Nonlinear Analysis 74 (2011) 1721–1730

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

The Calderón–Zygmund property for quasilinear divergence form equations over Reifenberg flat domains

Dian K. Palagachev [a,*], Lubomira G. Softova [b]

[a] Politecnico di Bari, Dipartimento di Matematica, Via E. Orabona, 4, 70 125 Bari, Italy
[b] Seconda Università di Napoli, Dipartimento di Ingegneria Civile, Via Roma, 29, 81 031 Aversa, Italy

ARTICLE INFO

Article history:
Received 14 July 2010
Accepted 21 October 2010

MSC:
primary 35J62
secondary 35R05
35B65
35K59

Keywords:
Divergence form quasilinear elliptic and parabolic equations
Weak solution
Hölder continuity
Reifenberg flat domain
BMO

ABSTRACT

The results by Palagachev (2009) [3] regarding global Hölder continuity for the weak solutions to quasilinear divergence form elliptic equations are generalized to the case of nonlinear terms with optimal growths with respect to the unknown function and its gradient. Moreover, the principal coefficients are discontinuous with discontinuity measured in terms of small *BMO* norms and the underlying domain is supposed to have fractal boundary satisfying a condition of Reifenberg flatness. The results are extended to the case of parabolic operators as well.

1. Introduction

The general aim of the paper is to study regularity issues regarding the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(a^{ij}(x,u)D_ju + a^i(x,u)\right) = b(x,u,Du) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

over a *Reifenberg flat* and bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. The nonlinear ingredients are of Carathéodory type (that is, measurable with respect to $x \in \Omega$ for all $(z,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and continuous with respect to $(z,\xi) \in \mathbb{R} \times \mathbb{R}^n$ for almost all (a.a.) $x \in \Omega$), and the principal coefficients $a^{ij}(x,\cdot)$ are allowed to be discontinuous with respect to $x \in \Omega$ with discontinuity measured in terms of appropriate smallness of their *bounded mean oscillation* (*BMO*). Regarding the lower-order terms, we assume that these satisfy *optimal growth conditions* with respect to u and Du, which are the sufficient ones to guarantee the relevant definition of weak solution to the problem (1.1). Precisely, we suppose $a^i(x,z) = \mathcal{O}\left(\varphi_1(x) + |z|^{\frac{n}{n-2}}\right)$ and $b(x,z,\xi) = \mathcal{O}\left(\varphi_2(x) + |z|^{\frac{n+2}{n-2}} + |\xi|^{\frac{n+2}{n}}\right)$ as $|z|, |\xi| \to \infty$ with $\varphi_1 \in L^p(\Omega)$, $p > n$ and $\varphi_2 \in L^q(\Omega)$, $q > n/2$.

Our main result asserts that the problem (1.1) supports the Calderón–Zygmund property. Namely, each $W_0^{1,2}(\Omega)$-weak solution to (1.1) gains better regularity from the data φ_1 and φ_2, and belongs to the Sobolev space $W^{1,\min\{p,q^*\}}(\Omega)$ where q^*

* Corresponding author. Tel.: +39 080 5963748; fax: +39 080 5963358.
E-mail addresses: dian@dm.uniba.it, palaga@poliba.it (D.K. Palagachev), Luba.Softova@unina2.it (L.G. Softova).

0362-546X/$ – see front matter
doi:10.1016/j.na.2010.10.044

is the Sobolev conjugate of q (see (2.7)). In particular, since $p > n$ and $q > n/2$, $\min\{p, q^*\} > n$ results, and therefore the $W_0^{1,2}(\Omega)$-weak solution of (1.1) is a globally Hölder continuous function in $\overline{\Omega}$ with exponent $\min\{1 - n/p, 2 - n/q\}$. Let us draw the reader's attention to the fact that Hölder continuity of the bounded weak solutions to (1.1) with *some* exponent $\alpha \in (0, 1)$ is not surprising, which is the essence of the famous De Giorgi–Nash result (cf. [1,2]). The advantage of our approach, apart from the improvement-of-Sobolev-regularity effect, is the *exact value* of the Hölder exponent, expressed in terms of the data of (1.1).

The present paper is a natural generalization of [3], where the problem (1.1) has been studied over C^1-smooth domains Ω, assuming *vanishing mean oscillation (VMO)* with respect to $x \in \Omega$ of the principal coefficients $a^{ij}(x, z)$ and *sub-critical* growths of the lower-order terms, $a^i(x, z) = \mathcal{O}\big(\varphi_1(x) + |z|^t\big)$, $b(x, z, \xi) = \mathcal{O}\big(\varphi_2(x) + |z|^s + |\xi|^r\big)$ with $t < \frac{n}{n-2}$, $s < \frac{n+2}{n-2}$ and $r < \frac{n+2}{n}$. It is worth noting that each continuous or *VMO*-function has automatically small *BMO*-norms (see (2.6)), while Ω is for sure *Reifenberg flat* (cf. Definition 2.1) when its boundary is C^1-smooth or even Lipschitz continuous with small enough Lipschitz constant. Anyway, the class of Reifenberg flat domains includes sets with very rough, fractal, boundaries such as the well-known Van Koch snowflake. This, together with the borderline values of the growths with respect to u and Du at the lower-order terms provides essential extension of the results from [3] to a more general situation.

In the last few years, there has been an increasing interest in boundary value problems with rough data as those considered here. Apart from the purely theoretical scopes to extend the general PDE theory to the limit cases (see the papers by Byun and Wang [4,5] and the most recent one of Milakis and Toro [6]), the importance of studying discontinuous problems of the type of (1.1) over rough domains is drawn by the fact that these arise naturally in models of real-world systems in media with fractal geometry such as blood vessels, the internal structure of lungs, clouds, bacteria growth, semiconductor devices and composite materials, optimal control, stock markets and the economic applications of the non-smooth variational analysis (see [7,8]). The *BMO*-discontinuity of the principal coefficients $a^{ij}(x, \cdot)$, instead, could be regarded to crack ruptures of the media (think, for instance, of small multipliers of the Heaviside step function).

The role played by the optimal growth conditions on the lower-order terms in proving the Calderón–Zygmund property for (1.1) is twofold. First of all, these ensure better regularity of the gradient of any $W_0^{1,2}(\Omega)$-weak solution to (1.1). Precisely, $u \in W^{1,r_0}(\Omega)$ for some $r_0 > 2$ through the Giaquinta–Gehring–Modica lemma. Second, the optimal growths of $a^i(x, z)$ and $b(x, z, \xi)$ are the minimal and sharp assumptions on the data implying essential boundedness of any $W_0^{1,2}(\Omega)$-weak solution to (1.1). In fact, gradient growths in $b(x, u, Du)$ greater than $\frac{n+2}{n}$ support *unbounded* weak solutions as shows the function $|x|^{\frac{r-2}{r-1}} - 1 \notin L^\infty$ with $r \in \big(\frac{n+2}{n}, 2\big)$ which is a $W_0^{1,2}$-weak solution to the Dirichlet problem for $\Delta u = C|Du|^r$ in the unit ball. Once having $u \in L^\infty(\Omega)$, the celebrated De Giorgi–Nash result (see [1,2]) ensures Hölder continuity of the weak solution with an exponent $\alpha \in (0, 1)$ and this allows control of the *BMO*-norm of the principal coefficients $a^{ij}(x, u(x))$ of the *linearized* at $u(x)$ problem (1.1). At this point the linear theory of divergence form elliptic equations over Reifenberg flat domains, recently developed by Byun and Wang in [4], applies to the linearized problem and the Calderón–Zygmund property of (1.1) follows by virtue of bootstrapping arguments.

The technique employed in the elliptic settings works also in the case of Cauchy–Dirichlet problem for quasilinear divergence form parabolic equations, and we propose in Section 4 the corresponding result with a brief sketch of the proof.

2. Hypotheses

Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. Throughout the paper $W^{1,p}(\Omega)$ stands for the Sobolev space of once weakly differentiable functions $u\colon \Omega \to \mathbb{R}$ belonging to $L^p(\Omega)$ together with the gradient Du, and the standard summation convention on repeated indices is adopted. We set $B_\rho(x)$ for the n-dimensional ball centered at x and of radius ρ, and $|E|$ for the Lebesgue measure of a measurable set $E \subseteq \Omega$.

Hereafter, we suppose that the domain Ω is *Reifenberg flat* in the sense of the following definition.

Definition 2.1. The domain Ω is said to be (δ, R)-Reifenberg flat if there exist positive constants δ and R with the property that for each $x \in \partial\Omega$ and each $\rho \in (0, R)$ there is a local coordinate system $\{y_1, \ldots, y_n\}$ such that

$$B_\rho(x) \cap \{y_n > \delta\rho\} \subset B_\rho(x) \cap \Omega \subset B_\rho(x) \cap \{y_n > -\delta\rho\}. \tag{2.1}$$

Pointing out that the definition is only significant for small values of δ, it means that the boundary $\partial\Omega$ is well approximated by hyperplanes at every point and at every scale. It was Reifenberg who arrived first at that concept of flatness in his studies [9] on Plateau problems and who proved that such a domain is locally a topological disc when δ is small enough. Moreover, he showed that a subset of $\mathbb{R}^n$ which is well approximated by m-dimensional hyperplanes at each point and at each scale is locally a bi-Hölder image of the unit ball in $\mathbb{R}^m$. These results have been extended very recently by David et al. [10] to area-minimizing cones, proving that a subset of $\mathbb{R}^3$ which is well approximated in the sense of Hausdorff distance by one of the three standard area-minimizing cones at each point and at each small scale is locally a bi-Hölder deformation of a minimal cone.

Loosely speaking, the Reifenberg flatness exhibits a sort of "minimal regularity requirement" on the boundary $\partial\Omega$ ensuring the validity in Ω of the main natural properties of the geometric analysis (we refer the reader to the excellent survey by Toro [11]). Indeed, each domain with at least C^1-smooth boundary is Reifenberg flat with δ vanishing when $R \searrow 0$. Another significative example is given by a domain Ω with boundary $\partial\Omega$ locally given as the graph of a Lipschitz continuous function with small ($<\delta$) Lipschitz constant. Actually, the class of Reifenberg flat domains extends beyond these common examples and contains domains with rough fractal boundaries. For instance, the Van Koch snowflake is a Reifenberg flat when the angle of the spike with respect to the horizontal is small enough (cf. [11] once again).

A remarkable feature of the Reifenberg flat domains, which follows immediately from (2.1), is the so-called (A) property of $\partial\Omega$ (cf. [12,13]). Precisely, the Lebesgue measure $|B_\rho(x) \cap \Omega|$ is δ-comparable to the measure of the ball $|B_\rho(x)|$,

$$|B_\rho(x) \cap \Omega| \le C(\delta)|B_\rho(x)| \quad \forall x \in \partial\Omega,\ \forall \rho \in (0, R). \tag{2.2}$$

As a consequence, the Reifenberg flat domains are $W^{1,p}$-extension domains, $1 \le p \le \infty$, which means that the usual extension theorems and the Sobolev inequalities hold true in Ω (cf. [4] and the references therein).

Turning back to the Dirichlet problem (1.1), we will suppose that the nonlinear terms $a^{ij}(x, z)$, $a^i(x, z)$: $\Omega \times \mathbb{R} \to \mathbb{R}$ and $b(x, z, \xi)$: $\Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions which satisfy the following structure and regularity conditions.

Uniform ellipticity: There is a positive constant λ such that

$$\lambda|\eta|^2 \le a^{ij}(x, z)\eta_i\eta_j \le \lambda^{-1}|\eta|^2, \quad a^{ij}(x, z) = a^{ji}(x, z) \tag{2.3}$$

for a.a. $x \in \Omega$, $\forall z \in \mathbb{R}$, $\forall \eta \in \mathbb{R}^n$. If follows in particular that $a^{ij}(x, z) \in L^\infty(\Omega \times \mathbb{R})$ and $\|a^{ij}\|_{L^\infty(\Omega\times\mathbb{R})} \le \lambda^{-1}$.

Optimal growth conditions: There exist a constant $\Lambda > 0$ and positive functions $\varphi_1 \in L^p(\Omega)$ with $p > n$ and $\varphi_2 \in L^q(\Omega)$ with $q > n/2$, such that

$$\begin{cases} |a^i(x, z)| \le \Lambda\left(\varphi_1(x) + |z|^{\frac{n}{n-2}}\right), \\ |b(x, z, \xi)| \le \Lambda\left(\varphi_2(x) + |z|^{\frac{n+2}{n-2}} + |\xi|^{\frac{n+2}{n}}\right) \end{cases} \tag{2.4}$$

for a.a. $x \in \Omega$, $\forall z \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^n$. In the particular case, $n = 2$ of 2D domains Ω, the powers of $|z|$ could be arbitrary positive numbers, while the growth of $|\xi|$ is quadratic.

For what concerns the regularity of the principal coefficients a^{ij} we assume the following.

Local uniform continuity of a^{ij} in z, uniformly in x: For each $M > 0$ there is a non-decreasing function $\mu_M : \mathbb{R}^+ \to \mathbb{R}^+$, $\lim_{t\downarrow 0} \mu_M(t) = 0$ and such that

$$|a^{ij}(x, z) - a^{ij}(x, z')| \le \mu_M(|z - z'|) \tag{2.5}$$

for a.a. $x \in \Omega$, $\forall z, z' \in [-M, M]$.

(δ, R)-vanishing property of $a^{ij}(x, z)$ with respect to x, locally uniformly in z: For each constant $M > 0$ there exist $R_M > 0$ and $\delta_M > 0$ such that

$$\sup_{0<\rho\le R_M} \sup_{x\in\Omega} \sup_{|z|\le M} \fint_{B_\rho(x)\cap\Omega} \left|a^{ij}(y, z) - \overline{a^{ij}}_{B_\rho(x)\cap\Omega}(z)\right|^2 dy \le \delta_M^2, \tag{2.6}$$

where $\overline{a^{ij}}_{B_\rho(x)\cap\Omega}(z)$ stands for the mean integral $\fint_{B_\rho(x)\cap\Omega} a^{ij}(y, z)dy = |B_\rho(x) \cap \Omega|^{-1} \int_{B_\rho(x)\cap\Omega} a^{ij}(y, z)dy$.

It is clear that (2.6) is a kind of small *BMO* condition regarding the behaviour of $a^{ij}(x, z)$ with respect to the variable x. For instance, (2.6) is surely satisfied when the coefficients a^{ij} are continuous with respect to x with δ_M being the modulus of continuity. Similarly, (2.6) holds true if the term on the left-hand side vanishes as $R_M \to 0$, that is, if $a^{ij}(x, \cdot) \in VMO(\Omega)$ and then δ_M is referred to as the *VMO*-modulus of a^{ij}'s (cf. [3]). Therefore, the assumption (2.6) generalizes both the cases of continuous or *VMO* principal coefficients $a^{ij}(x, \cdot)$.

Let us recall that a function $u \in W_0^{1,r}(\Omega)$ is a *weak solution* to the Dirichlet problem (1.1) if

$$\int_\Omega a^{ij}(x, u(x))D_ju(x)D_i\chi(x)\,dx + \int_\Omega a^i(x, u(x))D_i\chi(x)\,dx + \int_\Omega b(x, u(x), Du(x))\chi(x)\,dx = 0$$

for each $\chi \in W_0^{1,r'}(\Omega)$ with $r' = r/(r - 1)$. The convergence of the integrals here appearing is ensured by (2.3) and (2.4) provided

$$r \in \begin{cases} \left[2, \min\left\{p, \dfrac{nq}{n-q}\right\}\right] & \text{if } q < n, \\ [2, p] & \text{if } q \ge n. \end{cases}$$

In what follows, we set

$$s^* = \begin{cases} \dfrac{ns}{n-s} & \text{when } s < n, \\ \text{arbitrary large number} \ge 1 & \text{when } s \ge n \end{cases} \tag{2.7}$$

for the Sobolev conjugate of an arbitrary $s > 1$.

Throughout the article, the omnibus term "*known quantities*" means that a given constant depends on the data of (2.3)–(2.6), which include n, λ, Λ, p, q, $\|\varphi_1\|_{L^p(\Omega)}$, $\|\varphi_2\|_{L^q(\Omega)}$, Ω, δ, R, μ_M, R_M and δ_M and where the constant M will be clear from the context.

3. Calderón–Zygmund property and global Hölder continuity

Our main result asserts that the problem (1.1) supports the Calderón–Zygmund property when Ω is (δ, R)-Reifenberg flat and the coefficients $a^{ij}(\cdot, z)$ are (δ, R)-vanishing for all small enough values of δ. It generalizes Theorem 2.2 and Corollary 2.4 of [3] to the case of small *BMO* coefficients $a^{ij}(\cdot, z)$, optimal growth in (2.4) and Reifenberg flat domains Ω.

Theorem 3.1. *Assume (2.3)–(2.5) and let $u \in W_0^{1,2}(\Omega)$ be a weak solution of the problem (1.1).*

Then there exists a small $\delta_0 > 0$ such that if Ω is (δ, R)-Reifenberg flat and $a^{ij}(x, z)$ are (δ, R)-vanishing (2.6) with $\delta < \delta_0$, then

$$u \in W_0^{1,r}(\Omega) \cap L^\infty(\Omega) \quad \text{with } r = \min\{p, q^*\}. \tag{3.1}$$

Moreover, $r > n$ and therefore the $W_0^{1,2}(\Omega)$-weak solution u of (1.1) is globally Hölder continuous in $\overline{\Omega}$, $u \in C^{0,\gamma}(\overline{\Omega})$ with exponent $\gamma = \min\{1 - n/p, 2 - n/q\}$.

It is worth noting that, without loss of generality, the constant R in Definition 2.1 and in (2.6) can be taken equal to 1 by scaling the equation in (1.1), while δ is scaling invariant. It will be clear from the proof given below that the small constant δ_0 in the statement of Theorem 3.1 will be obtained through [4, Theorem 1.5] applied to the *linearized* problem (3.14), and therefore it depends on λ, n, p and q and the (δ, R)-vanishing property of the composition $A^{ij}(x) = a^{ij}(x, u(x))$, where $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is the fixed solution to (1.1). On the other hand, as shown in Lemma 3.2, the *BMO*-norm of $a^{ij}(x, u(x))$ estimates in terms of δ from (2.6), $\|u\|_{L^\infty(\Omega)}$ and the modulus of continuity of $u(x)$ over the balls of radius R, and the last two quantities are controlled through (3.8) and (3.9), respectively, by means of the data appearing in (2.4) and $\|Du\|_{L^2(\Omega)}$ (cf. (3.2)). This way, the constant δ_0 in Theorem 3.1 depends on λ, Λ, n, p, q, $|\Omega|$, $\partial\Omega$, $\|\varphi_1\|_{L^p(\Omega)}$, $\|\varphi_2\|_{L^q(\Omega)}$, $\mu(\cdot)$ and on the $L^2(\Omega)$-norm of the gradient of the fixed weak solution to (1.1).

Note further that in the particular case, when $\partial\Omega$ is C^1-smooth and $a^{ij}(\cdot, z)$ are *VMO* functions, the existence of δ_0 is automatic and therefore Theorem 3.1 holds for arbitrary C^1-smooth domains Ω and *VMO* principal coefficients $a^{ij}(\cdot, z)$ (compare with [3, Theorem 2.2]).

We split the *proof of* Theorem 3.1 in four steps, obtaining successively fine properties of the $W_0^{1,2}(\Omega)$-weak solution to the problem (1.1). Without loss of generality, we may assume hereafter that Ω is $(1, 1)$-Reifenberg flat, so that $\partial\Omega$ satisfies the (A) condition (2.2) with a constant $C(1)$.

Step 1: *Better integrability of the gradient Du*. Applying the Gehring–Giaquinta–Modica lemma (cf. [14, Chapter V], [13, Chapter III] or [15, Lemma 3.2.23]), the ellipticity condition (2.3) and the controlled growths (2.4) ensure existence of an exponent $r_0 > 2$ such that

$$Du \in L^{r_0}(\Omega) \quad \text{with } \|Du\|_{L^{r_0}(\Omega)} \le C \tag{3.2}$$

where r_0 depends on n, λ, Λ, p, q, $\|\varphi_1\|_{L^p(\Omega)}$, $\|\varphi_2\|_{L^q(\Omega)}$ and the (A) property (2.2) of the boundary $\partial\Omega$, and C depends on $\|Du\|_{L^2(\Omega)}$ in addition. With no loss of generality, we suppose hereafter that $r_0 > 2$ is close enough to 2.

Step 2: *Global boundedness of the weak solution*. We will show now that (2.3) and (2.4) are sufficient conditions ensuring essential boundedness of the $W_0^{1,2}(\Omega)$-weak solution u to (1.1). For, it follows from (3.2) and the Sobolev imbedding theorem that $u \in L^{\frac{nr_0}{n-r_0}}(\Omega)$ when $n > 2$. It is immediate that $u \in L^\infty(\Omega)$ if $n = 2$, so we will concentrate on the case $n > 2$. The global boundedness of $u \in W_0^{1,2}(\Omega) \cap L^{\frac{nr_0}{n-r_0}}(\Omega)$ will be obtained as consequence of [12, Chapter IV, Theorem 7.1]. For this goal, making use of the Young inequality, (2.4) and $r_0 > 2$, we get

$$\begin{aligned} \left|a^{ij}(x, z)\xi_j + a^i(x, z)\right| &\le \lambda^{-1}|\xi| + \Lambda\varphi_1(x) + \Lambda|z|^{\frac{n}{n-2}} \\ &\le \lambda^{-1}|\xi| + \Lambda|z|^{\frac{nr_0}{2(n-r_0)}} + \Lambda\big(1 + \varphi_1(x)\big), \quad \varphi_1 \in L^p(\Omega) \subset L^2(\Omega), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} |b(x, z, \xi)| &\le \Lambda\left(|\xi|^{\frac{n+2}{n}} + |z|^{\frac{n+2}{n-2}} + \varphi_2(x)\right) \\ &\le \Lambda\left(|\xi|^{\frac{2(nr_0-n+r_0)}{nr_0}} + |z|^{\frac{nr_0-n+r_0}{n-r_0}} + 2 + \varphi_2(x)\right), \quad \varphi_2 \in L^q(\Omega) \subset L^{\frac{n}{2}}(\Omega) \subset L^{\frac{nr_0}{nr_0-n+r_0}}(\Omega), \end{aligned} \tag{3.4}$$

for a.a. $x \in \Omega$, all $z \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$.

Further on, for a.a. $x \in \Omega$, all $z \in \mathbb{R}$ with $|z| \ge 1$ and all $\xi \in \mathbb{R}^n$, we have by (2.4) that

$$\begin{aligned} |a^i(x, z)\xi_i| &\le n\Lambda|\xi|\left(|z|^{\frac{n}{n-2}} + \varphi_1(x)\right) \\ &\le n\Lambda\left(2\varepsilon|\xi|^2 + \varepsilon^{-1}|z|^{\frac{2n}{n-2}} + \varepsilon^{-1}\varphi_1^2(x)\right) \\ &\le 2n\Lambda\varepsilon|\xi|^2 + n\Lambda\varepsilon^{-1}|z|^{\frac{2n}{n-2}} + n\Lambda\varepsilon^{-1}|z|^2\varphi_1^2(x) \end{aligned}$$

after applying the Young inequality ($ab \le \varepsilon a^{p'} + \varepsilon^{\frac{1}{1-p'}} b^{\frac{p'}{p'-1}}\,\forall\varepsilon > 0, \forall p' > 1$) with arbitrary $\varepsilon > 0$. This way, (2.3) yields

$$\begin{aligned}\left(a^{ij}(x,z)\xi_j + a^i(x,z)\right)\xi_i &\ge (\lambda - 2n\Lambda\varepsilon)|\xi|^2 - n\Lambda\varepsilon^{-1}|z|^{\frac{2n}{n-2}} - n\Lambda\varepsilon^{-1}|z|^2\varphi_1^2(x)\\ &= \frac{\lambda}{2}|\xi|^2 - \frac{4n^2\Lambda^2}{\lambda}|z|^{\frac{2n}{n-2}} - \frac{4n^2\Lambda^2}{\lambda}|z|^2\varphi_1^2(x)\end{aligned} \tag{3.5}$$

after choosing $\varepsilon = \frac{\lambda}{4n\Lambda}$.

Similarly,

$$\begin{aligned}-b(x,z,\xi)z &\le |b(x,z,\xi)|\,|z| \le \Lambda\left(|\xi|^{\frac{n+2}{n}}|z| + |z|^{\frac{2n}{n-2}} + |z|\varphi_2(x)\right)\\ &\le \varepsilon\Lambda|\xi|^2 + \Lambda\left(1 + \varepsilon^{-\frac{n+2}{n-2}}\right)|z|^{\frac{2n}{n-2}} + \Lambda|z|^2\varphi_2(x)\end{aligned}$$

as a consequence of (2.4), $|z| \ge 1$ and, taking $\varepsilon = \frac{\lambda}{4\Lambda}$, we obtain

$$-b(x,z,\xi)z \le \frac{\lambda}{4}|\xi|^2 + \Lambda\left(1 + \left(\frac{4\Lambda}{\lambda}\right)^{\frac{n+2}{n-2}}\right)|z|^{\frac{2n}{n-2}} + \Lambda|z|^2\varphi_2(x). \tag{3.6}$$

Setting

$$\nu = \frac{\lambda}{2}, \qquad \mu_0 = \frac{\lambda}{4}, \qquad \mu_1 = \max\left\{\frac{4n^2\Lambda^2}{\lambda}, \Lambda\left(1 + \left(\frac{4\Lambda}{\lambda}\right)^{\frac{n+2}{n-2}}\right)\right\},$$

$$\varphi_3(x) = \max\left\{\frac{4n^2\Lambda^2}{\lambda}\varphi_1^2(x), \Lambda\varphi_2(x)\right\} \in L^{\min\{\frac{p}{2},q\}}(\Omega), \quad \min\left\{\frac{p}{2}, q\right\} > \frac{n}{2},$$

(3.5) and (3.6) take on the form

$$\begin{aligned}&\left(a^{ij}(x,z)\xi_j + a^i(x,z)\right)\xi_i \ge \nu|\xi|^2 - \mu_1|z|^{\frac{2n}{n-2}} - |z|^2\varphi_3(x),\\ &-b(x,z,\xi)z \le \mu_0|\xi|^2 + \mu_1|z|^{\frac{2n}{n-2}} + |z|^2\varphi_3(x)\end{aligned} \tag{3.7}$$

for a.a. $x \in \Omega$, all $z \in \mathbb{R}$ with $|z| \ge 1$ and all $\xi \in \mathbb{R}^n$.

With (3.3), (3.4) and (3.7) at hand, we are in a position to apply [12, Chapter IV, Theorem 7.1] which asserts that the $W_0^{1,2}(\Omega)$-weak solution to the Dirichlet problem (1.1) is essentially bounded in Ω and

$$\|u\|_{L^\infty(\Omega)} \le M \tag{3.8}$$

where the constant M depends on known quantities and on $\|u\|_{L^{\frac{nr_0}{n-r_0}}(\Omega)}$ (that is, on $\|Du\|_{L^{r_0}(\Omega)}$) in addition (cf. (3.2)).

Step 3: *Hölder continuity of* u. It follows from (2.3), (2.4) and (3.7) that

$$\begin{aligned}&\left(a^{ij}(x,z)\xi_j + a^i(x,z)\right)\xi_i \ge \nu|\xi|^2 - \overline{\mu}(|z|)\left(1 + \varphi_3(x)\right),\\ &\left(|a^{ij}(x,z)|\,|\xi| + |a^i(x,z)|\right)(1 + |\xi|), |b(x,z,\xi)| \le \overline{\mu}(|z|)\left(\varphi_3(x) + |\xi|^2\right)\end{aligned}$$

with a non-decreasing function $\overline{\mu}(|z|) > 0$ whence, the celebrated De Giorgi–Nash result [1,2], [12, Chapter IV, Theorem 1.1] ensures Hölder continuity of u in $\overline{\Omega}$. Namely, there exist constants $\alpha \in (0,1)$ and H depending on known quantities, on M from (3.8) and on the (A) property of $\partial\Omega$ (see (2.2) and recall Ω is at least $(1,1)$-Reifenberg flat), such that

$$\sup_{x,y\in\overline{\Omega}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \le H. \tag{3.9}$$

Step 4: *Calderón–Zygmund property of* Du. We are going to prove now that the $W_0^{1,2}(\Omega)$-weak solution of the problem (1.1) gains better integrability from the data φ_1 and φ_2. Namely, we are going to show that, under the additional hypotheses (2.5) and (2.6), $u \in W_0^{1,r}(\Omega)$ with $r = \min\{p, q^*\}$ (cf. (2.7)).

For this purpose, fix the solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ into the nonlinear terms at (1.1). This linearizes the Dirichlet problem (1.1) into

$$\begin{cases}\operatorname{div}\left(A^{ij}(x)D_ju\right) = f(x) - \operatorname{div}(\mathbf{A}(x)) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases} \tag{3.10}$$

where

$$\begin{aligned}&A^{ij}(x) := a^{ij}(x,u(x)), \qquad f(x) := b(x,u(x),Du(x)),\\ &\mathbf{A}(x) := \left(a^1(x,u(x)), \ldots, a^n(x,u(x))\right).\end{aligned} \tag{3.11}$$

In order to get higher integrability of the gradient Du, we will apply to the linearized problem (3.10) the results of Byun and Wang from [4] regarding divergence form linear elliptic equations with small BMO coefficients over Reifenberg flat domains, and for this goal we have to be able to control the BMO-norms of the principal coefficients $A^{ij}(x)$ in (3.10). The required control is ensured by the following lemma which generalizes results from [16,15,17] about VMO-mapping properties of superposition operators.

Lemma 3.2. *Assume (2.5) and (2.6) and let $u \in C^0(\overline{\Omega})$. Then the composition $A^{ij}(x) = a^{ij}(x, u(x))$ is (δ, R)-vanishing, that is,*

$$\sup_{0<\rho\le R}\sup_{x\in\Omega} \fint_{B_\rho(x)\cap\Omega} \left|A^{ij}(y) - \overline{A^{ij}}_{B_\rho(x)\cap\Omega}\right|^2 \mathrm{d}y \le \delta^2, \tag{3.12}$$

with $\delta^2 = 6\mu_M^2(\omega_u(R)) + 3\delta_M^2$, M is the constant from (3.8) and $\omega_u(R)$ is the modulus of continuity of the function u.

Proof. Take an arbitrary point $x \in \Omega$ and set $\Omega_\rho(x) = B_\rho(x) \cap \Omega$ for the sake of brevity. Straightforward calculations based on the Young and the Jensen inequalities yield

$$\begin{aligned}
\fint_{\Omega_\rho(x)} \left|A^{ij}(y) - \overline{A^{ij}}_{\Omega_\rho(x)}\right|^2 \mathrm{d}y &= \fint_{\Omega_\rho(x)} \left|\fint_{\Omega_\rho(x)} \left[a^{ij}(y, u(y)) - a^{ij}(z, u(z))\right]\mathrm{d}z\right|^2 \mathrm{d}y \\
&= \fint_{\Omega_\rho(x)} \left|\fint_{\Omega_\rho(x)} \Big[a^{ij}(y, u(y)) - a^{ij}(y, u(x))\right. \\
&\qquad \left. +\ a^{ij}(z, u(x)) - a^{ij}(z, u(z)) + a^{ij}(y, u(x)) - a^{ij}(z, u(x))\Big]\mathrm{d}z\right|^2 \mathrm{d}y \\
&\le 6\fint_{\Omega_\rho(x)} \left|a^{ij}(y, u(y)) - a^{ij}(y, u(x))\right|^2 \mathrm{d}y \\
&\quad + 3\fint_{\Omega_\rho(x)} \left|\fint_{\Omega_\rho(x)} \left[a^{ij}(y, u(x)) - a^{ij}(z, u(x))\right]\mathrm{d}z\right|^2 \mathrm{d}y.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\fint_{\Omega_\rho(x)} \left|a^{ij}(y, u(y)) - a^{ij}(y, u(x))\right|^2 \mathrm{d}y &\le \fint_{\Omega_\rho(x)} \mu^2_{\|u\|_{L^\infty(\Omega)}}\big(|u(x) - u(y)|\big)\mathrm{d}y \\
&\le \mu_M^2(\omega_u(\rho))
\end{aligned}$$

in view of (2.5) and (3.8), while (2.6) yields

$$\begin{aligned}
\fint_{\Omega_\rho(x)} \left|\fint_{\Omega_\rho(x)} \left[a^{ij}(y, u(x)) - a^{ij}(z, u(x))\right]\mathrm{d}z\right|^2 \mathrm{d}y &= \fint_{\Omega_\rho(x)} \left|a^{ij}(y, u(x)) - \fint_{\Omega_\rho(x)} a^{ij}(z, u(x))\mathrm{d}z\right|^2 \mathrm{d}y \\
&= \fint_{\Omega_\rho(x)} \left|a^{ij}(y, u(x)) - \overline{a^{ij}}_{\Omega_\rho(x)}(u(x))\right|^2 \mathrm{d}y \le \delta_M^2.
\end{aligned}$$

This way

$$\sup_{0<\rho\le R}\sup_{x\in\Omega} \fint_{\Omega_\rho(x)} \left|A^{ij}(y) - \overline{A^{ij}}_{\Omega_\rho(x)}\right|^2 \mathrm{d}y \le 6\mu_M^2(\omega_u(R)) + 3\delta_M^2$$

and (3.12) follows. □

Before proceeding further, let us recall (cf. [3, Lemma 3.1]) that for each function $f \in L^s(\Omega)$, $s \in (1, \infty)$, there is a vector field $\mathbf{F}$: $\Omega \to \mathbb{R}^n$, $\mathbf{F}(x) = (F^1(x), \ldots, F^n(x))$, such that

$$\begin{aligned}
&f(x) = \operatorname{div}(\mathbf{F}(x)) \quad \text{for a.a. } x \in \Omega,\ \mathbf{F} \in (L^{s^*}(\Omega))^n, \\
&\|\mathbf{F}\|_{(L^{s^*}(\Omega))^n} \le C(n, s, \partial\Omega)\|f\|_{L^s(\Omega)},
\end{aligned} \tag{3.13}$$

where s^* is given by (2.7).

To get the higher integrability of the gradient Du, we set $\mathbf{F}(x)$ for the vector field corresponding to $f(x) = b(x, u(x), Du(x))$ where $u \in W_0^{1,2}(\Omega)$ is the weak solution to (1.1). This rewrites (3.10) into the form

$$\begin{cases} \operatorname{div}\big(A^{ij}(x)D_j u\big) = \operatorname{div}\big(\mathbf{F}(x) - \mathbf{A}(x)\big) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

where the matrix $\{A^{ij}(x)\}_{i,j=1}^n$ is uniformly elliptic. Moreover, it follows from (3.8), (3.9) and Lemma 3.2 that $A^{ij}(x)$'s are (δ, R)-*vanishing* (see (2.6) and (3.12)). We apply now [4, Theorem 1.5] which ensures existence of a positive δ' such that the *linear* problem (3.14) is well posed in $W^{1,s}(\Omega)$ for all $s \in (1,\infty)$ provided Ω is (δ, R)-Reifenberg flat and $A^{ij}(x)$ are (δ, R)-vanishing with $\delta < \delta'$. Namely, $\mathbf{F} - \mathbf{A} \in (L^s(\Omega))^n$ yields existence of a unique $W_0^{1,s}(\Omega)$-weak solution of (3.14) which coincides with the original $W_0^{1,2}(\Omega)$-solution u of (3.10) and (1.1), and $\|u\|_{W^{1,s}(\Omega)} \leq C\|\mathbf{F} - \mathbf{A}\|_{(L^s(\Omega))^n}$. Based on this result, we will get $u \in W_0^{1,r}(\Omega)$ with $r = \min\{p, q^*\}$ as claimed in Theorem 3.1 by modifying the bootstrapping procedure from [3] and taking into account the optimal values of the growths in (2.4).

To fix the ideas, let us concentrate to the case $n > 2$. It follows from (3.2) and (3.8) that $u \in W^{1,r_0}(\Omega) \cap L^\infty(\Omega)$ whence (2.4) implies $\mathbf{A} \in (L^p(\Omega))^n$ and $|Du|^{\frac{n+2}{n}} \in L^{\frac{r_0 n}{n+2}}(\Omega)$. Therefore $f(x) = b(x, u(x), Du(x)) \in L^{q_1}(\Omega)$ with $q_1 = \min\left\{q, \frac{r_0 n}{n+2}\right\}$. We get $\mathbf{F} \in (L^{q_1^*}(\Omega))^n$ from (3.13) and, since $\mathbf{F} - \mathbf{A} \in (L^{\min\{p, q_1^*\}}(\Omega))^n$, the above mentioned Byun–Wang result [4, Theorem 1.5] yields

$$Du \in L^{r_1}(\Omega) \quad \text{with } r_1 = \min\left\{p, q_1^*\right\} \tag{3.15}$$

for the weak solution u of (3.14), provided Ω is (δ, R)-Reifenberg flat and A^{ij}'s are (δ, R)-vanishing for δ less than some fixed $\delta_1 > 0$.

Let us observe that we have immediately $Du \in L^{\min\{p,q^*\}}(\Omega)$ as claimed in Theorem 3.1 when either $q \leq \frac{r_0 n}{n+2}$ or $\frac{r_0 n}{n+2} \geq n$. Actually, $q \leq \frac{r_0 n}{n+2}$ yields $q_1 = q$, $q_1^* = q^*$ whence $r_1 = \min\{p, q^*\}$. If instead $\frac{r_0 n}{n+2} \geq n$ and $q_1 = \frac{r_0 n}{n+2}$ then q_1^* is arbitrary large number and thus $r_1 = p = \min\{p, q_1^*\}$ once again.

This way, it remains the case

$$q > \frac{r_0 n}{n+2}, \qquad \frac{r_0 n}{n+2} < n$$

to be considered. We have now $q_1 = \frac{r_0 n}{n+2} < n$, $q_1 < q$, whence $q_1^* = \left(\frac{r_0 n}{n+2}\right)^* = \frac{r_0 n}{n+2-r_0} < q^*$ and $r_1 = \min\left\{p, \frac{r_0 n}{n+2-r_0}\right\}$. If $r_1 = p$ we have $p < q_1^* < q^*$ and therefore $Du \in L^p(\Omega)$ which gives (3.1). Otherwise, (3.15) becomes

$$Du \in L^{\frac{r_0 n}{n+2-r_0}}(\Omega).$$

The arguments already used give $|Du|^{\frac{n+2}{n}} \in L^{\frac{r_0 n^2}{(n+2)^2 - r_0(n+2)}}(\Omega)$, $f \in L^{q_2}(\Omega)$ with $q_2 = \min\left\{q, \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)}\right\}$, $\mathbf{F} - \mathbf{A} \in (L^{\min\{p, q_2^*\}}(\Omega))^n$ and therefore

$$Du \in L^{r_2}(\Omega) \quad \text{with } r_2 = \min\left\{p, q_2^*\right\} \tag{3.16}$$

as a consequence of the Byun–Wang result [4, Theorem 1.5], provided Ω is (δ, R)-Reifenberg flat and A^{ij}'s are (δ, R)-vanishing for δ less than some positive and fixed $\delta_2 < \delta_1$. The cases

$$q \leq \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)} \quad \text{or} \quad \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)} \geq n$$

imply (3.1) since $q_2^* = q^*$ as before, whence the remaining situation to be studied is

$$q > \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)} \quad \text{and} \quad \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)} < n.$$

We have now $q_2 = \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)} < n$, $q_2^* = \left(\frac{r_0 n^2}{(n+2)^2 - r_0(n+2)}\right)^* = \frac{r_0 n^2}{(n+2)^2 - r_0(n+2) - r_0 n}$, $r_2 = \min\left\{p, \frac{r_0 n^2}{(n+2)^2 - r_0(n+2) - r_0 n}\right\}$ and $Du \in L^p(\Omega)$ if $r_2 = p$, which completes the proof of Theorem 3.1 because of $p < q_2^* < q^*$ now. Alternatively, $r_2 = \frac{r_0 n^2}{(n+2)^2 - r_0(n+2) - r_0 n}$ and (3.16) becomes

$$Du \in L^{\frac{r_0 n^2}{(n+2)^2 - r_0(n+2) - r_0 n}}(\Omega).$$

Iterating k times this procedure yields $\mathbf{F}(x) - \mathbf{A}(x) \in (L^{\min\{p, q_k^*\}}(\Omega))^n$ with

$$q_k = \min\left\{q, \frac{r_0 n^k}{(n+2)^k - r_0 \sum_{i=0}^{k-2} n^i (n+2)^{k-i-1}}\right\}$$

and

$$Du \in L^{r_k}(\Omega) \quad \text{with } r_k = \min\left\{p, q_k^*\right\} \tag{3.17}$$

if Ω is (δ, R)-Reifenberg flat and A^{ij}'s are (δ, R)-vanishing for δ less than some fixed $\delta_k < \delta_{k-1}$.

Setting $N := \frac{n+2}{n} \in (1,2)$, direct calculations yield

$$\frac{r_0 n^k}{(n+2)^k - r_0 \sum_{i=0}^{k-2} n^i (n+2)^{k-i-1}} = \frac{r_0 n}{(n+2)N^{k-1} - r_0 \sum_{i=0}^{k-2} N^{k-i-1}}$$
$$= \frac{r_0 n}{(n+2)N^{k-1} - r_0 \frac{N^k - N}{N-1}} = \frac{r_0 n (N-1)}{(2-r_0)N^k + r_0 N} \geq n \tag{3.18}$$

if k is the smallest integer with the property $k \geq \log \frac{r_0}{r_0-2} \Big/ \log N$.

With this choice of k, we have $Du \in L^{\min\{p,q^*\}}(\Omega)$ from (3.17) if $q_k = q$ and this gives (3.1). Otherwise, $q_k \geq n$ by (3.18) which yields $r_k = p$ and we get (3.1) once again. It remains to point out that the constant δ_0 in the statement of Theorem 3.1 is exactly δ_k corresponding to the number k of iterations necessary for (3.18) to hold.

Finally, $p > n$ and $q > n/2$ give $r > n$ and therefore $u \in C^{0,\min\{1-n/p,2-n/q\}}(\overline{\Omega})$ by means of the Morrey lemma.

The two-dimensional case $n = 2$ is simpler and is left to the reader. Theorem 3.1 is completely proved. □

4. Quasilinear parabolic problems

The technique described above can be applied to get higher regularity for the weak solutions to the Cauchy–Dirichlet problem for divergence form quasilinear parabolic equations

$$\begin{cases} \operatorname{div}\big(\hat{a}^{ij}(x,t,u)D_j u + \hat{a}^i(x,t,u)\big) - u_t = \hat{b}(x,t,u,Du) & \text{in } Q,\\ u = 0 & \text{on } \partial_p Q, \end{cases} \tag{4.1}$$

where Q is the cylinder $\Omega \times (0,T)$, $\partial_p Q$ stands for its parabolic boundary $\Omega \cup S = (\Omega \times \{t=0\}) \cup (\partial\Omega \times [0,T])$, and the base $\Omega \subset \mathbb{R}^n$ is supposed to be *Reifenberg flat* domain in the sense of Definition 2.1.

Given a function $u: Q \to \mathbb{R}$, we denote $u_t = \partial u/\partial t$ and $Du = (D_1 u, \ldots, D_n u)$ stands for its *spatial* gradient. If instead $\mathbf{F}: Q \to \mathbb{R}^n$ is a given vector field $\mathbf{F}(x,t) = (F_1(x,t), \ldots, F_n(x,t))$ then the *divergence* operator is acting on the spatial variables x only, that is,

$$\operatorname{div} \mathbf{F}(x,t) = \sum_{i=1}^n \frac{\partial F_i(x,t)}{\partial x_i}.$$

The nonlinear terms

$$\hat{a}^{ij}(x,t,z), \hat{a}^i(x,t,z) \colon\ Q \times \mathbb{R} \to \mathbb{R}, \qquad \hat{b}(x,t,z,\xi) \colon\ Q \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

of the equation in (4.1) will be supposed to be of Carathéodory type and to verify the following equivalents of (2.3)–(2.6) for a.a. $(x,t) \in Q$, all $z \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$.

Uniformly parabolicity: There is a constant $\lambda > 0$ such that

$$\lambda|\xi|^2 \leq \hat{a}^{ij}(x,t,z)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2, \quad \hat{a}^{ij}(x,t,z) = \hat{a}^{ji}(x,t,z). \tag{4.2}$$

Optimal growth conditions: There exist a positive constant $\Lambda > 0$ and positive functions $\hat{\varphi}_1 \in L^{\hat{p}}(Q)$ with $\hat{p} > n+2$, and $\hat{\varphi}_2 \in L^{\hat{q}}(Q)$ with $\hat{q} > (n+2)/2$ such that

$$\begin{aligned} &|\hat{a}^i(x,t,z)| \leq \Lambda\Big(\hat{\varphi}_1(x,t) + |z|^{\frac{n+2}{n}}\Big),\\ &|\hat{b}(x,t,z,\xi)| \leq \Lambda\Big(\hat{\varphi}_2(x,t) + |z|^{\frac{n+4}{n}} + |\xi|^{\frac{n+4}{n+2}}\Big). \end{aligned} \tag{4.3}$$

Local uniform continuity of $\hat{a}^{ij}$ in z, uniformly in (x,t):

$$|\hat{a}^{ij}(x,t,z) - \hat{a}^{ij}(x,t,z')| \leq \mu_M(|z-z'|) \quad \forall z,z' \in [-M,M], \tag{4.4}$$

with a non-decreasing function $\mu_M : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{s\downarrow 0} \mu_M(s) = 0$.

(δ, R)-vanishing property of $\hat{a}^{ij}(x,t,z)$ with respect to (x,t), locally uniformly in z: For each $M > 0$ there exist $R_M > 0$ and $\delta_M > 0$ such that

$$\sup_{0<\rho\leq R_M} \sup_{(x,t)\in Q} \sup_{|z|\leq M} ⨍_{C_\rho(x,t)\cap Q} \Big|\hat{a}^{ij}(y,s,z) - \overline{\hat{a}^{ij}}_{C_\rho(x,t)\cap Q}(z)\Big|^2 \,dyds \leq \delta_M^2 \tag{4.5}$$

where $C_\rho(x,t)$ is the centered parabolic cylinder $B_\rho(x) \times \left(t-\rho^2/2, t+\rho^2/2\right]$ and $\overline{\hat{a}^{ij}}_{C_\rho(x,t)\cap Q}(z)$ stands for the mean integral

$$⨍_{C_\rho(x,t)\cap Q} \hat{a}^{ij}(y,s,z)\mathrm{d}y\mathrm{d}s = \frac{1}{|C_\rho(x,t)\cap Q|}\int_{C_\rho(x,t)\cap Q} \hat{a}^{ij}(y,s,z)\mathrm{d}y\mathrm{d}s.$$

As in the elliptic case, (4.5) expresses a *small BMO condition* for what concerns the regularity of $a^{ij}(x,t,z)$ with respect to the independent variables (x,t), and (4.5) automatically holds true for continuous with respect to (x,t) coefficients a^{ij} with modulus of continuity δ_M. Moreover, (4.5) is satisfied if $a^{ij}(x,t,\cdot) \in VMO(Q)$ when the term on the left-hand side is infinitesimal as $R_M \to 0$.

Let us introduce now the natural functional spaces for the solutions to (4.1).

Definition 4.1. Let $1 < r < \infty$ be arbitrary.

1. The Sobolev space $W_r^{1,0}(Q)$ (see [18]) consists of all functions $u: Q \to \mathbb{R}$ with finite norm

$$\|u\|_{W_r^{1,0}(Q)} = \left(\|u\|^r_{L^r(Q)} + \|Du\|^r_{L^r(Q)}\right)^{1/r}.$$

2. The space $W_*^{1,r}(Q)$ (cf. [5,18] if $r=2$) is defined as the collection of these functions $u \in W_r^{1,0}(Q)$ for which there exist a vector field $\mathbf{F} \in \left(L^r(Q)\right)^n$ and a function $g \in L^r(Q)$ such that

$$u_t = \operatorname{div}\mathbf{F} - g \quad \text{in } Q \tag{4.6}$$

in the sense of distributions, that is,

$$\int_Q u\chi_t \mathrm{d}x\mathrm{d}t = \int_Q \left(\mathbf{F}\cdot D\chi + g\chi\right)\mathrm{d}x\mathrm{d}t$$

for each $\chi \in C_0^\infty(Q)$ with $\chi = 0$ on $t = T$.

The norm in $W_*^{1,r}(Q)$ is defined by

$$\|u\|_{W_*^{1,r}(Q)} = \|u\|_{W_r^{1,0}(Q)} + \inf\left\{\left(\int_Q \left(|\mathbf{F}|^r + |g|^r\right)\mathrm{d}x\mathrm{d}t\right)^{1/r}\right\}$$

where the infimum is taken over all $\mathbf{F}$ and g satisfying (4.6).

The closure of $C_0^\infty(Q)$ with respect to this norm is denoted by $\mathring{W}_*^{1,r}(Q)$.

It is clear that

$$W_*^{1,r}(Q) = L^r\left(0,T; W^{1,r}(\Omega)\right) \cap W^{1,r}\left(0,T; W^{-1,r'}(\Omega)\right)$$

where $1/r + 1/r' = 1$ and $W^{-1,r'}(\Omega)$ stands for the dual of the usual Sobolev space $W^{1,r}(\Omega) = \Big\{v: \Omega \to \mathbb{R}:\ \|v\|_{L^r(\Omega)} + \|Dv\|_{L^r(\Omega)} < \infty\Big\}$.

3. $V_2(Q)$ stands for the Banach space (see [18]) of all functions $u \in W_2^{1,0}(Q)$ with finite norm

$$\|u\|_{V^2(Q)} = \operatorname*{ess\,sup}_{t\in[0,T]} \|u(\cdot,t)\|_{L^2(\Omega)} + \|Du\|_{L^2(Q)}.$$

4. The Banach space consisting of all elements $u \in V_2(Q)$ that are continuous in t with respect to the norm of $L^2(\Omega)$, that is,

$$\lim_{\Delta t\to 0} \|u(\cdot,t+\Delta t) - u(\cdot,t)\|_{L^2(\Omega)} = 0$$

is denoted by $V_2^{1,0}(Q)$ (see [18]). The norm in $V_2^{1,0}(Q)$ is given by

$$\|u\|_{V_2^{1,0}(Q)} = \max_{t\in[0,T]} \|u(\cdot,t)\|_{L^2(\Omega)} + \|Du\|_{L^2(Q)}.$$

5. $V_2^{1,1/2}(Q)$ is the subspace of those $u \in V_2^{1,0}(Q)$ for which

$$\lim_{h\to 0}\int_0^{T-h}\int_\Omega \frac{|u(x,t+h) - u(x,t)|^2}{h}\mathrm{d}x\mathrm{d}t = 0.$$

It follows from [18] that in the particular case $q = 2$ one has

$$W_*^{1,2}(Q) \equiv V_2^{1,1/2}(Q).$$

The weak solution to the problem (4.1) is now defined in a standard way. Namely, a function $u \in \mathring{W}^{1,r}_*(Q)$ is said to be a *r-weak solution* to the Cauchy–Dirichlet problem (4.1) if

$$\int_Q u(x,t)\chi_t(x,t)\mathrm{d}x\mathrm{d}t - \int_Q \left(\hat{a}^{ij}(x,t,u(x,t))D_j u(x,t) + \hat{a}^i(x,t,u(x,t))\right)D_i\chi(x,t)\mathrm{d}x\mathrm{d}t$$
$$= \int_Q \hat{b}(x,t,u(x,t),Du(x,t))\chi(x,t)\mathrm{d}x\mathrm{d}t$$

for each test function $\chi \in \mathring{W}^{1,r'}_*(Q)$, $r' = r/(r-1)$, such that $\chi(x,T) = 0$.

Finally, the parabolic Sobolev conjugate of a number $s > 1$ is defined by

$$s^{**} = \begin{cases} \dfrac{s(n+2)}{n+2-s} & \text{if } s < n+2,\\ \text{arbitrary large number} & \text{if } s \geq n+2. \end{cases}$$

The following result is a parabolic counterpart of Theorem 3.1.

Theorem 4.2. *Assume (4.2)–(4.4) and let $u \in \mathring{W}^{1,2}_*(Q)$ be a weak solution to the Cauchy–Dirichlet problem (4.1).*

Then there exists a small $\delta_0 > 0$ such that if Ω is (δ, R)-Reifenberg flat and $\hat{a}^{ij}(x,t,z)$ are (δ, R)-vanishing (4.5) with $\delta < \delta_0$, then

$$u \in \mathring{W}^{1,\hat{r}}_*(Q) \cap L^\infty(Q) \quad \text{with } \hat{r} = \min\{\hat{p}, \hat{q}^{**}\}.$$

Moreover, $\hat{r} > n+2$ and the 2-weak solution u of (4.1) is globally Hölder continuous in $\overline{Q}$ with exponent $\hat{\gamma} = \min\left\{1-(n+2)/\hat{p}, 2-(n+2)/\hat{q}\right\}$, that is,

$$\sup_{(x,t),(y,s)\in\overline{Q}} \frac{|u(x,t)-u(y,s)|}{\left(|x-y|^2+|t-s|\right)^{\hat{\gamma}/2}} \leq C.$$

The proof of Theorem 4.2 follows the approach used in Section 3. Precisely, (4.2), the optimal growth condition (4.3) and the reverse Hölder inequalities for nonlinear parabolic equations (cf. [19]) imply $Du \in L^{\hat{r}_0}(Q)$ with $\hat{r}_0 > 2$. Later on, (4.2), (4.3) and [18, Chapter V] yield essential boundedness of the 2-weak solution to (4.1), while the celebrated result of Nash [2] implies its Hölder continuity in $\overline{Q}$ with an exponent $\hat{\alpha} \in (0,1)$. Finally, a combination of the bootstrapping procedure from Section 3 with the L^p-theory of linear divergence form parabolic equations due to Byun and Wang (see [5]) leads to the claim of Theorem 4.2. We left the details to the interested reader.

References

[1] E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. (3) 3 (1957) 25–43.
[2] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. 80 (1958) 931–954.
[3] D.K. Palagachev, Global Hölder continuity of weak solutions to quasilinear divergence form elliptic equations, J. Math. Anal. Appl. 359 (2009) 159–167.
[4] S.-S. Byun, L. Wang, Elliptic equations with *BMO* coefficients in Reifenberg domains, Comm. Pure Appl. Math. 57 (2004) 1283–1310.
[5] S.-S. Byun, L. Wang, Parabolic equations in Reifenberg domains, Arch. Ration. Mech. Anal. 176 (2005) 271–301.
[6] E. Milakis, T. Toro, Divergence form operators in Reifenberg flat domains, Math. Z. 264 (2010) 15–41.
[7] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation. I. Basic Theory, in: Grundl. Math. Wissensch., vol. 330, Springer-Verlag, Berlin, 2006.
[8] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation. II. Applications, in: Grundl. Math. Wissensch., vol. 331, Springer-Verlag, Berlin, 2006.
[9] E.R. Reifenberg, Solution of the Plateau problem for m-dimensional surfaces of varying topological type, Acta Math. 104 (1960) 1–92.
[10] G. David, T. De Pauw, T. Toro, A generalization of Reifenberg's theorem in $\mathbb{R}^3$, Geom. Funct. Anal. 18 (2008) 1168–1235.
[11] T. Toro, Doubling and flatness: geometry of measures, Notices Amer. Math. Soc. 44 (1997) 1087–1094.
[12] O.A. Ladyzhenskaya, N.N. Ural'tseva, Linear and Quasilinear Equations of Elliptic Type, 2nd ed., Nauka, Moscow, 1973 (in Russian).
[13] S. Campanato, Sistemi ellittici in forma divergenza, in: Regolarità all'interno, Pubblicazioni della Classe di Scienze: Quaderni, Scuola Norm. Sup., Pisa, 1980.
[14] M. Giaquinta, Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, in: Annals of Mathematics Studies, vol. 105, Princeton University Press, Princeton, NJ, 1983.
[15] A. Maugeri, D.K. Palagachev, L.G. Softova, Elliptic and Parabolic Equations with Discontinuous Coefficients, in: Math. Res., vol. 109, Wiley-VCH, Berlin, 2000.
[16] D.K. Palagachev, Quasilinear elliptic equations with *VMO* coefficients, Trans. Amer. Math. Soc. 347 (1995) 2481–2493.
[17] D.K. Palagachev, L. Recke, L.G. Softova, Applications of the differential calculus to nonlinear elliptic operators with discontinuous coefficients, Math. Ann. 336 (2006) 617–637.
[18] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, in: Transl. Math. Monographs, vol. 23, Amer. Math. Soc., RI, 1967.
[19] A.A. Arkhipova, L_p-estimates of the gradients of solutions of initial/boundary-value problems for quasilinear parabolic systems, J. Math. Sci. 73 (1995) 609–617.