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SOME APPLICATIONS OF DIFFERENTIAL SUBORDINATION

K.S. PADMANABHAN AND R. PARVATHAM

Let $S_a(h)$ denote the class of analytic functions f on the unit disc E with $f(0) = 0 = f'(0)-1$ satisfying $\frac{z(K_a * f)'}{(K_a * f)} \prec h$, where $K_a(z) = \frac{z}{(1-z)^a}$ (a real), $K_a * f$ denotes the Hadamard product of K_a with f, and h is a convex univalent function on E, with Re $h > 0$. Let $F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\,dt$. It is proved that $F \in S_a(h)$ whenever $f \in S_a(h)$ and also that $S_{a+1}(h) \subset S_a(h)$ for $a \geq 1$. Three more such classes are introduced and studied here. The method of differential subordination due to Eenigenburg et al. is used.

Let $E = \{z \in C : |z| < 1\}$ and $H(E)$ be the set of all functions holomorphic in E. Let $A = \{f \in H(E) \ ; \ f(0) = f'(0) - 1 = 0\}$. By $f * g$ we denote the Hadamard product or convolution of $f, g \in H(E)$. That is,

$$f(z) = \sum_{j=0}^{\infty} a_j z^j \ , \ g(z) = \sum_{j=0}^{\infty} b_j z^j \ \text{ then } \ (f * g)(z) = \sum_{j=0}^{\infty} a_j b_j z^j \ .$$

Let g and G be two functions in $H(E)$. Then we say that $g(z)$ is subordinate to $G(z)$ (written $g(z) \prec G(z)$) if $G(z)$ is univalent, $g(0) = G(0)$ and $g(E) \subset G(E)$.

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We need the following theorem due to P. Eenigenburg, S.S. Miller, P. T. Mocanu and M.O. Reade [3].

THEOREM A. *Let* $\beta, \gamma \in C$, *let* $h \in H(E)$ *be convex univalent in* E *with* $h(0) = 1$ *and* $\mathrm{Re}(\beta h(z)+\gamma) > 0$, $z \in E$ *and let* $p \in H(E)$, $p(z) = 1 + p_1 z + \ldots$. *Then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)$$

implies that $p(z) \prec h(z)$.

We also require a modification of the above result which we will use very often in the sequel. We state it in the form of a lemma.

LEMMA 1. *Let* $\beta, \gamma \in C$, $h \in H(E)$ *be convex univalent in* E *with* $h(0) = 1$ *and* $\mathrm{Re}(\beta h(z)+\gamma) > 0$, $z \in E$ *and let* $q \in H(E)$ *with* $q(0) = 1$ *and* $q(z) \prec h(z)$, $z \in E$. *If* $p(z) = 1 + p_1 z + \ldots$ *is analytic in* E , *then*

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z) .$$

The proof is essentially the same as that of Theorem A. However we give the details below for the sake of completeness.

Proof of Lemma 1. We first suppose that all functions under consideration are analytic in $\overline{E}$, and show that if $p(z)$ is not subordinate to $h(z)$, then there is a z_o in E such that

$$p(z_o) + \frac{z_o p'(z_o)}{\beta q(z_o) + \gamma} \notin h(E) \tag{1}$$

which contradicts the hypothesis.

If $p(z)$ is not subordinate to $h(z)$, then by Lemma B in P. Eenigenburg, S.S. Miller, P.T. Mocanu and M.O. Reade [3], we conclude that there are $z_o \in E$, $\zeta_o \in \partial E$ and m , $m \geq 1$ such that $p(z_o) = h(\zeta_o)$, $\arg[z_o p'(z_o)] = \arg[\zeta_o p'(\zeta_o)]$ and $|z_o p'(z_o)| = m|\zeta_o h'(\zeta_o)| > 0$. Hence we can write

$$p(z_o) + \frac{z_o p'(z_o)}{\beta q(z_o) + \gamma} = h(\zeta_o) + m\frac{\zeta_o h'(\zeta_o)}{\beta q(z_o) + \gamma} . \tag{2}$$

Now $q(z) \prec h(z) \Rightarrow \beta q(z) + \gamma \prec \beta h(z) + \gamma$ and so $\mathrm{Re}(\beta h(z)+\gamma) > 0 \Rightarrow \mathrm{Re}(\beta q(z)+\gamma) > 0$, or equivalently

$$|\arg(\beta q(z)+\gamma)| < \frac{\pi}{2} \quad \text{for} \quad z \in E .$$

Hence $|\arg(\beta q(z_o)+\gamma)| < \frac{\pi}{2}$ and $\zeta_o h'(\zeta_o)$ is in the direction of the outer normal to the convex domain $h(E)$, so that the right-hand member of (2) is a complex number outside $h(E)$; that is, (1) holds. We conclude that $p(z) \prec h(z)$ provided all functions under consideration are analytic in $\bar{E}$.

To remove the restriction imposed, we need to replace $p(z)$ by $p_\rho(z) = p(\rho z)$ and $h(z)$ by $h_\rho(z) = h(\rho z)$, $0 < \rho < 1$. All the hypotheses of the lemma are satisfied and we conclude that $p_\rho(z) \prec h_\rho(z)$ for each ρ , $0 < \rho < 1$. By letting $\rho \to 1$ we obtain $p(z) \prec h(z)$.

Let $K_a(z) = \dfrac{z}{(1-z)^a}$ where a is any real number. We consider the convolution of $f \in A$ with K_a . We need the following easily verified result:

$$z(K_a * f)'(z) = a(K_{a+1} * f)(z) - (a-1)(K_a * f)(z) . \tag{3}$$

In the sequel $h \in H(E)$ is convex univalent in E and satisfies $h(0) = 1$ and $\mathrm{Re}(h(z)) > 0$ for $z \in E$.

DEFINITION 1. Let $S_a(h)$ denote the class of functions $f \in A$ such that $\dfrac{z(K_a * f)'(z)}{(K_a * f)(z)} \prec h(z)$ where $\dfrac{(K_a * f)(z)}{z} \neq 0$ for $z \in E$.

REMARK 1. If $a = 1$ and $h(z) = \dfrac{1 - z}{1 + z}$, then $(K_a * f)(z) \equiv f(z)$, and $S_a(h) = S^*$, the class of starlike univalent functions.

THEOREM 1. *If* $f \in S_{a+1}(h)$, *then* $f \in S_a(h)$ *holds for* $a \geq 1$ *provided* $\dfrac{(K_a * f)(z)}{z} \neq 0$ *for* $z \in E$.

Proof. Let $p(z) = \dfrac{z(K_a * f)'((z)}{(K_a * f)(z)}$. Using (3), we get

$$p(z) + (a-1) = \frac{a(K_{a+1} * f)(z)}{(K_a * f)(z)} .$$

Taking logarithmic derivatives and multiplying by z , we get

$$\frac{zp'(z)}{p(z)+(a-1)} = \frac{z(K_{a+1}*f)'(z)}{(K_{a+1}*f)(z)} - p(z) ,$$

which yields

$$\frac{z(K_{a+1}*f)'(z)}{(K_{a+1}*f)(z)} = \frac{zp'(z)}{p(z)+(a-1)} + p(z) . \tag{4}$$

This means that if $f \in S_{a+1}(h)$, then

$$\frac{zp'(z)}{p(z)+(a-1)} + p(z) \prec h(z) . \tag{5}$$

From Theorem A, it follows that for $a \geq 1$, $p(z) \prec h(z)$; that is $\frac{z(K_a*f)'(z)}{(K_a*f)(z)} \prec h(z)$ which means $f \in S_a(h)$ for all $a \geq 1$.

DEFINITION 2. Let $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. Define $h_r(z) = \sum_{j=1}^{\infty}\left(\frac{r+1}{r+j}\right)z^j$ for $\operatorname{Re} r > 0$ and $F(z) = (f*h_r)(z) = \sum_{j=1}^{\infty}\left(\frac{r+1}{r+j}\right)a_j z^j$

Then $F(z) = \frac{(r+1)}{z^r}\int_0^z t^{r-1}f(t)\,dt$.

THEOREM 2. *Suppose* $f \in S_a(h)$; *then* $F \in S_a(h)$ *provided* $\frac{(K_a*F)(z)}{z} \neq 0$ *for* $z \in E$.

Proof. We have $zF'(z) + rF(z) = (r+1)f(z)$; and so

$$(K_a*(zF'))(z) + r(K_a*F)(z) = (r+1)(K_a*f)(z) .$$

Using the fact

$$z(K_a*F)'(z) = (K_a*zF')(z) , \tag{6}$$

we obtain

$$z(K_a*F)'(z) + r(K_a*F)(z) = (r+1)(K_a*f)(z) . \tag{7}$$

Let $p(z) = \frac{z(K_a*F)'(z)}{(K_a*F)(z)}$. Then (7) yields

$$p(z) + r = (r+1)\frac{(K_a*f)(z)}{(K_a*F)(z)} ;$$

taking logarithmic derivatives and multiplying by z, we get

$$\frac{zp'(z)}{p(z)+r}+p(z)=\frac{z(K_a*f)'(z)}{(K_a*f)(z)}\ . \tag{8}$$

Since $f\in S_a(h)$, it follows that $\frac{zp'(z)}{p(z)+r}+p(z)\prec h(z)$ which implies that $p(z)\prec h(z)$ by Theorem A. Thus

$$\frac{z(K_a*F)'(z)}{(K_a*F)(z)}=p(z)\prec h(z)\ ;$$

that is $F\in S_a(h)$.

REMARK 2. If a is a positive integer ≥ 1 and $h(z)=\frac{1-z}{1+z}$, then the above theorem reduces to Theorem 5 in St. Ruscheweyh [8].

REMARK 3. If $a=1$, $r=1$ and $h(z)=\frac{1-z}{1+z}$, we deduce Theorem 1 of R.J. Libera (1965).

REMARK 4. If $a=1$ and $h(z)=\frac{1+Az}{1+Bz}$, $-1\leq A<B\leq 1$, we deduce Lemma 2 of R.M. Goel and B.S. Mehrok [4].

DEFINITION 3. Let $C_a(h)$ (h as above) denote the class of functions $f\in A$ such that $\frac{z(K_a*f)'(z)}{(K_a*\phi)(z)}\prec h(z)$ for some $\phi\in S_a(h)$.

REMARK 5. If $a=1$ and $h(z)=\frac{1-z}{1+z}$, then $C_a(h)=C$ the class of close-to-convex functions introduced by W. Kaplan [5].

THEOREM 3. *If* $f\in C_{a+1}(h)$, *then* $f\in C_a(h)$ *holds for* $a\geq 1$ *provided* $\frac{(K_a*\phi)(z)}{z}\neq 0$ *for* $z\in E$.

Proof. Let $p(z)=\frac{z(K_a*f)'(z)}{(K_a*\phi)(z)}$. Using (3) and simplifying

$$(K_a*\phi)(z)p(z)+(a-1)(K_a*f)(z)=a(K_{a+1}*f)(z)\ .$$

Differentiating and multiplying by z, we obtain

$$(K_a*\phi)(z)zp'(z)+zp(z)(K_a*\phi)'(z)+(a-1)z(K_a*f)'(z)=za(K_{a+1}*f)'(z)\ . \tag{9}$$

Let $q(z) = \dfrac{z(K_a * \phi)'(z)}{(K_a * \phi)(z)}$. Then once again using (3),

$$q(z) + (a-1) = \frac{a(K_{a+1} * \phi)(z)}{(K_a * \phi)(z)} . \tag{10}$$

From (9) and (10), we obtain

$$\frac{zp'(z)}{q(z) + (a-1)} + p(z) = \frac{z(K_{a+1} * f)'(z)}{(K_{a+1} * \phi)(z)} . \tag{11}$$

If $f \in C_{a+1}(h)$, then $\dfrac{zp'(z)}{q(z) + (a-1)} + p(z) \prec h(z)$. Applying Lemma 1 for $a \geq 1$, we get $\dfrac{z(K_a * f)'(z)}{(K_a * \phi)(z)} = p(z) \prec h(z)$. Since $\phi \in S_{a+1}(h)$, ϕ also belongs to $S_a(h)$ for $a \geq 1$ by Theorem 1. Hence $f \in C_a(h)$.

THEOREM 4. *Suppose* $f \in C_a(h)$ *with respect to the function* $\phi \in S_a(h)$. *Define* ψ *by* $\psi(z) = (\phi * h_r)(z)$. *Then* $F \in C_a(h)$ *with respect to* ψ , *provided* $\dfrac{(K_a * \psi)(z)}{z} \neq 0$ *for* $z \in E$.

Proof. Since $f \in C_a(h)$ with respect to $\phi \in S_a(h)$, we have $\dfrac{z(K_a * f)'(z)}{(K_a * \phi)(z)} \prec h(z)$. Since $\psi(z) = (\phi * h_r)(z)$, we have

$\psi(z) = \dfrac{(r+1)}{z^r} \displaystyle\int_0^z t^{r-1}\phi(t)\,dt$. By Theorem 2 $\psi \in S_a(h)$ provided $\dfrac{(K_a * \psi)(z)}{z} \neq 0$ for $z \in E$. Also

$$z(K_a * \psi)'(z) + r(K_a * \psi)(z) = (r+1)(K_a * \phi)(z) . \tag{12}$$

Also we have

$$z(K_a * F)'(z) + r(K_a * F)(z) = (r+1)(K_a * f)(z) . \tag{13}$$

Let $p(z) = \dfrac{z(K_a * F)'(z)}{(K_a * \psi)(z)}$. Then

$$p(z)(K_a * \psi)(z) + r(K_a * F)(z) = (r+1)(K_a * f)(z) ;$$

differentiating with respect to z , multiplying by z and dividing by $(K_a * \psi)(z)$, we get

$$zp'(z) + p(z)(q(z)+r) = (r+1)\frac{z(K_a * f)'(z)}{(K_a * \psi)(z)}$$

where $q(z) = \dfrac{z(K_a * \psi)'(z)}{(K_a * \psi)(z)} \prec h(z)$. Hence

$$\frac{zp'(z)}{q(z) + r} + p(z) = (r+1)\frac{z(K_a * f)'(z)}{z(K_a * \psi)'(z) + r(K_a * \psi)(z)}$$

$$= \frac{z(K_a * f)'(z)}{(K_a * \phi)(z)}$$

by (12). Since $f \in C_a(h)$, $\dfrac{zp'(z)}{q(z) + r} + p(z) \prec h(z)$. Once again by Lemma 1, $p(z) \prec h(z)$ and hence $F \in C_a(h)$.

REMARK 6. For $a = 1$, $r = 1$ and $h(z) = \dfrac{1 - z}{1 - z}$, this theorem reduces to Theorem 3 in R.J. Libera [6].

DEFINITION 4. Let $C_a^{\alpha}(h)$, $\alpha > 0$ denote the class of functions $f \in A$ such that

$$J_a(\alpha;f;\phi) = \alpha\frac{z(K_{a+1} * f)'(z)}{(K_{a+1} * \phi)(z)} + (1-\alpha)\frac{z(K_a * f)'(z)}{(K_a * \phi)(z)} \prec h(z)$$

for some $\phi \in S_a(h)$ satisfying $\dfrac{(K_{a+1} * \phi)(z)}{z} \neq 0$ for $z \in E$.

THEOREM 5. *If* $f \in C_a^{\alpha}(h)$ *, then* $f \in C_a^{\rho}(h) = C_a(h)$ *, for* $a \geq 1$.

Proof. Let $p(z) = \dfrac{z(K_a * f)'(z)}{(K_a * \phi)(z)}$ where $\phi \in S_a(h)$. By (11)

$$\frac{z(K_{a+1} * f)'(z)}{(K_{a+1} * \phi)(z)} = \frac{zp'(z)}{q(z) + (a-1)} + p(z)$$

where $q(z) = \dfrac{z(K_a * \phi)'(z)}{(K_a * \phi)(z)}$. Hence

$$J_a(\alpha;f;\phi) = \alpha\Big(\frac{zp'(z)}{q(z) + (a-1)} + p(z)\Big) + (1-\alpha)p(z)$$

$$= \frac{\alpha zp'(z)}{q(z) + (a-1)} + p(z) .$$

If $f \in C_a^\alpha(h)$, then $\dfrac{\alpha z p'(z)}{q(z) + (a-1)} + p(z) \prec h(z)$. By Lemma 1, $p(z) \prec h(z)$ which implies $f \in C_a(h)$.

THEOREM 6. *For* $\alpha > \beta \geq 0$ $C_a^\alpha(h) \subset C_a^\beta(h)$.

Proof. The case $\beta = 0$ was treated in Theorem 5. Hence we can assume that $\beta \neq 0$. Suppose that $f \in C_a^\alpha(h)$. Then $J_a(\alpha;f;\phi)(z) \prec h(z)$. Let z_1 be any arbitrary point in E . Then

(14) $$J_a(\alpha;f;\phi)(z_1) \in h(E) .$$

Also $\dfrac{z(K_a * f)'(z)}{(K_a * \phi)(z)} \prec h(z)$ by the previous theorem; so

(15) $$\frac{z_1(K_a * f)'(z_1)}{(K_a * \phi)(z_1)} \in h(E) .$$

Also $J_a(\beta;f;\phi)(z) = (1-\frac{\beta}{\alpha}) \dfrac{z(K_a * f)'(z)}{(K_a * \phi)(z)} + \frac{\beta}{\alpha} J_a(\alpha;f;\phi)(z)$.

Since $\frac{\beta}{\alpha} < 1$ and $h(E)$ is convex $J_a(\beta;f;\phi)(z_1) \in h(E)$ by virtue of (14) and (15). It follows that $J_a(\beta;f;\phi)(z) \prec h(z)$. That is, $f \in C_a^\beta(h)$.

REMARK 7. If $a = 1$ and $h(z) = \dfrac{1 - z}{1 + z}$, Theorem 5 and Theorem 6 reduce to Theorem 1 and Theorem 2 respectively of Pram Nath Chichra [2].

REMARK 8. If $a = 1$ and $h(z) = \dfrac{1 + (2^\rho-1)z}{1 + z}$, $0 \leq \rho \leq 1$, Theorem 5 reduces to Theorem 1 of V.A. Zmorovich and V.A. Pokhilevich [9].

DEFINITION 5. Let $S_a^\alpha(h)$, $\alpha \geq 0$ denote the class of functions $f \in A$ such that

$$J_a(\alpha,f) = \alpha\frac{z(K_{a+1} * f)'(z)}{(K_{a+1} * f)(z)} + (1-\alpha)\frac{z(K_a * f)'(z)}{(K_a * f)(z)} \prec h(z)$$

with $\dfrac{(K_{a+1} * f)(z)}{z} \neq 0$ and $\dfrac{(K_a * f)(z)}{z} \neq 0$ for $z \in E$.

THEOREM 7. *If* $f \in S_a^\alpha(h)$ *then* $f \in S_a^0(h) = S_a(h)$, *for* $a \geq 1$.

Proof. Let $p(z) = \dfrac{z(K_a * f)'(z)}{(K_a * f)(z)}$. Then using (4) we have

$$J_a(\alpha,f) = \alpha\left(\frac{zp'(z)}{p(z) + (a-1)}\right) + (1-\alpha)p(z)$$

$$= \frac{\alpha zp'(z)}{p(z) + (a-1)} + p(z) .$$

Since $f \in S_a(h)$, $\dfrac{\alpha zp'(z)}{p(z) + (a-1)} + p(z) \prec h(z)$ and Theorem A implies that $p(z) \prec h(z)$ for $a \geq 1$. Hence $f \in S_a(h)$.

THEOREM 8. *For* $\alpha > \beta \geq 0$, $S_a^{\alpha}(h) \subset S_a^{\beta}(h)$.

Proof. The proof of this theorem is similar to that of Theorem 6, and hence is omitted.

REMARK 9. If a is an integer ≥ 1 and $h(z) = \dfrac{1 - z}{1 + z}$ in Theorem 7, then we get Theorem 1 of Al-Amiri [1].

REMARK 10. If $a = 1$ and $h(z) = \dfrac{1 - z}{1 + z}$ then Theorem 7 reduces to the well-known result that all α-convex functions are starlike by S.S. Miller, P.T. Mocanu and M.O. Reade [7].

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Ramanujan Institute,
University of Madras,
Madras - 600 005,
India.