tag{5.6}$$

so that $G \in H^\infty(\varDelta)$. Let $c := \int_0^{2\pi} \left(2 + \hat{\partial}_r^- \,\mathrm{Re}\, G(\mathrm{e}^{\mathrm{i}t})\right) \mathrm{d}t > 0$, and define $F(z) := (2\pi/c)[2 + G(z)]$, $z \in \varDelta$. By (5.6), $F \in H^\infty(\varDelta)$ and $f := \hat{\partial}_r^- \,\mathrm{Re}\, F \in \mathscr{F}$. Moreover, if $\mathrm{e}^{\mathrm{i}x} \neq 1$, then $\hat{\gamma}'_f(x) = f(\mathrm{e}^{\mathrm{i}x}) = (2\pi/c)(2 + \cos(-\cot(x/2)))$. Hence $\hat{\gamma}'_f$ is discontinuous at $x = 0$, and so the function $|\gamma'_f|$ is not continuous at $1 \in \boldsymbol{T}$.

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