Super efficient solutions for set-valued maps

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Super efficient solutions for set-valued maps

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Abstract

Some properties for K-preinvex set-valued maps in terms of normal coderivative are obtained. Furthermore, some sufficient conditions for existence of super minimal points and necessary optimality conditions for a general kind of super efficiency are established.

Keywords: Set-valued maps; K-preinvex functions; Normal coderivative; Set-valued optimization; Super minimizers

1 Introduction

In recent years, extending and characterizing generalized convex scalar-valued functions to set-valued maps have been studied by many authors; see, e.g. [4, 5, 22, 29, 31] and references therein. Sach and Yen [29] gave some necessary and sufficient conditions for a set-valued map F to be K-convex, in terms of contingent derivative of the epigraphical multifunction of F with respect to the ordering cone K. Also, Yang [31] introduced Dini directional derivative for set-valued mappings and used it to obtain some properties of K-convex maps.

Vector optimization problems involving set-valued maps have been studied in many works in recent years; see, e.g. [7, 8, 9, 12, 13, 19, 20, 23]. Super efficiency for the first time was introduced by Borwein and Zhuang [6] in normed linear space and then studied in several papers [2, 9, 18, 22, 27, 34, 35]. Super efficiency refines the concept of efficiency and other kinds of proper efficiency. In this way, Rong and Wu [27] presented some characterizations of super efficiency in terms of scalarization, Lagrange multipliers and super duality under cone-convexlike assumptions. Zaffaroni [34] used different scalarizing functions in order to characterize super minimizers and other various solutions of vector optimization problems. Also, Bao and Mordukhovich [2] obtained some necessary conditions of super efficiency in constrained problems of multiobjective optimization by advanced tools of variational analysis.

Here, we obtain a characterization of K-preinvex maps by using normal coderivative of their marginal functions. Moreover, two sufficient conditions for existence of super minimal points are obtained under K-preinvexity condition. Also, some necessary optimality conditions for a general kind of super efficiency are deduced. The paper is organized as follows: Section 2 contains some basic definitions and preliminary results. In Section 3, some properties of K-preinvex maps in terms of normal coderivative are established. Some necessary and sufficient optimality conditions are given for their efficiency and a general kind of super efficiency in Section 4.

2 Preliminaries

Let X be a Banach space and X^* its topological dual space. The norm in X and X^* will be denoted by $||.||$. We denote $\langle .,. \rangle$, $[x,y]$ and $]x,y[$ the dual pair between X and X^*, the line segment for $x, y \in X$, and the interior of $[x,y]$, respectively. Now, we recall some concepts of subdifferentials and coderivatives that we need in the next sections.

Definition 2.1. *Let X be a normed vector space, Ω be a nonempty subset of X, $x \in \Omega$ and $\varepsilon > 0$. The set of ε-normals to Ω at x is*

$$\widehat{N}_\varepsilon(x;\Omega) := \{x^* \in X^* | \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{||u - x||} \leq \varepsilon\},$$

where $u \xrightarrow{\Omega} x$ means that $u \to x$ with $u \in \Omega$. If $\varepsilon = 0$, the above set is denoted by $\widehat{N}(x;\Omega)$ and called regular normal cone to Ω at x. Let $\overline{x} \in \Omega$, the basic normal cone to Ω at $\overline{x}$ is

$$N(\bar{x};\Omega) := \limsup_{x \to \bar{x}, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x;\Omega).$$

Let $f : X \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in X$. The regular(Fréchet) subdifferential and basic(limiting) subdifferential due to [24] of f at $\bar{x}$ are defined as follows

$$\widehat{\partial} f(\bar{x}) := \{x^* \in X^* | (x^*, -1) \in \widehat{N}((\bar{x}, f(\bar{x})); \text{epi} f)\},$$

$$\partial_M f(\bar{x}) := \{x^* \in X^* | (x^*, -1) \in N((\bar{x}, f(\bar{x})); \text{epi} f)\}.$$

When the Banach space X is Asplund, i.e., every continuous convex function defined on X is Fréchet differentiable on a dense set of points, one has

$$\partial_M f(\bar{x}) = \limsup_{x \xrightarrow{f} \bar{x}} \widehat{\partial} f(x).$$

Theorem 2.1. *[25] Suppose that X is Asplund, f_1 is Lipschitz around $\bar{x}$ and f_2 is l.s.c. around this point. Then the limiting subdifferential satisfies the sum rule:*

$$\partial_M (f_1 + f_2)(\bar{x}) \subset \partial_M f_1(\bar{x}) + \partial_M f_2(\bar{x}).$$

Theorem 2.2. *[25] Suppose that X is a Banach space, f_1, f_2 are arbitrary extended-real-valued functions that are finite at $\bar{x}$ and*

$$\widehat{\partial}^+ f_2(\bar{x}) := -\widehat{\partial}(-f_2)(\bar{x})$$

is nonempty, then

$$\widehat{\partial}(f_1 + f_2)(\bar{x}) \subset \bigcap_{x^* \in \widehat{\partial}^+ f_2(\bar{x})} [x^* + \widehat{\partial} f_1(\bar{x})].$$

Mean-value theorems are important and useful tools in nonsmooth analysis. We here present a mean value theorem for limiting subdifferential.

Theorem 2.3. *[25] Let X be an Asplund space and φ be Lipschitz continuous on an open set containing $[a, b]$ in X. Then one has*

$$\langle x^*, b - a \rangle \geq \varphi(b) - \varphi(a), \qquad \text{for some } x^* \in \partial_M \varphi(c);\ c \in [a, b[.$$

Given a set-valued mapping $F : X \rightrightarrows Y$ between Banach spaces with the rang space Y partially ordered by a nonempty closed and convex cone K. Denoting the ordering relation on Y by " $\leq$ ", we have

$$y_1 \leq y_2 \quad \text{if and only if} \quad y_2 - y_1 \in K.$$

Define $\text{dom}F := \{x \in X | F(x) \neq \emptyset\}$, $\text{gr}F := \{(x, y) | x \in \text{dom}F, y \in F(x)\}$ and $\text{epi}F := \{(x, y) | x \in X, y \in F(x) + K\}$. Now, we present some definitions and results concerning coderivatives and subdifferentials of set-valued mappings.

Definition 2.2. *[25] Let $F : X \rightrightarrows Y$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \text{gr}F$. Then the Fréchet coderivative of F at $(\bar{x}, \bar{y})$ is the set-valued mapping $\widehat{D}^* F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ given by*

$$\widehat{D}^* F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* | (x^*, -y^*) \in \widehat{N}((\bar{x}, \bar{y}); \text{gr}F)\},$$

and the normal coderivative of F at $(\bar{x}, \bar{y})$ is the set-valued mapping $D^*_N F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ given by

$$D^*_N F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* | (x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gr}F)\}.$$

If $F = f : X \to Y$ is single-valued, we denote its Fréchet and normal coderivatives at $(\bar{x}, f(\bar{x}))$ by $\widehat{D}^* f(\bar{x})$ and $D^*_N f(\bar{x})$, respectively.

Using coderivative of the epigraphical multifunction, Bao and Mordukhovich [1] defined appropriate extensions of the subdifferential notion from extended-real-valued functions to vector-valued and set-valued maps with values in partially ordered spaces and gave some of their applications to multiobjective optimization problems in [2, 3].

Definition 2.3. *[1] Let $F : X \rightrightarrows Y$ be a set-valued mapping. Then the epigraphical multifunction $\mathcal{E}_F : X \rightrightarrows Y$ is defined by*

$$\mathcal{E}_F(x) := \{y \in Y | y \in F(x) + K\}.$$

The Fréchet subdifferential and normal subdifferential of F at the point $(\bar{x}, \bar{y}) \in \text{epi}F$ in the direction $y^* \in Y^*$ are definded, respectively, by

$$\widehat{\partial} F(\bar{x}, \bar{y})(y^*) := \widehat{D}^* \mathcal{E}_F(\bar{x}, \bar{y})(y^*),$$

$$\partial F(\bar{x}, \bar{y})(y^*) := D^*_N \mathcal{E}_F(\bar{x}, \bar{y})(y^*).$$

The Fréchet subdifferential and normal subdifferential of F at $(\bar{x}, \bar{y}) \in \text{epi}F$ is given, respectively, by

$$\widehat{\partial} F(\bar{x}, \bar{y}) := \{x^* \in X^* | x^* \in \widehat{D}^* \mathcal{E}_F(\bar{x}, \bar{y})(y^*), -y^* \in \widehat{N}(0; K), ||y^*|| = 1\},$$

$$\partial F(\bar{x}, \bar{y}) := \{x^* \in X^* | x^* \in D^*_N \mathcal{E}_F(\bar{x}, \bar{y})(y^*), -y^* \in N(0; K), ||y^*|| = 1\}.$$

Remark 2.1. [1] If $F = f : X \to \bar{\mathbb{R}}$ and $K = \mathbb{R}_+$, then Fréchet subdifferential and normal subdifferential defined in above for set-valued mappings, respectively, reduce to Fréchet subdifferential and limiting subdifferential defined in Definition 2.1.

Definition 2.4. *[25] Let $F : \Omega \subset X \rightrightarrows Y$ with $\text{dom}F \neq \emptyset$.*
(i) F is said to be Lipschitz around $\bar{x} \in \text{dom}F$, if there are a neighborhood U of $\bar{x}$ and $\ell \geq 0$ such that

$$F(x) \subset F(u) + \ell ||x - u|| B_Y, \qquad \text{for all } x, u \in \Omega \cap U.$$

(ii) F is said to be epi-Lipschitz around $\bar{x} \in \text{dom}F$, if $\mathcal{E}_F$ is Lipschitz around this point.
(iii) F is said to be Lipschitz-like around $(\bar{x}, \bar{y}) \in \text{gr}F$, if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and number $\ell \geq 0$ such that

$$F(x) \cap V \subset F(u) + \ell ||x - u|| B_Y, \qquad \text{for all } x, u \in \Omega \cap U.$$

(iv) F is said to be epi-Lipschitz-like around $(\bar{x}, \bar{y}) \in \text{epi}F$, if $\mathcal{E}_F$ is Lipschitz-like around this point.

Let Ω be a subset of X and $F : X \rightrightarrows Y$. We associate to Ω and F, the multifunction $F_\Omega : X \rightrightarrows Y$ given by

$$F_\Omega(x) = \begin{cases} F(x) & \text{if } x \in \Omega, \\ \emptyset & \text{if } x \notin \Omega. \end{cases}$$

In the next result, a relation between normal coderivatives of F_Ω and F is given (see Proposition 3.12 in [25]).

Proposition 2.4. *[25] Let X, Y be Asplund spaces, $\Omega \subset X$ a closed set and $F : X \rightrightarrows Y$ be locally closed and Lipschitz-like around $(\bar{x}, \bar{y}) \in \text{gr}F$. Then, for every $y^* \in Y^*$, we have the inclusion*

$$D^*_N F_\Omega(\bar{x}, \bar{y})(y^*) \subset D^*_N F(\bar{x}, \bar{y})(y^*) + N(\bar{x}; \Omega).$$

Let K be a closed convex pointed cone in Y with $K^+ := \{y^* \in Y^* | y^*(k) \geq 0, \ \forall k \in K\}$. Our next object is the marginal functions associated to F. We associate to F and $y^* \in Y^*$, the marginal function

$$f_{y^*}(x) := \inf\{y^*(y) | y \in F(x)\},$$

and the minimum set

$$M_{y^*}(x) := \{y \in F(x) | f_{y^*}(x) = y^*(y)\}.$$

The next proposition presents some properties of f_{y^*} and M_{y^*}. Since the proof is direct, it is omitted.

Proposition 2.5. *Suppose that $\bar{x} \in \mathrm{dom}F$. If $F(\bar{x})$ is compact, then for every $y^* \in Y^*$, we have $M_{y^*}(\bar{x}) \neq \emptyset$. Furthermore, if $M_{y^*}(x)$ is nonempty around $\bar{x}$ and F is u.s.c. set-valued map, then f_{y^*} is a l.s.c. real-valued function at this point.*

Throughout this paper, we suppose that $\mathrm{gr}F$ is closed and for all $x \in \mathrm{dom}F$ and $y^* \in K^+$, $M_{y^*}(x)$ is nonempty.

Theorem 2.6. *(see [16]) Suppose that $F : X \rightrightarrows Y$ is u.s.c. and has convex and closed graph. Then for any $x \in X$, $y^* \in Y^*$ and $y \in M_{y^*}(x)$, one has*

$$\partial_M f_{y^*}(x) = D_N^* F(x, y)(y^*).$$

Definition 2.5. *[30] Let $\eta : X \times X \to X$. A subset Ω of X is said to be invex with respect to η if, for any $x, y \in \Omega$ and $\lambda \in [0, 1]$, $y + \lambda\eta(x, y) \in \Omega$.*

We present now a property of the basic normal cone for invex set. The proof is similar to those of Propositions 1.3 and 1.5 in [25]. We give it for the sake of completeness.

Lemma 2.7. *Let X be a Banach space, $\Omega \subset X$ be a closed invex set with respect to η such that η is continuous with respect to the second argument and $\bar{x} \in \Omega$. Then for any $x^* \in N(\bar{x}; \Omega)$, one has*

$$\langle x^*, \eta(x, \bar{x})\rangle \leq 0, \qquad \text{for any} \quad x \in \Omega.$$

Proof. Fix $x \in \Omega$ and set $x(t) = \bar{x} + t\eta(x, \bar{x})$. By invexity of Ω with respect to η, we have $x(t) \in \Omega$ for all $t \in [0, 1]$. Consider $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega)$. Since $x(t) \to \bar{x}$ as $t \downarrow 0$, for an arbitrary $\gamma > 0$, we easily conclude from Definition 2.1, that

$$\langle x^*, x(t) - \bar{x}\rangle \leq (\varepsilon + \gamma)||x(t) - \bar{x}|| \quad \text{for small } t > 0,$$

hence,

$$\langle x^*, \eta(x, \bar{x})\rangle \leq \varepsilon||\eta(x, \bar{x})|| \quad \text{for all } x \in \Omega. \tag{2.1}$$

Now, take any $x^* \in N(\bar{x};\Omega)$, and by Definition 2.1, find sequences of $(\varepsilon_k, x_k, x_k^*)$ such that $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$ and $x_k^* \xrightarrow{w^*} x^*$ that $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)$ for all $k \in \mathbb{N}$. Then (2.1) ensures that for any k,

$$\langle x_k^*, \eta(x, x_k)\rangle \le \varepsilon_k ||\eta(x, x_k)|| \text{ for all } x \in \Omega.$$

Passing to the limit as $k \to \infty$ and by continuity of η with respect to the second argument, we complete the proof. □

Definition 2.6. *Let Ω be an invex set with respect to η and $f : \Omega \to \mathbb{R}$.*

(1) f is said to be preinvex with respect to η on Ω, if for any $x_1, x_2 \in \Omega$ and $\lambda \in [0,1]$, one has

$$f(x_2 + \lambda\eta(x_1, x_2)) \le \lambda f(x_1) + (1-\lambda)f(x_2);$$

(2) f is said to be invex with respect to η on Ω, if for any $x_1, x_2 \in \Omega$ and $\xi \in \partial_M f(x_2)$, one has

$$\langle \xi, \eta(x_1, x_2)\rangle \le f(x_1) - f(x_2);$$

(3) f is said to be weakly invex with respect to η on Ω, if for any $x_1, x_2 \in \Omega$, there exists $\xi \in \partial_M f(x_2)$, such that

$$\langle \xi, \eta(x_1, x_2)\rangle \le f(x_1) - f(x_2);$$

(4) $\partial_M f$ is said to be invariant monotone on Ω with respect to η, if for any $x_i \in \Omega$ and $\xi_i \in \partial_M f(x_i)$, $(i = 1, 2)$, one has

$$\langle \xi_1, \eta(x_2, x_1)\rangle + \langle \xi_2, \eta(x_1, x_2)\rangle \le 0.$$

Definition 2.7. *[22] Let $\Omega \subset X$ be an invex set with respect to η and $F : \Omega \subset X \rightrightarrows Y$. F is said to be K-preinvex with respect to η on Ω, if for any $x_1, x_2 \in \Omega$ and $\lambda \in [0,1]$, one has*

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(x_2 + \lambda\eta(x_1, x_2)) + K.$$

Remark 2.2. In the above definition, if $\eta(x_1, x_2) = x_1 - x_2$, then we obtain the definition of K-convexity in [31].

The following conditions are useful in the sequel.

Condition *A*. [22] A mapping $F : \Omega \rightrightarrows Y$ from an invex set Ω with respect to η to an ordered Banach space is said to enjoy Condition A if

$$F(x_1) \subset F(x_2 + \eta(x_1, x_2)) + K, \quad \text{for all } x_1, x_2 \in \Omega.$$

Remark 2.3. (i) If $F = f : \Omega \to \mathbb{R}$ and $K = \mathbb{R}_+$, then we obtain Condition A for real single-valued functions in [32]

$$f(x_2 + \eta(x_1, x_2)) \le f(x_1), \quad \text{for all } x_1, x_2 \in \Omega.$$

(ii) It is easy to see that if $F : \Omega \subset X \rightrightarrows Y$ satisfies Condition A, then for any $y^* \in K^+$, f_{y^*} satisfies Condition A for real single-valued functions.

Condition C. [26] Let $\eta : X \times X \to X$. Then for any $x, y \in X, \lambda \in [0,1]$

$$\eta(y, y + \lambda\eta(x,y)) = -\lambda\eta(x,y);$$

$$\eta(x, y + \lambda\eta(x,y)) = (1-\lambda)\eta(x,y).$$

Remark 2.4. Yang et al. [33] have shown that if $\eta : X \times X \to X$ satisfies Condition C, then

$$\eta(y + \lambda\eta(x,y), y) = \lambda\eta(x,y).$$

For unspecified terms, we refer to [25].

3 K-preinvex maps

In this section, we introduce concepts of K-invexity, weakly K-invexity and K-monotonicity for set-valued maps in terms of normal subdifferential, then the relations between them and K-preinvex maps are established.

Definition 3.1. *Let $\Omega \subset X$ be an invex set with respect to η and $F : \Omega \subset X \rightrightarrows Y$.*
(1) F is said to be K-invex with respect to η on Ω, if for any $y^* \in K^+, x_i \in \Omega, y_i \in M_{y^*}(x_i), (i = 1,2)$ and $\xi \in \partial F(x_1, y_1)(y^*)$, one has

$$\langle \xi, \eta(x_2, x_1)\rangle \leq y^*(y_2) - y^*(y_1);$$

(2) F is said to be weakly K-invex with respect to η on Ω, if for any $y^* \in K^+, x_i \in \Omega, y_i \in M_{y^*}(x_i), (i = 1,2)$, there exists $\xi \in \partial F(x_1, y_1)(y^*)$, such that

$$\langle \xi, \eta(x_2, x_1)\rangle \leq y^*(y_2) - y^*(y_1);$$

(3) The set-valued map $\partial F : X \times Y \times Y^* \rightrightarrows X^*$ is said to be invariant K-monotone on Ω with respect to η if for any $y^* \in K^+, x_i \in \Omega, y_i \in M_{y^*}(x_i)$ and $\xi_i \in \partial F(x_i, y_i)(y^*)$, (i=1,2), one has

$$\langle \xi_1, \eta(x_2, x_1)\rangle + \langle \xi_2, \eta(x_1, x_2)\rangle \leq 0.$$

Remark 3.1. If $F(x) := \{f(x)\}$ is a real-valued function, then the above definition reduces to [2.6, Definition 2.6] of invexity, weakly invexity and invariant monotonicity, respectively, for real-valued functions.

Lemma 3.1. *Suppose that $F : \Omega \subset X \rightrightarrows Y$ is a set-valued map and $\bar{x} \in \mathrm{dom}F$. If F is epi-Lipschitz around $\bar{x}$ and $y^* \in K^+$, then the scalar-function f_{y^*} is locally Lipschitz at $\bar{x}$.*

Proof. Suppose that F is epi-Lipschitz around $\bar{x}$, hence there exist a neighborhood U of $\bar{x}$ and $\ell \geq 0$ such that

$$F(x) + K \subset F(u) + K + \ell||x-u||B_Y, \quad \text{for all } x, u \in U.$$

Let $y^* \in K^+$ be fixed, so we have

$$\inf(y^*(F(u)+K+\ell||x-u||B_Y)) \leq \inf(y^*(F(x)+K)) \leq \inf(y^*(F(x))).$$

Since $y^* \in K^+$ and

$$\inf(y^*(F(u))) - \ell||y^*||||x-u|| \leq \inf(y^*(F(u)+K+\ell||x-u||B_Y)),$$

we get

$$f_{y^*}(u) \leq f_{y^*}(x) + \ell||y^*||||x-u||, \quad \text{for all } x, u \in U.$$

By changing the role of x and u, the above relation shows that f_{y^*} is locally Lipschitz at $\bar{x}$. □

Lemma 3.2. *Let $F : X \rightrightarrows Y$ and $0 \neq y^* \in K^+$. Suppose that $\bar{x} \in \mathrm{dom}F$ and $\bar{y} \in M_{y^*}(\bar{x})$. Then, one has*

$$\widehat{\partial} f_{y^*}(\bar{x}) \subseteq \widehat{\partial} F(\bar{x},\bar{y})(y^*) \subseteq \widehat{D}^* F(\bar{x},\bar{y})(y^*).$$

Proof. Suppose that $x^* \in \widehat{\partial} F(\bar{x},\bar{y})(y^*)$. Hence, we get $(x^*, -y^*) \in \widehat{N}((\bar{x},\bar{y}); \mathrm{gr}\mathcal{E}_F)$. Since $\mathrm{gr}F \subset \mathrm{gr}\mathcal{E}_F = \mathrm{epi}F$, by relation (1.5) in [25], we obtain

$$\widehat{N}((\bar{x},\bar{y}); \mathrm{gr}\mathcal{E}_F) \subseteq \widehat{N}((\bar{x},\bar{y}); \mathrm{gr}F).$$

Therefore, $x^* \in \widehat{D}^* F(\bar{x},\bar{y})(y^*)$, i.e. $\widehat{\partial} F(\bar{x},\bar{y})(y^*) \subseteq \widehat{D}^* F(\bar{x},\bar{y})(y^*)$.
Now, suppose that $x^* \in \widehat{\partial} f_{y^*}(\bar{x})$. Hence

$$\limsup_{(x,r) \xrightarrow{\mathrm{epi} f_{y^*}} (\bar{x}, f_{y^*}(\bar{x}))} \frac{\langle x^*, x-\bar{x}\rangle - (r - f_{y^*}(\bar{x}))}{||x-\bar{x}|| + |r - f_{y^*}(\bar{x})|} \leq 0. \tag{3.1}$$

Notice that if $(x,y) \xrightarrow{\mathrm{gr}\mathcal{E}_F} (\bar{x},\bar{y})$, then $y \to \bar{y}$ and $y \in F(x)+K$. Hence

$$y^*(y) \in y^*(F(x)) + [0,+\infty[\subset f_{y^*}(x) + [0,+\infty[.$$

This shows that $(x, y^*(y)) \xrightarrow{\mathrm{epi} f_{y^*}} (\bar{x}, y^*(\bar{y}))$. Now, by using (3.1), we obtain

$$\limsup_{(x,y) \xrightarrow{\mathrm{gr}\mathcal{E}_F} (\bar{x},\bar{y})} \frac{\langle x^*, x-\bar{x}\rangle - (y^*(y) - y^*(\bar{y}))}{||x-\bar{x}|| + |y^*(y) - y^*(\bar{y})|} \leq 0.$$

Therefore,

$$\limsup_{(x,y) \xrightarrow{\mathrm{gr}\mathcal{E}_F} (\bar{x},\bar{y})} \frac{\langle x^*, x-\bar{x}\rangle - (y^*(y) - y^*(\bar{y}))}{||x-\bar{x}|| + ||y-\bar{y}||} \leq 0.$$

Hence, $(x^*, -y^*) \in \widehat{N}((\bar{x},\bar{y}); \mathrm{gr}\mathcal{E}_F)$, that it shows that $x^* \in \widehat{\partial} F(\bar{x},\bar{y})(y^*)$. □

Now, we present a relation between normal coderivative of F and limiting subdifferential of its marginal functions.

Theorem 3.3. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ and $y^* \in K^+$. Suppose that $\bar{x} \in \mathrm{dom}F$, $\bar{y} \in M_{y^*}(\bar{x})$ and F is Lipschitz around $\bar{x}$. Then*

$$\partial_M f_{y^*}(\bar{x}) \subseteq D_N^* F(\bar{x}, \bar{y})(y^*).$$

Proof. It is enough to use Theorem 3.38 in [25]. □

By using the above theorem for $\mathcal{E}_F$, we get the following result.

Corollary 3.4. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ and $y^* \in K^+$. Suppose that $\bar{x} \in \mathrm{dom}F$, $\bar{y} \in M_{y^*}(\bar{x})$ and F is epi-Lipschitz around $\bar{x}$. Then*

$$\partial_M f_{y^*}(\bar{x}) \subseteq \partial F(\bar{x}, \bar{y})(y^*).$$

Lemma 3.5. *Let $F : X \rightrightarrows Y$ be K-preinvex with respect to η. Then, for every $y^* \in K^*$, f_{y^*} is preinvex with respect to η.*

Proof. By some modifications in the proof of Lemma 1.1 in [4] and Proposition 2.1 in [4], we can deduce the proof. □

Lemma 3.6. *Let $F : X \rightrightarrows Y$ be K-invex with respect to η. Then ∂F is invariant K-monotone.*

Proof. The proof deduces easily from Definition 3.1. □

Theorem 3.7. *Suppose that X, Y are Asplund spaces and $F : X \rightrightarrows Y$ is a locally epi-Lipschitz map satisfying Condition* A*. If η satisfies Condition* C *and ∂F is invariant K-monotone with respect to η, then F is K-invex with respect to η.*

Proof. Let ∂F be invariant K-monotone with respect to η and $x_1, x_2 \in X$. Let $z = x_2 + \frac{1}{2}\eta(x_1, x_2)$ and fix $y^* \in K^+$ be arbitrary. By Lemma 3.1, f_{y^*} is locally Lipschitz. Now, By Theorem 2.3, there exist λ_1, λ_2 such that $0 < \lambda_2 \leq \frac{1}{2} < \lambda_1 \leq 1$, $\xi_1 \in \partial_M f_{y^*}(u_1)$ and $\xi_2 \in \partial_M f_{y^*}(u_2)$ such that

$$f_{y^*}(x_2 + \eta(x_1, x_2)) - f_{y^*}(z) \geq \frac{1}{2}\langle \xi_1, \eta(x_1, x_2)\rangle, \tag{3.2}$$

and

$$f_{y^*}(z) - f_{y^*}(x_2) \geq \frac{1}{2}\langle \xi_2, \eta(x_1, x_2)\rangle, \tag{3.3}$$

where $u_1 = x_2 + \lambda_1\eta(x_1, x_2)$ and $u_2 = x_2 + \lambda_2\eta(x_1, x_2)$. By using Corollary 3.4, $\xi_i \in \partial_M f_{y^*}(u_i) \subseteq \partial F(u_i, z_i)(y^*)$ that $z_i \in M_{y^*}(u_i), (i = 1, 2)$. Since ∂F is invariant K-monotone, we have

$$\langle \xi_1, \eta(x_2, u_1)\rangle + \langle w, \eta(u_1, x_2)\rangle \leq 0, \tag{3.4}$$

for any $y_2 \in M_{y^*}(x_2)$ and $w \in \partial F(x_2, y_2)(y^*)$. Now, by Condition C, we get

$$\eta(u_1, x_2) = \lambda_1\eta(x_1, x_2), \quad \eta(x_2, u_1) = -\lambda_1\eta(x_1, x_2).$$

If we replace these relations in (3.4), we obtain

$$\langle \xi_1, \eta(x_1, x_2) \rangle \geq \langle w, \eta(x_1, x_2) \rangle.$$

Now, by (3.2), we have

$$f_{y^*}(x_2 + \eta(x_1, x_2)) - f_{y^*}(z) \geq \frac{1}{2}\langle w, \eta(x_1, x_2) \rangle.$$

In a similar way, we can derive

$$f_{y^*}(z) - f_{y^*}(x_2) \geq \frac{1}{2}\langle w, \eta(x_1, x_2) \rangle.$$

By adding these two relations, we obtain

$$f_{y^*}(x_2 + \eta(x_1, x_2)) - f_{y^*}(x_2) \geq \langle w, \eta(x_1, x_2) \rangle.$$

Since by Remark 2.3(i), f_{y^*} is a real single-valued function that satisfies Condition A, we get

$$f_{y^*}(x_1) - f_{y^*}(x_2) \geq \langle w, \eta(x_1, x_2) \rangle,$$

for any $y_i \in M_{y^*}(x_i), (i = 1, 2)$ and $w \in \partial F(x_2, y_2)(y^*)$. Therefore,

$$y^*(y_1) - y^*(y_2) \geq \langle w, \eta(x_1, x_2) \rangle,$$

which completes the proof. □

Theorem 3.8. *Suppose that $F : X \rightrightarrows Y$ is a locally epi-Lipschitz set-valued map that satisfies Condition* A *and $\mathcal{E}_F$ is closed convex-valued for every x and η satisfies Condition* C. *If F is K-invex with respect to η, then F is K-preinvex.*

Proof. Suppose that F is K-invex. By Corollary 3.4, we can see easily that f_{y^*} is invex for all $y^* \in K^+$. In a similar way of Lemma 3.2 in [21], f_{y^*} is preinvex for all $y^* \in K^+$. Now, from Theorem 3.1 in [22], we can deduce that F is K-preinvex. □

Theorem 3.9. *Suppose that X, Y are Asplund spaces and $F : X \rightrightarrows Y$ is a locally epi-Lipschitz map. If F is K-preinvex with respect to η, then F is weakly K-invex.*

Proof. By Lemmas 3.1 and 3.5, for any $y^* \in K^+$, f_{y^*} is a locally Lipschitz preinvex function. Now, by using Theorem 3.1 in [28], we can deduce that f_{y^*} is weakly invex for any $y^* \in K^+$. Hence, Corollary 3.4 implies that F is weakly K-invex. □

Theorem 3.10. *Suppose that $F : X \rightrightarrows Y$ is a set-valued map that is K-preinvex. If η is continuous with respect to the second argument, then for any $x_i \in X, y_i \in F(x_i)$, $(i = 1, 2)$, $y^* \in Y^*$, and $x^* \in \partial F(x_1, y_1)(y^*)$, one has*

$$\langle x^*, \eta(x_2, x_1) \rangle \leq y^*(y_2) - y^*(y_1).$$

Proof. Since F is K-preinvex, we can deduce that epi$F$ is an invex set with respect to $\bar{\eta} : X \times Y \to X \times Y$ where $\bar{\eta}((x_1, y_1), (x_2, y_2)) := (\eta(x_1, x_2), y_1 - y_2)$. Since $x^* \in \partial F(x_1, y_1)(y^*)$, we get $(x^*, -y^*) \in N((x_1, y_1); \text{epi}F)$. Hence, by Lemma 2.7, we obtain

$$\langle (x^*, -y^*), \bar{\eta}((x_2, y_2), (x_1, y_1)) \rangle = \langle x^*, \eta(x_2, x_1) \rangle - \langle y^*, y_2 - y_1 \rangle \leq 0.$$

Therefore, for any $x_i \in X, y_i \in F(x_i)$, $(i = 1, 2)$, $y^* \in Y^*$, and $x^* \in \partial F(x_1, y_1)(y^*)$, we have

$$\langle x^*, \eta(x_2, x_1) \rangle \leq y^*(y_2) - y^*(y_1).$$

□

Remark 3.2. The above results generalize well known earlier works from scalar-valued functions to set-valued maps. Moreover, from Theorems 3.7 and 3.8, we obtain that for a K-preinvex map, invariant K-monotonicity of its normal coderivative is a sufficient condition for K-preinvexity. Also, from Theorem 3.9, we deduce that weak K-invexity is a necessary condition for K-preinvexity. In the case of K-convex set-valued maps the Conditions A and C are trivially satisfied.

4 Super efficient solutions

This section is devoted to get sufficient conditions for existence of super minimizers of set-valued problems. Also, we introduce a generalization of super minimizers and give some necessary optimality conditions for it.

Suppose that $F : X \rightrightarrows Y$ is a set-valued map between Banach spaces. We consider the following set-valued optimization problem

$$\text{minimize} \;\; F(x), \qquad \text{subject to} \;\; x \in \Omega \subset X. \tag{4.1}$$

The notion of super minimal points to arbitrary subsets of partially ordered spaces was introduced by Borwein and Zhuang [6]. Given a subset A of a Banach space Y which is ordered by a closed and convex cone $K \subset Y$, we say that $\bar{a} \in A$ is a super minimal point of A ($\bar{a} \in SE(A; K)$), if there is a number $M > 0$ such that

$$\text{cl}[\text{cone}(A - \bar{a})] \cap (\mathbb{B}_Y - K) \subset M\mathbb{B}_Y,$$

where $\mathbb{B}_Y$ signifies the closed unit ball of Y. Let $(\bar{x}, \bar{y}) \in \text{gr}F$ with $\bar{x} \in \Omega$, then $(\bar{x}, \bar{y})$ is a local super minimizer to problem (4.1), if there is a neighborhood U of $\bar{x}$ such that $\bar{y} \in SE(F(\Omega \cap U); K)$.

Now, we present a necessary condition for super minimizers that has been proved in [2] (see Theorem 3.8 in [2]).

Theorem 4.1. *[2] Let X, Y be Asplund spaces and F be Lipschitz-like around $(\bar{x}, \bar{y})$. Suppose that Ω and $\mathrm{gr}F$ are locally closed around $\bar{x}$ and $(\bar{x}, \bar{y})$, respectively, $K \neq \{0\}$ and $\mathrm{int}K^+ \neq \emptyset$. If $(\bar{x}, \bar{y})$ is a local super minimizer for problem (4.1), then there is $y^* \in \mathrm{int}K^+$ with $||y^*|| = 1$ such that*

$$0 \in D_N^* F(\bar{x}, \bar{y})(y^*) + N(\bar{x}; \Omega).$$

It is known that K has a bounded base, if and only if $\mathrm{int}K^+ \neq \emptyset$. By this assumption, we prove a lemma that we need in sequel.

Lemma 4.2. *Suppose that K is a convex pointed ordering cone in Y, Ω is invex and F is K-preinvex map with respect to η. If there exist $y^* \in \mathrm{int}K^+$ and $\bar{y} \in F(\bar{x})$ such that*

$$y^*(y) \geq y^*(\bar{y}),$$

for all $x \in \Omega$ and $y \in F(x)$, then $(\bar{x}, \bar{y})$ is a local super minimal point to problem (4.1).

Proof. By some modifications in proof of Theorem 6.1 in [22] and Remark 6.1 in [22], we can deduce the proof. □

Remark 4.1. Rong and Wu [27] have proved that, if convex pointed cone K has a closed bounded base, then $SE(A; K) = SE(A + K; K)$, for all nonempty set $A \subset Y$. By this requirement and assumptions of Theorem 4.1, we can conclude that

$$0 \in \partial F(\bar{x}, \bar{y})(y^*) + N(\bar{x}; \Omega) \tag{4.2}$$

is a necessary optimality condition for super minimal points. Recently, Bao and Mordukhovich [2] showed that the relation (4.2) holds under normality property of the ordering cone K.

Now, we prove that under K-preinvexity of F, the converse of Theorem 4.1 in the presence of (4.2) holds.

Theorem 4.3. *Let Ω be a closed and invex set, K be a closed convex pointed ordering cone in Y and $F : \Omega \subset X \rightrightarrows Y$ be a K-preinvex map with respect to η which is continuous with respect to the second argument. Suppose that $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ and there is a $y^* \in \mathrm{int}K^+$ such that*

$$0 \in \partial F(\bar{x}, \bar{y})(y^*) + N(\bar{x}; \Omega), \tag{4.3}$$

then $(\bar{x}, \bar{y})$ is a local super minimizer to problem (4.1).

Proof. By using the relation (4.3), there exist $x_1^* \in \partial F(\bar{x}, \bar{y})(y^*)$ and $x_2^* \in N(\bar{x}; \Omega)$ such that $x_1^* + x_2^* = 0$. Since $\mathrm{gr}\mathcal{E}_F = \mathrm{epi}F$, we obtain

$$(x_1^*, -y^*) \in N((\bar{x}, \bar{y}); \mathrm{gr}\mathcal{E}_F) = N((\bar{x}, \bar{y}); \mathrm{epi}F).$$

From K-preinvexity of F, we can conclude that epi$F$ is an invex set with respect to $\bar{\eta} : X \times Y \to X \times Y$, where $\bar{\eta}((x_1, y_1), (x_2, y_2)) := (\eta(x_1, x_2), y_1 - y_2)$. Now, by Lemma 2.7, we get

$$\langle (x_1^*, -y^*), \bar{\eta}((x, y), (\bar{x}, \bar{y})) \rangle = \langle x_1^*, \eta(x, \bar{x}) \rangle - \langle y^*, y - \bar{y} \rangle \leq 0 \tag{4.4}$$

for any $x \in \Omega$ and $y \in F(x)$. Since Ω is invex and $-x_1^* = x_2^* \in N(\bar{x}; \Omega)$, by Lemma 2.7, we deduce that

$$\langle x_1^*, \eta(x, \bar{x}) \rangle \geq 0 \qquad \text{for all} \quad x \in \Omega. \tag{4.5}$$

By (4.4) and (4.5), we obtain

$$\langle y^*, y - \bar{y} \rangle \geq 0 \qquad \text{for all} \quad x \in \Omega \quad \text{and} \quad y \in F(x). \tag{4.6}$$

From (4.6) and Lemma 4.2, we get $(\bar{x}, \bar{y})$ is a local super minimal point to problem (4.1). □

Remark 4.2. In Theorem 4.3, only the existence of an η such that F is K-preinvex with respect to η is important. If $\eta(x, y) = x - y$, then Theorem 4.3 is an existence result for K-convexity.

Example 4.1. *Suppose that* $X = \Omega = \mathbb{R}^2, Y = \mathbb{R}, K = [0, +\infty[, (\bar{x}, \bar{y}) = ((0, 0), 0)$ *and* $F : X \rightrightarrows Y$ *given by* $F(x_1, x_2) = [\sqrt{x_1^2 + x_2^2}, +\infty[$. *By some computation we deduce that* F *is* K*-convex map (since* epi$F$ *is convex) and* $\partial F((0, 0), 0)(1) = \{(x_1, x_2) | x_1^2 + x_2^2 \leq 1\}$. *Therefore,* $0 \in \partial F(\bar{x}, \bar{y})(1) + N(\bar{x}; \Omega)$ *and all assumptions of Theorem 4.3 are fulfilled. Hence,* $(\bar{x}, \bar{y})$ *is a local super minimizer to problem (4.1).*

Theorem 4.4. *Let* $\Omega \subset X$ *be a closed and invex set and* $F : \Omega \subset X \rightrightarrows Y$ *be u.s.c.* K*-preinvex map with respect to* η *such that* η *is continuous with respect to the second argument. Suppose that* $\bar{x} \in \Omega$ *and there is a* $y^* \in \text{int}K^+$ *such that*

$$0 \in \partial_M f_{y^*}(\bar{x}) + N(\bar{x}; \Omega). \tag{4.7}$$

Then, for all $\bar{y} \in M_{y^*}(\bar{x})$, $(\bar{x}, \bar{y})$ *is a local super minimizer to problem (4.1).*

Proof. By Lemma 3.5 and a similar proof as that of Theorem 4.3, we can deduce the proof. □

For Gâteaux differentiable vector-valued functions, a similar theorem has been proved in [14] under K-convexity condition. Hence, this theorem generalizes Theorem 5.14 in [14] to u.s.c. K-preinvex set-valued maps.

Bao and Mordukhovich [2] has proved that $(\bar{x}, \bar{y})$ is a local super minimizer to problem (4.1) with constant $M > 0$ if and only if there is a neighborhood U of $\bar{x}$ such that for all $x \in \Omega \cap U$ and $y \in F(x)$, one has

$$Md(y - \bar{y}, -K) \geq ||y - \bar{y}||.$$

Now, we introduce a general kind of super minimal points that is weaker than super minimality. For this purpose, we use the concept of the oriented distance function in [17], that is defined as

$\Delta(y,A) := d(y,A) - d(y,Y\backslash A)$, where A is a subset of Y and $y \in Y$. Readers can find main properties and applications of Δ in [34].

Definition 4.1. *Let $\psi : [0,+\infty[\to \mathbb{R}$ be a nondecreasing function with the property that $\psi(t) = 0$ if and only if $t = 0$. One says that a point $(\bar{x},\bar{y}) \in \mathrm{gr}F \cap (\Omega \times Y)$ is a local ψ-super minimizer for problem (4.1) if there exist a neighborhood U of $\bar{x}$ and $\alpha > 0$ such that for any $x \in \Omega \cap U$ and $y \in F(x)$, one has*

$$\Delta(y - \bar{y}, -K) \geq \alpha\psi(d(y, F(\bar{x}))).$$

Remark 4.3. (1) In the above definition, if instead of right hand side we set $\alpha\psi(d(x,W))$ where $W = \Omega \cap F^{-1}(\bar{y})$, then it is reduced to the definition of weak ψ-sharp local Pareto minimizer, that has been introduced and investigated by Durea and Strugariu [11].

(2) Notice that if $(\bar{x},\bar{y})$ is a local super minimizer of problem (4.1), then it is also a local ψ-super minimizer to it, when $\psi(t) = t$.

Example 4.2. *Consider problem (4.1) with $X = \mathbb{R}, Y = \mathbb{R}^2, K = \mathbb{R}^2_+, \Omega = [\frac{\pi}{6}, \frac{\pi}{2}]$ and the set-valued map $F_1 : \Omega \rightrightarrows Y$ given by*

$$F_1(x) = \begin{cases}]0,+\infty[(\cos x, \sin x) & \text{if } x \neq \frac{\pi}{6}, \\ [0,+\infty[(\cos x, \sin x) & \text{if } x = \frac{\pi}{6}. \end{cases}$$

Set $\psi(t) = t$. It is easy to check that $(\bar{x},\bar{y}) = (\frac{\pi}{6}, (0,0))$ is the unique local ψ-super minimizer to the problem (4.1) for $\alpha = 1$. This point is also a local super minimal point but is not a weak ψ-sharp local Pareto minimizer for problem (4.1). Now, suppose that $\Omega = [0, \frac{\pi}{2}]$ and

$$F_2(x) = \begin{cases}]0,+\infty[(\cos x, \sin x) & \text{if } x \neq 0, \\ [0,+\infty[(1,0) & \text{if } x = 0. \end{cases}$$

Then it is easy to see that $(\bar{x},\bar{y}) = (0,(1,0))$ is a local ψ-super minimizer of problem (4.1) for $\psi(t) = t$ and constant $\alpha = 1$, but is not a local super minimizer of it for all $\alpha > 0$.

Now, we prove some necessary optimality conditions in terms of normal coderivative and normal subdifferential for local ψ-super minimal points to problem (4.1).

Theorem 4.5. *(Coderivative conditions for local ψ-super minimizers) Let X, Y be Asplund spaces, F be Lipschitz-like around $(\bar{x},\bar{y})$ and K be a closed convex pointed cone in Y. Suppose that Ω is closed, ψ is Fréchet differentiable at 0, $\nabla\psi(0) > 0$ and $\mathrm{gr}F$ is locally closed around $(\bar{x},\bar{y})$. If $(\bar{x},\bar{y}) \in \mathrm{gr}F$ is a local ψ-super minimizer with the constant α for problem (4.1), then for any $y_1^* \in \alpha\nabla\psi(0)\mathbb{B}_{Y^*} \cap \widehat{N}(\bar{y}; F(\bar{x}))$, we can find $y_2^* \in K^+ \cap \mathbb{B}_{Y^*}$ such that*

$$0 \in D^*_N F(\bar{x},\bar{y})(y_2^* - y_1^*) + N(\bar{x};\Omega).$$

Proof. Suppose that $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ is a local ψ-super minimizer of problem (4.1) with constant α. Hence, there exists a neighborhood U of $\bar{x}$ such that

$$\Delta(y - \bar{y}, -K) \geq \alpha\psi(d(y, F(\bar{x}))),$$

for every $x \in \Omega \cap U$ and $y \in F(x)$. Because $d(y - \bar{y}, -K) \geq \Delta(y - \bar{y}, -K)$, the above inequality shows that $(\bar{x}, \bar{y})$ is a local minimum point for the penalty function

$$(x, y) \longmapsto f(x, y) := d(y - \bar{y}, -K) - \alpha\psi(d(y, F(\bar{x}))) + \delta((x, y); \mathrm{gr}F_\Omega).$$

By generalized Fermat rule, we obtain

$$(0, 0) \in \widehat{\partial} f(\bar{x}, \bar{y}).$$

Now, by Theorems 2.1 and 2.2, we get

$$\{0\} \times \alpha\widehat{\partial}\psi(d(\cdot, F(\bar{x})))(\bar{y}) \subset \{0\} \times \partial d(\cdot - \bar{y}, -K)(\bar{y}) + N((\bar{x}, \bar{y}); \mathrm{gr}F_\Omega).$$

By computation of the above subdifferentials, we conclude

$$\{0\} \times [\alpha\nabla\psi(0)\mathbb{B}_{Y^*} \cap \widehat{N}(\bar{y}, F(\bar{x}))] \subset \{0\} \times [\mathbb{B}_{Y^*} \cap (-N(0; K))] + N((\bar{x}, \bar{y}); \mathrm{gr}F_\Omega).$$

Since $-N(0; K) = K^+$, hence for any $y_1^* \in \alpha\nabla\psi(0)\mathbb{B}_{Y^*} \cap \widehat{N}(\bar{y}; F(\bar{x}))$, we can find $y_2^* \in K^+ \cap \mathbb{B}_{Y^*}$ such that

$$0 \in D_N^* F_\Omega(\bar{x}, \bar{y})(y_2^* - y_1^*).$$

Since F is Lipschitz-like around $(\bar{x}, \bar{y})$, Proposition 2.4 implies that

$$0 \in D_N^* F(\bar{x}, \bar{y})(y_2^* - y_1^*) + N(\bar{x}; \Omega).$$

This completes the proof. □

The above theorem generalizes Theorem 3.3 in [2] to local ψ-super minimal points.

Remark 4.4. Notice that in the above theorem, if $\Omega = X$, then we have $F_\Omega(x) = F(x)$ for all $x \in X$. Therefore, in this case we do not need to impose that F is Lipschitz-like around $(\bar{x}, \bar{y})$.

Example 4.3. *Consider problem (4.1) with $X = \Omega = Y = \mathbb{R}, K = [0, +\infty[, \alpha = 1$ and $F : X \rightrightarrows Y$ be defined as*

$$F(x) = \begin{cases} [x + 1, +\infty[& \text{if } x > 0, \\ [-x, +\infty[& \text{if } x \leq 0. \end{cases}$$

Take $\psi(t) = 2t$ and $(\bar{x}, \bar{y}) = (0, 0)$. With some computation we have $\alpha\nabla\psi(0)\mathbb{B}_{Y^*} \cap \widehat{N}(\bar{y}, F(\bar{x})) = [-2, 0]$ and $K^+ \cap \mathbb{B}_{Y^*} = [0, 1]$. It is easy to see that $(\bar{x}, \bar{y})$ is a local ψ-super minimizer of problem (4.1) and for any $a \in \alpha\nabla\psi(0)\mathbb{B}_{Y^*} \cap \widehat{N}(\bar{y}, F(\bar{x}))$, we can find $b \in K^+ \cap \mathbb{B}_{Y^*}$ such that $0 \in D_N^* F(\bar{x}, \bar{y})(b - a) + N(\bar{x}; \Omega) = [a - b, +\infty[$ (notice that $a - b \leq 0$ and F is not Lipschitz-like around $(\bar{x}, \bar{y})$).

Theorem 4.6. *(Subdifferential conditions for local ψ-super minimizers) Let X, Y be Asplund spaces, F be epi-Lipschitz-like around $(\bar{x}, \bar{y})$ and K be a closed convex pointed cone in Y. Suppose that Ω is closed, ψ is Fréchet differentiable at 0, $\nabla\psi(0) > 0$ and* epi$F$ *is locally closed around $(\bar{x}, \bar{y})$. If $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ is a local ψ-super minimizer with the constant α for problem (4.1), then for any $y_1^* \in \alpha\nabla\psi(0)\mathbb{B}_{Y^*} \cap \widehat{N}(\bar{y}; F(\bar{x}) + K)$, we can find $y_2^* \in K^+ \cap \mathbb{B}_{Y^*}$ such that*

$$0 \in \partial F(\bar{x}, \bar{y})(y_2^* - y_1^*) + N(\bar{x}; \Omega).$$

Proof. First, we show that if $(\bar{x}, \bar{y})$ is a local ψ-super minimizer for problem (4.1), then it is also a local ψ-super minimizer for problem

$$\text{minimize } \mathcal{E}_F(x), \quad \text{subject to } x \in \Omega \subset X. \tag{4.8}$$

Suppose that $(\bar{x}, \bar{y})$ is a local ψ-super minimizer with the constant α for problem (4.1). By Definition 4.1, there exists a neighborhood U of $\bar{x}$ such that for any $x \in \Omega \cap U$ and $y \in F(x)$, one has

$$\Delta(y - \bar{y}, -K) \geq \alpha\psi(d(y, F(\bar{x}))).$$

Hence, by using the definition of Δ and ψ, for any $k \in K$, we get

$$\Delta(y + k - \bar{y}, -K) \geq \Delta(y - \bar{y}, -K) \geq \alpha\psi(d(y, F(\bar{x}))) \geq \alpha\psi(d(y, F(\bar{x}) + K)).$$

It shows that $(\bar{x}, \bar{y})$ is a local ψ-super minimizer for problem (4.8). Applying now Theorem 4.5 to the problem (4.8) and taking into account the subdifferential constructions for set-valued mapping presented in section 2, we can drive the optimality condition. □

Now we consider that the ordering cone of the output space is solid, i.e. it has nonempty topological interior and under this assumption by using the Gerstewitz's scalarizing functional, we get better necessary optimality conditions. We first recall the main properties of this scalar function.

Lemma 4.7. *[15] Let $K \subset Y$ be a closed convex cone with nonempty interior and $e \in \mathrm{int}K$. Define the functional $\varphi_e : Y \to \mathbb{R}$ as*

$$\varphi_e(y) = \inf\{\lambda \in \mathbb{R} | y \in \lambda e - K\}.$$

This map is continuous, convex, Lipschitz and for every $\lambda \in \mathbb{R}$

$$\{y | \varphi_e(y) \leq \lambda\} = \lambda e - K, \quad \{y | \varphi_e(y) < \lambda\} = \lambda e - \mathrm{int}K.$$

Moreover, for every $u \in Y$

$$\partial\varphi_e(u) = \{v^* \in K^+ | v^*(e) = 1, v^*(u) = \varphi_e(u)\},$$

where ∂ means the convex subdifferential.

In the next lemma we get a characterization of local ψ-super minimizers by using the functional φ_e. This technique has been used in [10, 11].

Lemma 4.8. *The point $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ is a local ψ-super minimizer for problem (4.1) if and only if for every $e \in \mathrm{int}K$, there exists $\delta > 0$ such that for every $x \in \Omega \cap U$ and $y \in F(x)$, one has*

$$\varphi_e(y - \bar{y}) \geq \delta\psi(d(y, F(\bar{x}))).$$

Proof. With some minor modifications in the proof of Lemma 3.2 in [11], we can deduce the proof. □

Theorem 4.9. *(Coderivative conditions for local ψ-super minimizers in the solid cone) Suppose that X, Y are Asplund spaces, F is Lipschitz-like around $(\bar{x}, \bar{y})$ and K is a closed convex pointed cone with $\mathrm{int}K \neq \emptyset$. Suppose that Ω is closed, ψ is Fréchet differentiable at 0, $\nabla\psi(0) > 0$ and $\mathrm{gr}F$ is locally closed around $(\bar{x}, \bar{y})$. If $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ is a local ψ-super minimizer for problem (4.1), then for any $e \in \mathrm{int}K$, there exists $\gamma > 0$ such that for any $y_1^* \in \gamma\nabla\psi(0)\mathbb{B}_{Y^*} \cap \widehat{N}(\bar{y}; F(\bar{x}))$ we can find $y_2^* \in K^+$ with $\langle y_2^*, e\rangle = 1$ such that*

$$0 \in D_N^* F(\bar{x}, \bar{y})(y_2^* - y_1^*) + N(\bar{x}; \Omega).$$

Proof. By using Lemma 4.8 and a similar proof as that of Theorem 4.5, we can deduce the proof. □

Theorem 4.10. *(Subdifferential conditions for local ψ-super minimizers in the solid cone) Suppose that X, Y are Asplund spaces, F is epi-Lipschitz-like around $(\bar{x}, \bar{y})$ and K is a closed convex pointed cone with $\mathrm{int}K \neq \emptyset$. Suppose that Ω is closed, ψ is Fréchet differentiable at 0, $\nabla\psi(0) > 0$ and $\mathrm{epi}F$ is locally closed around $(\bar{x}, \bar{y})$. If $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ is a local ψ-super minimizer for problem (4.1), then for any $e \in \mathrm{int}K$, there exists $\gamma > 0$ such that for any $y_1^* \in \gamma\nabla\psi(0)\mathbb{B}_{Y^*} \cap \widehat{N}(\bar{y}; F(\bar{x}) + K)$ we can find $y_2^* \in K^+$ with $\langle y_2^*, e\rangle = 1$ such that*

$$0 \in \partial F(\bar{x}, \bar{y})(y_2^* - y_1^*) + N(\bar{x}; \Omega).$$

Proof. By a similar way of Theorems 4.6 and 4.9, we can deduce the proof. □

Remark 4.5. In the case that $\mathrm{int}K \neq \emptyset$, if we compare Theorem 4.5 with Theorem 4.9 (also, Theorem 4.6 with Theorem 4.10), by choosing $e \in \mathrm{int}K$ such that $d(e, \mathrm{bd}K) \geq 1$ then, we have

$$\{y^* \in K^+ | \langle y^*, e\rangle = 1\} \subset K^+ \cap \mathbb{B}_{Y^*}.$$

Therefore, in the solid case, Theorem 4.9 (resp. Theorem 4.10) is stronger than Theorem 4.5 (resp. Theorem 4.6).

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