COMPLEMENTARY LAGRANGIANS IN INFINITE DIMENSIONAL SYMPLECTIC HILBERT SPACES

PAOLO PICCIONE AND DANIEL V. TAUSK

ABSTRACT. Using the Spectral Theorem for unbounded self-adjoint operators we prove that any countable family of Lagrangian subspaces of a symplectic Hilbert space admits a common complementary Lagrangian.

1. INTRODUCTION

A *symplectic Hilbert space* is a real Hilbert space $(V, \langle\cdot,\cdot\rangle)$ endowed with a symplectic form; by a *symplectic form* we mean a bounded anti-symmetric bilinear form $\omega : V\times V \to \mathbb{R}$ that is represented by a (anti-self-adjoint) linear isomorphism H of V, i.e., $\omega = \langle H\cdot,\cdot\rangle$. If $H = PJ$ is the polar decomposition of H then P is a positive isomorphism of V and J is a orthogonal complex structure on V; the inner product $\langle P\cdot,\cdot\rangle$ on V is therefore equivalent to $\langle\cdot,\cdot\rangle$ and ω is represented by J with respect to $\langle P\cdot,\cdot\rangle$. We may therefore replace $\langle\cdot,\cdot\rangle$ with $\langle P\cdot,\cdot\rangle$ and assume from the beginning that ω is represented by a orthogonal complex structure J on V. A subspace S of V is called *isotropic* if ω vanishes on S or, equivalently, if $J(S)$ is contained in $S^\perp$. A *Lagrangian subspace* of V is a maximal isotropic subspace of V. We have that $L \subset V$ is Lagrangian if and only if $J(L) = L^\perp$. If $L \subset V$ is Lagrangian then a Lagrangian $L' \subset V$ such that $V = L \oplus L'$ is called a *complementary Lagrangian* to L. Obviously every Lagrangian L has a complementary Lagrangian, namely, its orthogonal complement $L^\perp$. Given a pair L_1, L_2 of Lagrangians, there are known sufficient conditions for the existence of a common complementary Lagrangian to L_1 and L_2 (see, for instance, [1]). In this paper we prove the following:

Theorem. *If $(V, \langle\cdot,\cdot\rangle, \omega)$ is a Hilbert symplectic space then any countable family of Lagrangian subspaces of V admits a common complementary Lagrangian.*

The existence of a common complementary Lagrangian is proven first in the case of two Lagrangians L and L_1 such that $L \cap L_1 = \{0\}$ (Corollary 4). In this case L is the graph of a densely defined self-adjoint operator on $L_1^\perp$ (Lemma 1), and the result is obtained as an application of the spectral theorem (Lemma 2 and

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Lemma 3). The existence of a common complementary Lagrangians is then proven in the general case by a reduction argument (Proposition 5), and the final result is an application of Baire's category theorem.

2. Proof of the Result

In what follows, $(V, \langle\cdot,\cdot\rangle, \omega)$ will denote a Hilbert symplectic space such that ω is represented by a orthogonal complex structure J on V. We will denote by $\Lambda(V)$ the set of all Lagrangian subspaces of V. It follows from Zorn's Lemma that V indeed has Lagrangian subspaces, i.e., $\Lambda(V) \neq \emptyset$. Given $L_0, L_1 \in \Lambda(V)$ then $(L_0 + L_1)^\perp = J(L_0 \cap L_1)$; in particular, $L_0 \cap L_1 = \{0\}$ if and only if $L_0 + L_1$ is dense in V. For $L \in \Lambda(V)$, we denote by $\mathcal{O}(L)$ the subset of $\Lambda(V)$ consisting of Lagrangians complementary to L. Given a real Hilbert space $\mathcal{H}$, we denote by $\mathcal{H}^\mathbb{C}$ the orthogonal direct sum $\mathcal{H} \oplus \mathcal{H}$ endowed with the orthogonal complex structure J defined by $J(x, y) = (-y, x)$. If $A : D \subset \mathcal{H} \to \mathcal{H}$ is a densely defined linear operator on $\mathcal{H}$ then $J\big(\mathrm{gr}(A)^\perp\big) = \mathrm{gr}(A^*)$. It follows that $\mathrm{gr}(A)$ is Lagrangian in $\mathcal{H}^\mathbb{C}$ if and only if A is self-adjoint; in this case, $\mathrm{gr}(A)$ is complementary to $\{0\} \oplus \mathcal{H}$ if and only if A is bounded.

Lemma 1. *Given $L \in \Lambda(\mathcal{H}^\mathbb{C})$ with $L \cap \big(\{0\} \oplus \mathcal{H}\big) = \{0\}$ then L is the graph of a densely defined self-adjoint operator $A : D \subset \mathcal{H} \to \mathcal{H}$.*

Proof. The sum $L + \big(\{0\} \oplus \mathcal{H}\big)$ is dense in $\mathcal{H}^\mathbb{C}$; thus, denoting by $\pi_1 : \mathcal{H}^\mathbb{C} \to \mathcal{H}$ the projection onto the first summand, we have that $D = \pi_1(L) = \pi_1\big(L + \big(\{0\} \oplus \mathcal{H}\big)\big)$ is dense in $\mathcal{H}$. Hence L is the graph of a densely defined operator $A : D \to \mathcal{H}$, which is self-adjoint by the remarks above. □

Given Lagrangians $L_0, L_1 \in \Lambda(V)$ with $V = L_0 \oplus L_1$ then we have an isomorphism $\rho_{L_1,L_0} : L_1 \to L_0$ defined by $\rho_{L_1,L_0} = P_{L_0} \circ J|_{L_1}$, where P_{L_0} denotes the orthogonal projection onto L_0. The map:

$$V = L_0 \oplus L_1 \ni x + y \longmapsto \big(x, -\rho_{L_1,L_0}(y)\big) \in L_0 \oplus L_0 = L_0^\mathbb{C} \tag{1}$$

is a symplectomorphism, i.e., it is an isomorphism that preserves the symplectic forms. Thus, we get a one-to-one correspondence φ_{L_0,L_1} between Lagrangian subspaces L of V with $L \cap L_1 = \{0\}$ and densely defined self-adjoint operators $A : D \subset L_0 \to L_0$; more explicitly, we set $A = \varphi_{L_0,L_1}(L)$ if the map (1) carries L to the graph of $-A$.

Lemma 2. *Let $L_0, L_1, L, L' \in \Lambda(V)$ be Lagrangians such that L_0 and L' are complementary to L_1 and $L \cap L_1 = \{0\}$. Set $\varphi_{L_0,L_1}(L) = A : D \subset L_0 \to L_0$ and $\varphi_{L_0,L_1}(L') = A' : L_0 \to L_0$. Then L' is complementary to L if and only if $(A - A') : D \to L_0$ is an isomorphism.*

Proof. The map (1) carries L and L' respectively to $\mathrm{gr}(-A)$ and $\mathrm{gr}(-A')$. We thus have to show that $L_0^\mathbb{C} = \mathrm{gr}(-A) \oplus \mathrm{gr}(-A')$ if and only if $A - A'$ is an isomorphism. This follows by observing that $(x, y) = (u, -Au) + (u', -A'u')$ is equivalent to $\big(u + u', (A' - A)u\big) = (x, y + A'x)$, for all $x, y, u' \in L_0$, $u \in D$. □

Lemma 3. *If $A : D \subset \mathcal{H} \to \mathcal{H}$ is a densely defined self-adjoint operator then for every $\varepsilon > 0$ there exists a bounded self-adjoint operator $A' : \mathcal{H} \to \mathcal{H}$ with $\|A'\| \le \varepsilon$ and such that $(A - A') : D \to \mathcal{H}$ is an isomorphism.*

Proof. By the Spectral Theorem for unbounded self-adjoint operators, we may assume that $\mathcal{H} = L^2(X, \mu)$ and $A = M_f$, where (X, μ) is a measure space, $f : X \to I\!R$ is a measurable function and M_f denotes the multiplication operator by f defined on $D = \{\phi \in L^2(X,\mu) : f\phi \in L^2(X,\mu)\}$. In this situation, the operator A' can be defined as $A' = M_g$, where $g = \varepsilon \cdot \chi_\varepsilon$ and χ_ε is the characteristic function of the set $f^{-1}\big([-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]\big)$; clearly $\|A'\| \le \|g\|_\infty = \varepsilon$. The conclusion follows by observing that $A - A' = M_{f-g}$, and $|f - g| \ge \frac{\varepsilon}{2}$ on X. □

Corollary 4. *Given $L_1, L \in \Lambda(V)$ with $L_1 \cap L = \{0\}$ then there exists a common complementary Lagrangian $L' \in \Lambda(V)$ to L_1 and L.*

Proof. Set $L_0 = L_1^\perp$ and $A = \varphi_{L_0,L_1}(L)$. Lemma 3 gives us a bounded self-adjoint operator $A' : L_0 \to L_0$ with $A - A'$ an isomorphism. Set $L' = \varphi_{L_0,L_1}^{-1}(A')$; L' is a Lagrangian complementary to L_1, because A' is bounded. It is also complementary to L, by Lemma 2. □

If $V = V_1 \oplus V_2$ is a orthogonal direct sum decomposition into J-invariant subspaces V_1 and V_2, then V_1 and V_2 are symplectic Hilbert subspaces of V. Given subspaces $L_1 \subset V_1$ and $L_2 \subset V_2$ then $L_1 \oplus L_2$ is Lagrangian in V if and only if L_i is Lagrangian in V_i, for $i = 1, 2$. A Lagrangian subspace $L \in \Lambda(V)$ is of the form $L = L_1 \oplus L_2$ with $L_i \in \Lambda(V_i)$, $i = 1, 2$, if and only if L is invariant by the orthogonal projection P_{V_1} onto V_1. In this case, $L_i = P_{V_i}(L) = L \cap V_i$, $i = 1, 2$. If S is a closed isotropic subspace of V then a decomposition $V = V_1 \oplus V_2$ of the type above can be obtained by setting $V_1 = S \oplus J(S)$ and $V_2 = V_1^\perp$. Then, if $L \in \Lambda(V)$ contains S, it follows that $P_{V_1}(L) = S$; namely, $S \subset L$ implies $L \subset J(S)^\perp$ and $J(S)^\perp$ is invariant by P_{V_1}. Hence $L = S \oplus P_{V_2}(L)$.

Proposition 5. *Given $L, L' \in \Lambda(V)$ then $\mathcal{O}(L) \cap \mathcal{O}(L') \neq \emptyset$.*

Proof. Set $S = L \cap L'$, $V_1 = S \oplus J(S)$, and $V_2 = V_1^\perp$. Then $L = S \oplus P_{V_2}(L)$, $L' = S \oplus P_{V_2}(L')$, and $P_{V_2}(L) \cap P_{V_2}(L') = (L \cap V_2) \cap (L' \cap V_2) = \{0\}$. By Corollary 4, there exists a Lagrangian $R \in \Lambda(V_2)$ complementary to both $P_{V_2}(L)$ and $P_{V_2}(L')$ in V_2. Hence $J(S) \oplus R \in \Lambda(V)$ is in $\mathcal{O}(L) \cap \mathcal{O}(L')$. □

The map $L \mapsto P_L$ is a bijection from $\Lambda(V)$ onto the space of bounded self-adjoint maps $P : V \to V$ with $P^2 = P$ and $PJ + JP = J$. Such bijection induces a topology on $\Lambda(V)$ which makes it homeomorphic to a complete metric space. Moreover, for any $L_0, L_1 \in \Lambda(V)$ with $V = L_0 \oplus L_1$, the set $\mathcal{O}(L_1)$ is open in $\Lambda(V)$ and the map $\mathcal{O}(L_1) \ni L \mapsto \varphi_{L_0,L_1}(L)$ is a homeomorphism onto the space of bounded self-adjoint operators on L_0.

Lemma 6. *For any $L_0 \in \Lambda(V)$, the set $\mathcal{O}(L_0)$ is dense in $\Lambda(V)$.*

Proof. Given $L \in \Lambda(V)$, Proposition 5 gives us $L_1 \in \mathcal{O}(L_0) \cap \mathcal{O}(L)$. By Lemma 3, the bounded self-adjoint operator $A = \varphi_{L_0,L_1}(L)$ on L_0 is the limit of a sequence of bounded self-adjoint isomorphisms $A_n : L_0 \to L_0$. Hence the sequence $\varphi_{L_0,L_1}^{-1}(A_n)$ is in $\mathcal{O}(L_0)$ and it tends to L. □

Proof of Theorem. Let $(L_n)_{n\geq 1}$ be a sequence in $\Lambda(V)$. Each $\mathcal{O}(L_n)$ is open and dense in $\Lambda(V)$, hence $\bigcap_{n=1}^{\infty} \mathcal{O}(L_n)$ is dense in $\Lambda(V)$, by Baire's category theorem. □

References

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Departamento de Matemática,
Instituto de Matemática e Estatística
Universidade de São Paulo,
Caixa Postal 66281, CEP 05315–970, SP
Brazil
E-mail address: piccione@ime.usp.br, tausk@ime.usp.br
URL: http://www.ime.usp.br/˜piccione
URL: http://www.ime.usp.br/˜tausk