Robin Hartshorne's Algebraic Geometry Solutions

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Chapter II Section 7, Projective Morphisms

7.1.

7.9. Let $r+1$ be the rank of $\mathcal{E}$.

(a). There are several ways to prove it.

Proof 1 We assume the following result from Chow group theory: (See Appendix A section2 A11 and section 3. The group $A(X)$ is here $CH(X)$.)

$$CH^*(\mathbb{P}(\mathcal{E})) \simeq \left(\mathbb{Z}[\xi] / \sum_{i=0}^{r} (-1)^i c_i(\mathcal{E}) \xi^{r-i} \right) \otimes_{\mathbb{Z}} CH^*(X)$$

as graded rings. If we look at the grade 1 part, as $\mathbb{Z}$-modules,

$$CH^1(\mathbb{P}(\mathcal{E})) \simeq (\mathbb{Z} \otimes_{\mathbb{Z}} CH^0(X)) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}} CH^1(X))$$

and $CH^1(-) = \mathrm{Pic}(-)$ so that $\mathrm{Pic}(\mathbb{P}(\mathcal{E})) \simeq \mathbb{Z} \oplus \mathrm{Pic}(X)$ as desired.

Proof 2 We can use the Grothendieck groups, i.e. K-theory to do so. Note that

$$K(\mathbb{P}(\mathcal{E})) \simeq \left(\mathbb{Z}[\xi] / \sum_{i=0}^{r} (-1)^i c_i(\mathcal{E}) \xi^{r-i} \right) \otimes_{\mathbb{Z}} K(X)$$

as rings. For the detail, see Yuri Manin *Lectures on the K-functor in Algebraic Geometry*, Russian Mathematical Surveys, 24 (1969) 1-90, in particular, p. 44, from Prop (10.2) to Cor. (10.5).

Proof 3 Here we give a direct proof. In fact, it adapts a way from Proof 1. It can also use the method from Proof 2. Totally your choice.

Define a map $\phi : \mathbb{Z} \oplus \mathrm{Pic}(X) \to \mathrm{Pic}(\mathbb{P}(\mathcal{E}))$ by $(n, \mathcal{L}) \mapsto (\pi^*\mathcal{L})(n) := (\pi^*\mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)$.

Claim. *This map is injective.*

Assume that $\phi(n, \mathcal{L}) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}$, i.e. $\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}$. Apply π_* to it. From II (7.11), recall thet

$$\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)) = \begin{cases} 0 & n < 0 \\ \mathcal{O}_X & n = 0 \\ \mathrm{Sym}^n(\mathcal{E}) & n > 0 \end{cases} .$$

So, by applying the projection formula (Ex. II (5.1)-(d)), we obtain, $\mathcal{L} \otimes \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \simeq \mathcal{O}_X$, i.e.

$$\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})} \simeq \mathcal{L}^{-1}.$$

Note that it is a line bundle and $\mathrm{rk}(\mathrm{Sym}^n(\mathcal{E})) \geq r+1 \geq 2$ if $n > 0$ by the given assumption, so that the only possible choice for n is $n = 0$. Then, it implies that $\mathcal{L} \simeq \mathcal{O}_X$. Hence ϕ is injective.

Claim. *This map is surjective.*

In case $\mathcal{E}$ is a trivial bundle, then $\mathbb{P}(\mathcal{E}) \simeq X \times \mathbb{P}^r$ so that we already know the result.

In general, choose an open subset $U \subset X$ over which $\mathcal{E}$ is trivial and let $Z = X - U$. Then, we have a closed immersion $\mathbb{P}(\mathcal{E}|_Z) \hookrightarrow \mathbb{P}(\mathcal{E})$ and an open immersion $\mathbb{P}(\mathcal{E}_U) \hookrightarrow \mathbb{P}(\mathcal{E}|_U) \simeq U \times \mathbb{P}^r$. Let $m = \dim X$. Then we have

$$\begin{array}{ccccccc} CH_{m+r-1}(\mathbb{P}(\mathcal{E}|_Z)) & \longrightarrow & \mathrm{Pic}(\mathbb{P}(\mathcal{E})) & \longrightarrow & \mathrm{Pic}(\mathbb{P}(\mathcal{E}|_U)) & \longrightarrow & 0\ . \\ \uparrow{\scriptstyle \phi_Z} & & \uparrow{\scriptstyle \phi_X} & & \uparrow{\scriptstyle \phi_U} & & \\ \mathbb{Z} \oplus CH_{m-1}(Z) & \longrightarrow & \mathbb{Z} \oplus \mathrm{Pic}X & \longrightarrow & \mathbb{Z} \oplus \mathrm{Pic}U & \longrightarrow & 0 \end{array}$$

By induction on the dimension, ϕ_Z is surjective and we already know that ϕ_U is an isomorphism. Hence, by a simple diagram chasing, we have the surjectivity of ϕ_X.

(b). Let $\pi : \mathbb{P}(\mathcal{E}) \to X$, $\pi' : \mathbb{P}(\mathcal{E}') \to X$ be the structure morphisms and let $\phi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ be the given isomorphism over X:

$$\begin{array}{ccc} & & \mathbb{P}(\mathcal{E})\ . \\ & \overset{\phi}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \mathbb{P}(\mathcal{E}') & \xrightarrow{\pi'} & X \end{array}$$

$\phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is an invertible sheaf on $\mathbb{P}(\mathcal{E})$ so that by part (a), we have

$$(1) : \phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \simeq \pi'^*\mathcal{L}' \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(n')$$

for some $\mathcal{L}' \in \mathrm{Pic}X$ and $n' \in \mathbb{Z}$. Similarly, ϕ^{-1} being a morphism, we have

$$(2) : \phi^{-1*}\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \simeq \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)$$

for some $\mathcal{L} \in \mathrm{Pic}X$ and $n \in \mathbb{Z}$. By applying ϕ^* to (2), we have

$$\begin{aligned} \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) &\simeq \phi^*\phi^{-1*}\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \simeq \phi^*\pi^*\mathcal{L} \otimes \phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \simeq \pi'^*\mathcal{L} \otimes \left(\phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\right)^{\otimes n} \\ &\simeq \pi'^*\mathcal{L} \otimes \left(\pi'^*\mathcal{L}' \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n')\right)^{\otimes n} \simeq \pi'^*\left(\mathcal{L} \otimes \mathcal{L}'^{\otimes n}\right) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(nn'). \end{aligned}$$

Recall that

$$\pi'_*\left(\mathcal{P}_{\mathbb{P}(\mathcal{E}')}(n)\right) = \begin{cases} 0 & m < 0 \\ \mathcal{O}_X & m = 0 \\ \mathrm{Sym}^m\mathcal{E}' & m > 0 \end{cases}$$

so that if we apply π'_* to the above, then by the projection formula, we will have

$$\mathcal{O}_X \simeq \mathcal{L} \otimes \mathcal{L}'^{\otimes n} \otimes \pi'_*\left(\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(nn')\right).$$

Since $\mathcal{O}_X$, $\mathcal{L} \otimes \mathcal{L}'^{\otimes n}$ are invertible sheaves, it makes sense only when $nn' = 1$. Hence we have either $(n, n') = (-1, -1)$ or $(n, n') = (1, 1)$.

If $(n, n') = (-1, -1)$, then, we have $\mathcal{L} \simeq \mathcal{L}'$ and (2) becomes $\phi^{-1*}\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \simeq \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)$. ϕ being an isomorphism, $\phi^{-1*} = \phi_*$, so that $\pi'_* = \pi_*\phi_* = \pi_*\phi^{-1*}$ and the projection formula gives $\mathcal{E}' \simeq \mathcal{L} \otimes 0 \simeq 0$ which is not possible. Hence $(n, n') = (1, 1)$.

Hence, we have (2): $\phi^{-1*}\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \simeq \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and as above, noting that $\phi^{-1*} = \phi_*$, applying π_* and using the projection formula, we will have $\mathcal{E}' \simeq \mathcal{L} \otimes \mathcal{E}$ as desired.