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Bounded log-harmonic functions with positive real part

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ABSTRACT

Let $H(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D} = \{z \mid |z| < 1\}$, and let B be the set of all functions $w(z) \in H(\mathbb{D})$ such that $|w(z)| < 1$ for all $z \in \mathbb{D}$. A log-harmonic mapping is a solution of the non-linear elliptic partial differential equation $\overline{f_{\bar{z}}} = w(z)\left(\overline{f}/f\right) f_z$, where $w(z)$ is the second dilatation of f and $w(z) \in B$. In the present paper we investigate the set of all log-harmonic mappings R defined on the unit disc $\mathbb{D}$ which are of the form $R = H(z)\overline{G(z)}$, where $H(z)$ and $G(z)$ are in $H(\mathbb{D})$, $H(0) = G(0) = 1$, and $Re(R) > 0$ for all $z \in \mathbb{D}$. The class of such functions is denoted by $\mathcal{P}_{LH}$.

1. Introduction

Let $H(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D}$, and let B be the set of all functions $w(z) \in H(\mathbb{D})$ such that $|w(z)| < 1$ for all $z \in \mathbb{D}$. A log-harmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = w(z)\left(\frac{\overline{f}}{f}\right) f_z, \tag{1.1}$$

where $w(z)$ is the second dilatation of f and $w(z) \in B$. It has been shown [1] that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f = h(z)\overline{g(z)} \tag{1.2}$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$, i.e., $h(z), g(z) \in H(\mathbb{D})$. On the other hand, if f vanishes at $z = 0$ but is not identically zero, then f admits the following representation:

$$f = z|z|^{2\beta}h(z)\overline{g(z)} \tag{1.3}$$

where $Re\beta > -1/2$, $H(z), G(z) \in H(\mathbb{D})$ and $h(0) \neq 0$, $g(0) = 1$. In general, the class of log-harmonic mappings is denoted by $\mathcal{S}_{LH}$. Univalent log-harmonic mappings and log-harmonic mappings have been studied extensively (for details see [1–5]).

Let $\mathcal{P}_{LH}$ be the set of all log-harmonic mappings R defined on the unit disc $\mathbb{D}$ which are of the form

$$R = H(z)\overline{G(z)}, \tag{1.4}$$

where $H(z)$ and $G(z)$ are in $H(\mathbb{D})$, $H(0) = G(0) = 1$ and $Re(R) > 0$ for all $z \in \mathbb{D}$. In particular, the set $\mathcal{P}$ of all analytic functions $p(z)$ in $\mathbb{D}$ with $p(0) = 1$ and $Rep(z) > 0$ in $\mathbb{D}$ is a subset of $\mathcal{P}_{LH}$ [2].

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Finally, let Ω be the family of functions $\phi(z)$ which are analytic in $\mathbb{D}$, and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$, and let $F_1(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$ and $F_2(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ be analytic functions in $\mathbb{D}$. We say that $F_1(z)$ is subordinate to $F_2(z)$ if there exists $\phi(z) \in \Omega$ such that $F_1(z) = F_2(\phi(z))$. We denote it by $F_1(z) \prec F_2(z)$ [6]. The following theorem was proved by Abdulhadi [2], and plays an important role in our study.

Theorem 1.1. *Let $R(z) = H(z)\overline{G(z)} \in \mathcal{P}_{LH}$. Then $p(z) = \frac{H(z)}{G(z)} \in \mathcal{P}$. Conversely, given $p(z) \in \mathcal{P}$ and $w(z) \in B$, there exist non-vanishing functions $H(z)$ and $G(z)$ in $H(\mathbb{D})$ such that $p(z) = \frac{H(z)}{G(z)}$, $R = H(z)\overline{G(z)} \in \mathcal{P}_{LH}$, and R is a solution of (1.1) with respect to the given $w(z)$.*

In this paper, we will investigate the class of log-harmonic mappings defined by

$$\mathcal{P}_{LH(M)} = \left\{ R \mid R = H(z)\overline{G(z)} \in \mathcal{P}_{LH}, \ \left| \frac{H(z)}{G(z)} - M \right| < M, M \geq 1 \right\}.$$

2. The main results

Theorem 2.1. *$R(z) = H(z)\overline{G(z)} \in \mathcal{P}_{LH(M)}$ if and only if $\frac{H(z)}{G(z)} \prec \frac{1+z}{1-\left(1-\frac{1}{M}\right)z}$.*

Proof. Let $R(z) = H(z)\overline{G(z)}$ be an element of $\mathcal{P}_{LH(M)}$; then we have

$$\left| \frac{H(z)}{G(z)} - M \right| < M \Leftrightarrow \left| \frac{1}{M} \frac{H(z)}{G(z)} - 1 \right| < 1.$$

Therefore the function

$$\psi(z) = \frac{1}{M} \frac{H(z)}{G(z)} - 1$$

has modulus at most 1 in the unit disc $\mathbb{D}$ and so

$$\phi(z) = \frac{\psi(z) - \psi(0)}{1 - \psi(0)\psi(z)} = \frac{\left(\frac{1}{M}\frac{H(z)}{G(z)} - 1\right) - \left(\frac{1}{M} - 1\right)}{1 - \left(\frac{1}{M} - 1\right)\left(\frac{1}{M}\frac{H(z)}{G(z)} - 1\right)}, \tag{2.1}$$

and then $\phi(0) = 0$, $|\phi(z)| < 1$; therefore by the Schwarz lemma,

$$|\phi(z)| \leq |z|. \tag{2.2}$$

From (2.1) and (2.2) we obtain

$$\frac{H(z)}{G(z)} = \frac{1 + \phi(z)}{1 - \left(1 - \frac{1}{M}\right)\phi(z)}. \tag{2.3}$$

The equality (2.3) shows that

$$\frac{H(z)}{G(z)} \prec \frac{1 + z}{1 - \left(1 - \frac{1}{M}\right)z}.$$

Conversely, suppose that the functions $H(z)$ and $G(z)$ are analytic in $\mathbb{D}$, and satisfy the conditions $H(0) = G(0) = 1$, $\frac{H(z)}{G(z)} \prec \frac{1+z}{1-\left(1-\frac{1}{M}\right)z}$; then we have

$$\frac{H(z)}{G(z)} \prec \frac{1 + z}{1 - \left(1 - \frac{1}{M}\right)z} \Rightarrow \frac{H(z)}{G(z)} = \frac{1 + \phi(z)}{1 - \left(1 - \frac{1}{M}\right)\phi(z)} \Rightarrow$$

$$\frac{H(z)}{G(z)} - M = M\frac{\frac{1-M}{M} + \phi(z)}{1 + \frac{1-M}{M}\phi(z)}.$$

On the other hand, the function $\left(\frac{\frac{1-M}{M}+\phi(z)}{1+\frac{1-M}{M}\phi(z)}\right)$ maps the unit circle onto itself; then we have

$$\left| \frac{H(z)}{G(z)} - M \right| = \left| M\frac{\frac{1-M}{M} + \phi(z)}{1 - \frac{1-M}{M}\phi(z)} \right| \leq M,$$

and at the same time $Re\left(\frac{H(z)}{G(z)}-M\right)\geq-\left|\frac{H(z)}{G(z)}-M\right|\geq-M\Rightarrow Re\left(\frac{H(z)}{G(z)}\right)>0\Rightarrow Re\left(H(z)\overline{G(z)}\right)>0$, and thus $R(z)=H(z)\overline{G(z)}$ belongs to $\mathcal{P}_{LH(M)}$. □

Corollary 2.2. *Let $R(z)=H(z)\overline{G(z)}$ be an element of $\mathcal{P}_{LH(M)}$; then*

$$\frac{1-r}{1+\left(1-\frac{1}{M}\right)r}\leq\left|\frac{H(z)}{G(z)}\right|\leq\frac{1+r}{1-\left(1-\frac{1}{M}\right)r}. \tag{2.4}$$

Proof. The function $w(z)=\frac{1+z}{1-\left(1-\frac{1}{M}\right)z}$ maps $|z|=r$ onto the circle with the centre $C(r)=\left(\frac{1+\left(1-\frac{1}{M}\right)^2r^2}{1-\left(1-\frac{1}{M}\right)^2r^2},0\right)$ and the radius $\rho(r)=\frac{2\left(1-\frac{1}{M}\right)r}{1-\left(1-\frac{1}{M}\right)^2r^2}$ again using the subordination principle; then we have

$$\left|\frac{H(z)}{G(z)}-\frac{1+\left(1-\frac{1}{M}\right)^2r^2}{1-\left(1-\frac{1}{M}\right)^2r^2}\right|\leq\frac{2\left(1-\frac{1}{M}\right)r}{1-\left(1-\frac{1}{M}\right)^2r^2}.$$

The last inequality gives (2.4). □

Corollary 2.3. *Let $R(z)=H(z)\overline{G(z)}$ be an element of $\mathcal{P}_{LH(M)}$; then*

$$\left|\frac{G'(z)}{H'(z)}\right|\leq\frac{1+\left(1-\frac{1}{M}\right)r}{1-r}. \tag{2.5}$$

Proof. Since $w(z)$ is the second dilatation of the class $\mathcal{P}_{LH}$, we then have

$$w(z)=\frac{\overline{R_{\bar{z}}}}{\overline{R}}\cdot\frac{R}{R_z}=\frac{\frac{G'(z)}{G(z)}}{\frac{H'(z)}{H(z)}}=\frac{G'(z)}{H'(z)}\cdot\frac{H(z)}{G(z)},$$

and thus

$$|w(z)|<1\Rightarrow\left|\frac{G'(z)}{H'(z)}\right|<\left|\frac{G(z)}{H(z)}\right|$$

and using Corollary 2.2 we obtain (2.5). □

Theorem 2.4. *If $R=H(z)\overline{G(z)}\in\mathcal{P}_{LH(M)}$, then*

$$\sum_{k=1}^{n}|a_k-b_k|^2\leq\left(2-\frac{1}{M}\right)^2+\sum_{k=1}^{n-1}\left|b_k^2+\left(1-\frac{1}{M}\right)a_k\right|^2.$$

Proof. The proof of this theorem is based on Clunie's method [7]. Begin with the equality

$$\frac{H(z)}{G(z)}=\frac{1+\phi(z)}{1-\left(1-\frac{1}{M}\right)\phi(z)}\Leftrightarrow H(z)-G(z)=\left[\left(1-\frac{1}{M}\right)H(z)+G(z)\right]\phi(z) \tag{2.6}$$

Let $H(z)=1+\sum_{n=1}^{\infty}a_nz^n$, $G(z)=1+\sum_{n=1}^{\infty}b_nz^n$, $\phi(z)=1+\sum_{n=1}^{\infty}c_nz^n$ and note that $|\phi(z)|<1$, for all $z\in\mathbb{D}$. Therefore Eq. (2.6) now takes the form

$$\sum_{k=1}^{n}(a_k-b_k)z^k+\sum_{k=n+1}^{\infty}(a_k-b_k)z^k=\left[(a+1)+\sum_{k=1}^{n-1}(b_k+aa_k)z^k+\sum_{k=n}^{\infty}(b_k+aa_k)z^k\right]\left(\sum_{k=1}^{\infty}c_kz^k\right)$$

where $a=1-\frac{1}{M}$, or

$$\sum_{k=1}^{n}(a_k-b_k)z^k+\sum_{k=n+1}^{\infty}(a_k-b_k)z^k-\left(\sum_{k=n}^{\infty}(b_k+aa_k)z^k\right)\left(\sum_{k=1}^{\infty}c_kz^k\right)=\left[(1+a)+\sum_{k=1}^{n-1}(b_k+aa_k)z^k\right]\phi(z). \tag{2.7}$$

The equality (2.7) can be written in the following form:

$$\sum_{k=1}^{n}(a_k-b_k)z^k+\sum_{k=n+1}^{\infty}d_kz^k=\left[(1+a)+\sum_{k=1}^{n-1}(b_k+aa_k)z^k\right]\phi(z). \tag{2.8}$$

Since (2.8) has the form $F_1(z)=F_2(z).\phi(z)$, where $|\phi(z)|<1$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|F_1(re^{i\theta})\right|^2d\theta\le\int_0^{2\pi}\left|F_2(re^{i\theta})\right|^2d\theta \tag{2.9}$$

for each $r(0<r<1)$. Expressing (2.9) in terms of the coefficients in (2.8) we obtain the inequality

$$\sum_{k=1}^{n}|a_k-b_k|^2r^{2k}+\sum_{k=n+1}^{\infty}|d_k|^2r^{2k}\le|1+a|^2+\sum_{k=1}^{n-1}|b_k+aa_k|^2r^{2k}. \tag{2.10}$$

From letting $r\to 1$ in (2.10) we conclude that

$$\sum_{k=1}^{n}|a_k-b_k|^2\le\left(2-\frac{1}{M}\right)^2+\sum_{k=1}^{n-1}\left|\left(1-\frac{1}{M}\right)a_k+b_k\right|^2. \quad\square$$

Corollary 2.5. *Let $R(z)=H(z)\overline{G(z)}$ be an element of $\mathcal{P}_{LH(M)}$; then*

$$|a_n-b_n|\le\left[(a+1)+\sum_{k=1}^{n-1}(a|a_k|+b_k)\right](1-|c_0|),$$

where $H(z)=1+\sum_{n=1}^{\infty}a_nz^n$, $G(z)=1+\sum_{n=1}^{\infty}b_nz^n$, $\phi(z)=\sum_{n=0}^{\infty}c_nz^n$, $a=1-\frac{1}{M}$.

Proof. The proof of this corollary can be derived from Theorem 2.4 by using the method in [8, p. 173] given as a problem, although another proof can be found in [9]. $\square$

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