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Generalized Minty vector variational-like inequalities and vector optimization problems in Asplund spaces

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Abstract

We consider two generalized Minty vector variational-like inequalities and investigate the relations between their solutions and vector optimization problems for non-differentiable α-invex functions.

Keywords: Vector variational-like inequalities · Nonsmooth functions · α-invex functions · Vector optimization problems

1 Introduction

The concept of vector variational inequality was first introduced by Giannessi [11] in finite dimensional spaces. Ceng and Yao [7] introduced the class of generalized α-monotone multifunctions and established some existence results for generalized variational-like inequalities on an arbitrary reflexive Banach space. The equivalence between the solutions of Minty vector variational-like inequalities and vector optimization problems has been studied in many works; see, e.g. [1, 2, 3, 4, 12, 15, 16, 24]. Giannessi [12] showed the equivalence between efficient solution of differentiable, convex optimization problem and solution of a variational inequality of Minty type. Yang et al. [26] extended this result to differentiable and pseudoconvex functions and Yang and Yang [24] proved it for differentiable pseudoinvex functions. Afterward, some authors focussed their works to nonsmooth functions. Fang and Hu [9] obtained similar results for pseudoconvex functions with lower Dini directional derivative and Rezaie and Zafarani [21] obtained the above results for pseudoinvex functions with Clarke's subdifferential. Also, Al-Homidan and Ansari [1] obtained these results for invex functions with Clarke's generalized directional derivative and Ansari and Lee [2] showed similar results hold for pseudoconvex functions with upper Dini directional derivative. Very recently, Long et al. [16] and Oveisiha and Zafarani [19] obtained similar results for pseudoinvex functions and Chen and Huang [4] for invex functions with limiting subdifferential. Here, we consider two generalized Minty vector variational-like inequalities and compare their solutions with solutions of vector optimization problems. The paper is organized as follows: In section 2, some basic definitions and preliminary results are given. In section 3, we consider two generalized Minty vector variational-like inequalities and investigate the relations between their solutions and vector optimization problems solutions under α-invexity condition. We show that a solution of our generalized Minty vector variational-like inequalities can be a solution of vector optimization problem (VOP), while it is not a solution of Minty vector variational-like inequalities. We obtain also an existence result for the generalized Minty vector variational-like inequalities.

2 Preliminaries

This section deals with some known definitions and results which will be used in the sequel. Let X be a Banach space endowed with a norm $||.||$ and X^* its dual space with a norm $||.||_*$. We denote $\langle .,. \rangle, [x,y]$ and $]x,y[$ the dual pair between X and X^*, the line segment for $x,y \in X$, and the interior of $[x,y]$, respectively. Let K be a nonempty open subset of X and $f : K \subset X \to \mathbb{R}$ be a non-differentiable real valued function. Now, we recall some concepts of subdifferentials that we need in the next sections.

Definition 2.1. [17] Let X be a normed vector space, Ω be a nonempty subset of X, $x \in \Omega$ and $\varepsilon \geq 0$. The set of ε-normals to Ω at x is

$$\widehat{N}_\varepsilon(x;\Omega) := \{x^* \in X^* | \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u-x\rangle}{||u-x||} \leq \varepsilon\}.$$

Let $\overline{x} \in \Omega$, the limiting normal cone to Ω at $\overline{x}$ is

$$N(\bar{x};\Omega) := \limsup_{x \to \bar{x}, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x;\Omega).$$

Let $f : X \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in X$; the limiting subdifferential of f at $\bar{x}$ is defined as follows

$$\partial_M f(\bar{x}) := \{x^* \in X^* | (x^*, -1) \in N((\bar{x}, f(\bar{x})); \text{epi} f)\}.$$

Remark 2.1. One can see easily that the set-valued mapping $x \mapsto \partial_M f(x)$ has closed graph for locally Lipschitz functions.

Definition 2.2. [23] Let $\eta : X \times X \to X$. A subset K of X is said to be invex with respect to η if for any $x, y \in K$ and $\lambda \in [0, 1]$, $y + \lambda\eta(x, y) \in K$.

Now, we present the concept of α-preinvex functions that are a natural generalization of strong convexity that has been introduced by Polyak [20].

Definition 2.3. Let K be an invex set with respect to η and $f : K \to \mathbb{R}$.
(i) f is said to be α-preinvex with respect to η on K, if there exists a constant α such that for any $x, y \in K$ and $\lambda \in [0, 1]$, one has

$$f(y + \lambda\eta(x, y)) \leq \lambda f(x) + (1 - \lambda)f(y) - \alpha\lambda(1 - \lambda)||\eta(x, y)||^2;$$

(ii) f is said to be α-invex with respect to η on K, if there exists a constant α such that for any $x, y \in K$ and $\zeta \in \partial_M f(y)$, one has

$$\langle \zeta, \eta(x, y)\rangle + \alpha||\eta(x, y)||^2 \leq f(x) - f(y).$$

(iii) f is said to be α-weakly invex with respect to η on K, if there exists a constant α such that for any $x, y \in K$ there exist $\zeta \in \partial_M f(y)$ such that

$$\langle \zeta, \eta(x, y)\rangle + \alpha||\eta(x, y)||^2 \leq f(x) - f(y).$$

Remark 2.2. [13, 14, 22] If $\alpha > 0$, then case(i) reduces to strong preinvexity and case(ii) to strong invexity. If $\alpha = 0$, then case(i) reduces to preinvexity, case(ii) to invexity and case(iii) to weak invexity.

Definition 2.4. Let $K \subset X$, $T : K \to 2^{X^*}$ be a set-valued mapping. T is said to be invariant α-monotone on K with respect to η if there exists a constant α such that for any $x, y \in K$ and any $u \in T(x), v \in T(y)$, one has

$$\langle v, \eta(x, y)\rangle + \langle u, \eta(y, x)\rangle \leq -\alpha(||\eta(x, y)||^2 + ||\eta(y, x)||^2).$$

Remark 2.3. If $\alpha > 0$ then above definition reduces to strong invariant monotonicity that has been studied in [14] for locally Lipschitz functions with Clarke's subdifferential. Furthermore, if $\eta(x, y) = x - y$, then we have definition of strong monotonicity.

Mean-value theorems are important and useful tools in nonsmooth analysis and we mention the following mean-value result for limiting subdiferential.

Theorem 2.1. *[17] Let X be an Asplund space and φ be a Lipschitz continuous function on an open set containing $[a, b]$ in X. Then, one has*

$$\langle x^*, b - a\rangle \geq \varphi(b) - \varphi(a) \quad \textit{for some} \quad x^* \in \partial_M\varphi(c); c \in [a, b[.$$

The following conditions are useful in the sequel.

Condition A [25]. Let K be an invex set with respect to η and $f : K \to \mathbb{R}$. Then,

$$f(y + \eta(x, y)) \leq f(x) \qquad for\ any \qquad x, y \in K.$$

Condition C [18]. Let η: $X \times X \to X$. Then for any $x, y \in X, \lambda \in [0, 1]$

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y), \qquad \eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

Remark 2.4. Yang et al. [27] have shown that if $\eta : X \times X \to X$ satisfies Condition C, then

$$\eta(y + \lambda\eta(x, y), y) = \lambda\eta(x, y).$$

Let K be a convex subset of a vector space X. Then a mapping $F : K \rightrightarrows X$ is called a KKM mapping iff for each nonempty finite subset A of K, conv(A) $\subseteq F(A)$, where conv(A) denotes the convex hull of A, and $F(A) = \bigcup\{F(x)|x \in A\}$.

Lemma 2.2. *(see e.g. [10]) Let K be a nonempty and convex subset of a Hausdorff topological vector space X. Suppose that $\Gamma, \widehat{\Gamma} : K \rightrightarrows K$ are two set-valued mappings such that the following conditions are satisfied:*

(A1) $\widehat{\Gamma}(x) \subseteq \Gamma(x), \quad \forall x \in K,$

(A2) $\widehat{\Gamma}$ *is a KKM map,*

(A3) Γ *is closed-valued,*

(A4) *there is a nonempty compact convex set $B \subseteq K$, such that $\mathrm{cl}_K(\bigcap_{x\in B} \Gamma(x))$ is compact.*

Then, $\bigcap_{x\in K} \Gamma(x) \neq \emptyset$.

For undefined terminologies, we refer to [17].

3 Main Result

In this section, we consider two generalized Minty vector variational-like inequalities and compare their solutions with vector optimization problems's solutions under α-invexity assumption.

Suppose $f_i : K \longrightarrow \mathbb{R}$, the components of $f : K \longrightarrow \mathbb{R}^n$ are non-differentiable functions, $C := \mathbb{R}^n_+ \backslash \{0\}$ and int $C :=$ int $\mathbb{R}^n_+$. In the sequel we have the following ordering relation:

$$x \geq_C y \Leftrightarrow x - y \in C; \qquad x \ngeq_C y \Leftrightarrow x - y \notin C, \quad \forall x, y \in K.$$

Now, we consider the following vector-minimization problem (VOP):

$$\min_{y \in K_C} f(y).$$

Solving a (VOP) means finding all the (weakly) efficient solutions, which are defined as follows.

Definition 3.1. (1) $x \in K$ is said to be an efficient solution (Pareto solution) of (VOP) iff

$$f(x) - f(y) \notin C, \quad \forall y \in K;$$

(2) $x \in K$ is said to be a weak efficient solution (weak Pareto solution) of (VOP) iff

$$f(x) - f(y) \notin \text{int } C, \quad \forall y \in K.$$

The definition of generalized subdifferential can be extended to vector-valued functions. Generalized limiting subdifferential of f at $x \in X$ is the set

$$\partial_M f(x) = \partial_M f_1(x) \times \partial_M f_2(x) \times \cdots \times \partial_M f_n(x).$$

Let K be an invex set with respect to η and $f : K \to \mathbb{R}^n$.

Problem (I): Generalized Minty vector variational-like inequality consists of finding a vector $x \in K$ and a real constant β such that

$$\langle \partial_M f(y), \eta(x, y)\rangle + \beta ||\eta(x, y)||^2 e \nsubseteq C;$$

Problem (II): Generalized weak Minty vector variational-like inequality consists of finding a vector $x \in K$ and a real constant β such that

$$\langle \partial_M f(y), \eta(x, y)\rangle + \beta ||\eta(x, y)||^2 e \nsubseteq \text{int } C;$$

where, $e = (\underbrace{1, 1, \cdots, 1}_{n})$.

Remark 3.1. (1) In Problem (I) (resp. (II)) if $\beta = 0$, then it reduces to Minty (resp. weak) vector variational-like inequality (MVLI) (resp. (MWVLI)) that has been studied in many papers; see, e.g. [1, 2, 4, 12, 24].

(2) Notice that, if x_0 is a solution of Problem (I), then it is also a solution of Problem (II); furthermore, if x_0 is either a solution of Problem (I) or (II) with constant β, then x_0 is also a solution of them for all parameters $\beta' \leq \beta$. Hence, we have

$$\begin{matrix}\text{solution set of Problem (I)} \\ \text{(resp. (II)) for } \beta > 0\end{matrix} \subseteq \begin{matrix}\text{solution set of} \\ \text{(MVLI) (resp. (MWVLI))}\end{matrix} \subseteq \begin{matrix}\text{solution set of Problem (I)} \\ \text{(resp. (II)) for } \beta < 0.\end{matrix}$$

The equivalence between optimization problems and Minty variational inequalities for the first time has been considered by Giannessi [12] and then studied and generalized under different conditions by many authors; see, e.g. [2, 9, 16, 19, 24]. Very recently, for invex functions Al-Homidan and Ansari [1] with Clarke's generalized directional derivative and Chen and Huang [4] with limiting subdifferential obtained the equivalence between vector optimization problems and Minty vector variational-like inequalities. But there are many examples of vector optimization problems that their solutions are not a solution of Minty vector variational-like inequality (e.g. 3.3). Hence by a minor modification in Minty vector variational-like inequalities, we get Problems (I) and (II). Notice that the role of term $\beta||\eta(x,y)||^2$ in Problems (I) and (II) is similar to a kind of perturbation in Minty vector variational-like inequalities. Because β can be choosed in $\mathbb{R}$, the solution set of Problems (I) and (II) are larger than the solution set of Minty vector variational-like inequalities. In this section, we obtain an equivalence between vector optimization problems's solutions and solutions of Problem (I) and also its weak version under α-invexity condition.

Example 3.2. Let $X = \mathbb{R},\ K =]-1,1[$ and f be defined as $f = (f_1, f_2)$ that:

$$f_1(x) = \begin{cases} -\frac{1}{4}x^4 & \text{if } x \geq 0, \\ -x & \text{if } x < 0, \end{cases}$$

and

$$f_2(x) = \begin{cases} x^2 - 4x & \text{if } x \geq 0, \\ x^2 & \text{if } x < 0. \end{cases}$$

The limiting subdifferential of f is

$$\partial_M f(x) = \begin{cases} (-x^3, 2x-4) & \text{if } x > 0, \\ \{(k,t) : k \in [-1,0], t \in \{-4,0\}\} & \text{if } x = 0, \\ (-1, 2x) & \text{if } x < 0. \end{cases}$$

Let $\eta : X \times X \to X$ be defined as $\eta(x,y) := \frac{x^3-y^3}{3}$. Then, by some computation we can see $x = 0$ is a solution of Problem (I) and (II) for $\beta \leq -3$.

Example 3.3. Let $X = \mathbb{R},\ K =]-1,1[$ and $f : K \to \mathbb{R}$ be defined as $f(x) = (f_1(x), f_2(x))$ such that:

$$f_1(x) = \begin{cases} -x^2 + x & \text{if } x > 0, \\ 2x & \text{if } x \leq 0, \end{cases}$$

and

$$f_2(x) = \begin{cases} x + 1 & \text{if } x > 0, \\ x^2 & \text{if } x \leq 0. \end{cases}$$

Let $\eta : X \times X \to X$ be defined as

$$\eta(x,y) = \begin{cases} x - y & \text{if } x > 0, y > 0 \ \ or \ \ x < 0, y < 0, \\ 1 - y & \text{otherwise.} \end{cases}$$

The limiting subdifferential of f is

$$\partial_M f(x) = \begin{cases} (-2x+1, 1) & \text{if } x > 0, \\ \{(k,t) : k \in \{1,2\}, t \in [0, +\infty[\} & \text{if } x = 0, \\ (2, 2x) & \text{if } x < 0. \end{cases}$$

Then, by some computation we can see $x = 0$ is a solution of (VOP) and Problem (I) but is not a solution of Minty vector variational-like inequality.

We present now a necessary and sufficient condition for being an efficient solution of (VOP) a solution of Problem (I).

Theorem 3.1. *Let X be an Asplund space, $K \subset X$ be invex with respect to η and $f = (f_1, \ldots, f_n) : K \to \mathbb{R}^n$ be l.s.c. on K. If each f_i, $1 \leq i \leq n$ is α-weakly invex and $x_0 \in K$ is an efficient solution of (VOP), then it is also a solution of Problem* (I). *Conversely, suppose that η satisfies Condition* C, *f is locally Lipschitz on K, each f_i satisfies Condition* A *and is α-invex with constant $\alpha_i > 0$ and x_0 is a solution of Problem* (I), *then x_0 is a solution of (VOP).*

Proof. Assume that x_0 is a solution of (VOP). Suppose to the contrary that x_0 is not a solution of Problem (I). Hence, for any β there exists $x_\beta \in K$ such that

$$\langle x_i^*, \eta(x_0, x_\beta)\rangle + \beta||\eta(x_0, x_\beta)||^2 \geq 0 \quad i = 1, 2 \ldots, n, \tag{3.1}$$

for all $x_i^* \in \partial_M f_i(x_\beta)$ and there is a strict inequality for some k with $1 \leq k \leq n$. Since each f_i is α-weakly invex with constant α_i, hence for any $x \in K$ there exists $x_i^* \in \partial_M f_i(x)$ such that

$$\langle x_i^*, \eta(x_0, x)\rangle \leq f_i(x_o) - f_i(x) - \alpha_i||\eta(x_0, x)||^2,$$

therefore

$$\langle x_i^*, \eta(x_0, x)\rangle \leq f_i(x_o) - f_i(x) - \beta_0||\eta(x_0, x)||^2, \tag{3.2}$$

where $\beta_0 = \min\{\alpha_1, \ldots, \alpha_n\}$. Now, by using (3.1) and (3.2) we obtain

$$f_i(x_0) - f_i(x_{\beta_0}) \geq 0$$

that we have a strict inequality for $i = k$. This contradicts that x_0 is a solution of (VOP).

Conversely, assume that $x_0 \in K$ is a solution of Problem (I) with constant β. Suppose to the contrary that x_0 is not a solution of (VOP). Hence, there exists a $x \in K$ such that

$$f_i(x_0) \geq f_i(x), \quad i = 1, 2, \ldots, n, \tag{3.3}$$

where (3.3) is satisfied as a strict inequality for some k with $1 \leq k \leq n$. Set

$$x(t) = x_0 + t\eta(x, x_0), \quad \forall t \in [0, 1].$$

Choose $t' \in]0,1[$ arbitrary. Now, by Theorem 2.1, there exist $t_i \in]0,t']$ and $\xi_i^* \in \partial_M f_i(x_0 + t_i\eta(x,x_0))$ such that

$$t'\langle \xi_i^*, \eta(x,x_0)\rangle \leq f_i(x_0 + t'\eta(x,x_0)) - f_i(x_0), \quad i = 1,2,\ldots,n. \tag{3.4}$$

By a similar proof as that of Lemma 3.2 in [14], α-invexity of f_i implies its α-preinvexity. Hence, we obtain

$$f_i(x_0 + t'\eta(x,x_0)) - f_i(x_0) \leq t'(f_i(x) - f_i(x_0) - \alpha_i(1-t')||\eta(x,x_0)||^2). \tag{3.5}$$

Now, by using (3.3), (3.4) and (3.5) we have

$$\langle \xi_i^*, \eta(x,x_0)\rangle \leq -\alpha_i(1-t')||\eta(x,x_0)||^2, \quad i = 1,2,\ldots,n, \tag{3.6}$$

which is satisfied as a strict inequality for $i = k$. Because $t_i \in]0,1[$ for $i = 1,2,\ldots,n$, we can choose $t^* \in]0,1[$ such that $t^* < \min\{t_i | 1 \leq i \leq n\}$ and to be sufficiently small. Now by Condition C, for any $1 \leq i \leq n$, we have

$$\begin{aligned}
\eta(x(t^*), x(t_i)) &= \eta(x_0 + t^*\eta(x,x_0), x_0 + t_i\eta(x,x_0)) \\
&= \eta(x_0 + t^*\eta(x,x_0), x_0 + t^*\eta(x,x_0) + (t_i - t^*)\eta(x,x_0)) \\
&= \eta(x_0 + t^*\eta(x,x_0), x_0 + t^*\eta(x,x_0) + \frac{t_i - t^*}{1-t^*}\eta(x, x_0 + t^*\eta(x,x_0))) \\
&= \frac{t^* - t_i}{1-t^*}\eta(x, x_0 + t^*\eta(x,x_0)) = (t^* - t_i)\eta(x,x_0),
\end{aligned}$$

therefore,

$$\eta(x(t^*), x(t_i)) = (t^* - t_i)\eta(x,x_0). \tag{3.7}$$

By a similar way and using Remark 2.4, we have

$$\eta(x(t_i), x(t^*)) = (t_i - t^*)\eta(x,x_0). \tag{3.8}$$

By relations (3.6) and (3.7), we obtain

$$\langle \xi_i^*, \eta(x(t^*), x(t_i))\rangle \geq \alpha_i(1-t')(t_i - t^*)||\eta(x,x_0)||^2, \quad i = 1,2,\ldots,n, \tag{3.9}$$

where is satisfied as a strict inequality for $i = k$. Since f_i is α-invex, one can easily show that $\partial_M f_i$ is invariant α-monotone. Hence, we get

$$\langle \xi_i^*, \eta(x(t^*), x(t_i))\rangle + \langle \zeta_i^*, \eta(x(t_i), x(t^*))\rangle \leq -\alpha_i(||\eta(x(t^*), x(t_i))||^2 + ||\eta(x(t_i), x(t^*))||^2),$$

for any $1 \leq i \leq n$ and $\zeta_i^* \in \partial_M f_i(x(t^*))$. If we use relations (3.7) and (3.8), we have

$$\langle \xi_i^*, \eta(x(t^*), x(t_i))\rangle + \langle \zeta_i^*, \eta(x(t_i), x(t^*))\rangle \leq -2\alpha_i(t_i - t^*)^2||\eta(x,x_0)||^2. \tag{3.10}$$

Now, by relations (3.9) and (3.10) we deduce that

$$\langle \zeta_i^*, \eta(x(t_i), x(t^*)) \rangle \le -\alpha_i(t_i - t^*)||\eta(x, x_0)||^2(2(t_i - t^*) + (1 - t')), \qquad i = 1, 2, \dots, n,$$

that we have a strict inequality for $i = k$. Since

$$\eta(x_0, x(t^*)) = -t^*\eta(x, x_0),$$

we obtain

$$\langle \zeta_i^*, \eta(x_0, x(t^*)) \rangle \ge \frac{\alpha_i}{t^*}(2(t_i - t^*) + (1 - t'))||\eta(x_0, x(t^*))||^2, \qquad i = 1, 2, \dots, n.$$

Set $\beta_0 = \min\{\alpha_1, \dots, \alpha_n\}$ and $t_0 = \min\{t_1, \dots, t_n\}$. Hence, we get

$$\langle \zeta_i^*, \eta(x_0, x(t^*)) \rangle + \beta'||\eta(x_0, x(t^*))||^2 \ge 0, \qquad i = 1, 2, \dots, n,$$

where $\beta' = -\frac{\beta_0}{t^*}(2(t_0 - t^*) + (1 - t')) < 0$ and a strict inequality for $i = k$. Notice that if $t^* \to 0^+$ then $\beta' \to -\infty$. Hence, we can suppose that $\beta' \le \beta$. Therefore, we have

$$\langle \partial_M f(x(t^*)), \eta(x_0, x(t^*)) \rangle + \beta'||\eta(x_0, x(t^*))||^2 \subseteq C.$$

It says that x_0 is not a solution of Problem (I) with constant β'. Now, by Remark 3.1 this contradicts with x_0 is a solution of Problem (I) with constant β.

□

Remark 3.4. Theorem 3.1 generalizes and improves Theorem 3.1 in [1] and Theorem 3.1 in [4] to α-(weakly) invex functions and generalized Minty vector variational-like inequalities.

In second part of Theorem 3.1, only existence a β such that x_0 to be a solution of Problem (I) is important. Hence, by using Remark 3.1, it is easier to investigate Problem (I) for negative constants.

Example 3.5. Let $X = \mathbb{R}$, $K =]-1, 1[$ and $f : K \to \mathbb{R}^2$ be defined as $f(x) = (f_1(x), f_2(x))$ such that:

$$f_1(x) = \begin{cases} -x^2 + 1 & \text{if } x > 0, \\ -x & \text{if } x \le 0, \end{cases}$$

and

$$f_2(x) = \begin{cases} -x^2 + x & \text{if } x > 0, \\ -x & \text{if } x \le 0. \end{cases}$$

The limiting subdifferential of f is

$$\partial_M f(x) = \begin{cases} (-2x, -2x+1) & \text{if } x > 0, \\ \{(k,t) : k \in [-1, \infty[, t \in [-1,1]\} & \text{if } x = 0, \\ (-1,1) & \text{if } x < 0. \end{cases}$$

Let $\eta : X \times X \to X$ be defined as in the Example 3.3, then by some computation we can see that f_1 and f_2 are α-weakly invex with constant $\alpha_1 = \alpha_2 = -1$ and $x = 0$ is a solution of (VOP) and therefore is a solution of Problem (I). Now, suppose that $f(x) = (f_3(x), f_4(x))$ such that:

$$f_3(x) = \begin{cases} x^2 + 2x & \text{if } x \geq 0, \\ x^2 & \text{if } x < 0, \end{cases}$$

and

$$f_4(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ x^2 - x & \text{if } x < 0. \end{cases}$$

Let $\eta : X \times X \to X$ be defined as

$$\eta(x,y) = \begin{cases} x - y & \text{if } x \geq 0, y \geq 0 \;\; \textit{or} \;\; x \leq 0, y \leq 0, \\ -y & \text{otherwise.} \end{cases}$$

Then, η satisfies Condition C, f_i $(i = 3, 4)$ satisfies Condition A and is α-invex with constant $\alpha_1 = \alpha_2 = 1$. Hence, by some computation we can see that all assumptions of Theorem 3.1 are fulfilled and $x = 0$ is a solution of Problem (I) and therefore is a solution of (VOP).

By a similar way of Theorem 3.1, we can deduce the relation between solutions of Problem (II) and weak efficient solutions of (VOP), hence the proof is omitted.

Theorem 3.2. *Let X be an Asplund space, $K \subset X$ be invex with respect to η and $f = (f_1, \ldots, f_n) : K \to \mathbb{R}^n$ be l.s.c. on K. If each f_i, $1 \leq i \leq n$ is α-weakly invex and $x_0 \in K$ is a weak efficient solution of (VOP), then it is also a solution of Problem* (II). *Conversely, suppose that η satisfies Condition* C, *f is locally Lipschitz on K, each f_i satisfies Condition* A *and also is α-invex with constant $\alpha_i > 0$, then x_0 is a weak efficient solution of (VOP), if it is a solution of Problem* (II).

Remark 3.6. Theorems 3.1 and 3.2 also hold for the Clarke 's subdifferential in any Banach space.

Now, we obtain an existence result for the solution of Problem (II) and therefore a weak efficient solution of (VOP).

Theorem 3.3. *Let X be an Asplund space and $f = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ be locally Lipschitz such that each f_i, $1 \leq i \leq n$ is α-invex with constant α_i. Assume that the following conditions are* *satisfied:*

(1) η *is affine and continuous in the first argument and skew,*

(2) *there are a nonempty compact set* $M \subset X$ *and a nonempty compact convex set* $B \subset X$ *such that for each* $x \in X \setminus M$*, there exists* $y \in B$ *such that*

$$\langle \partial_M f(y), \eta(x,y) \rangle + \beta ||\eta(x,y)||^2 \subseteq \text{int } C,$$

where $\beta = 2 \min\{\alpha_1, \ldots, \alpha_n\}$*. Then Problem* (II) *has a solution and the set of solutions is compact.*

Proof. Define two set-valued mappings $\Gamma, \widehat{\Gamma} : X \rightrightarrows X$ by

$$\Gamma(y) := \{x \in X : \langle \partial_M f(y), \eta(x,y) \rangle + \beta ||\eta(x,y)||^2 e \nsubseteq \text{int } C\},$$

$$\widehat{\Gamma}(y) := \{x \in X : \langle \partial_M f(x), \eta(y,x) \rangle \nsubseteq -\text{int } C\},$$

for each $y \in X$. $\Gamma(x)$ and $\widehat{\Gamma}(x)$ are nonempty because they contain x. The proof is divided in the following steps.

(i) $\widehat{\Gamma}$ is a KKM mapping on X. Suppose that $\widehat{\Gamma}$ is not a KKM mapping. Then, there exist $\{y_1, y_2, \ldots, y_m\}$ and $\lambda \geq 0$, $i = 1, \ldots, m$ with $\sum_{i=1}^m \lambda_i = 1$ such that $y_0 = \sum_{i=1}^m \lambda_i y_i \notin \cup_{i=1}^m \widehat{\Gamma}(y_i)$. Hence, it follows that $y_0 \notin \widehat{\Gamma}(y_i)$ for all $i = 1, \ldots, m$, i.e.

$$\langle \partial_M f(y_0), \eta(y_i, y_0) \rangle \subseteq -\text{int } C, \tag{3.11}$$

for each $i = 1, \ldots, m$. Since η is affine in the first argument, one has

$$0 = \langle \partial_M f(y_0), \eta(y_0, y_0) \rangle = \langle \partial_M f(y_0), \eta(\sum_{i=1}^m \lambda_i y_i, y_0) \rangle = \sum_{i=1}^m \lambda_i \langle \partial_M f(y_0), \eta(y_i, y_0) \rangle \subseteq -\text{int } C,$$

which yields a contradiction. Hence $\widehat{\Gamma}$ is a KKM mapping.

(ii) By Theorem **??** α-invexity of f_i, $i = 1, \ldots, n$ implies that $\partial_M f_i$ is invariant α-monotone, hence we obtain $\widehat{\Gamma}(y) \subseteq \Gamma(y)$ and therefore, Γ is also a KKM mapping.

(iii) Γ is closed valued: Let $\{x_n\}$ be a sequence in $\Gamma(y)$ which converges to a x_0. Therefore, there exists $\xi_n \in \partial_M f(y)$ such that

$$z_n = \langle \xi_n, \eta(x_n, y) \rangle + \beta ||\eta(x_n, y)||^2 e \notin \text{int } C. \tag{3.12}$$

Since $\partial_M f(y)$ is w^*-compact, $\{\xi_n\}$ has a convergent subsequence $\{\xi_m\}$ in $\partial_M f(y)$. We denote its limit by $\xi_0 \in \partial_M f(y)$. Since η is continuous in the first argument $\{\eta(x_n, y)\}$ is a convergent sequence. Hence, we obtain

$$z_0 = \lim_m z_m = \langle \xi_0, \eta(x_0, y) \rangle + \beta ||\eta(x_0, y)||^2 e.$$

From (3.12), we deduce that $z_0 \notin \text{int } C$ and therefore,

$$\langle \xi_0, \eta(x_0, y)\rangle + \beta||\eta(x_0, y)||^2 e \notin \text{int } C.$$

Thus, $x_0 \in \Gamma(y)$, this means that Γ is closed valued.

(iv) From condition (2), there exists a nonempty compact convex set B, such that $\text{cl}(\bigcap_{x\in B} \Gamma(x))$ is compact.

(v) Thus, all of the conditions of Lemma 2.2 are fulfilled by mapping Γ. Therefore,

$$\bigcap_{y\in X} \Gamma(y) \neq \emptyset.$$

Hence, there exists x such that

$$\langle \partial_M f(y), \eta(x, y)\rangle + \beta||\eta(x, y)||^2 e \nsubseteq \text{int } C, \qquad \forall y \in X.$$

Thus, Problem (II) has a solution. From (iii), Γ is closed valued and therefore, the set of solutions of Problem (II), i.e. $\bigcap_y \Gamma(y)$ is closed. Now, from (2), the set of solutions must be contained in the compact set M, hence it is compact. □

Remark 3.7. If $K \subseteq X$ is convex and also invex with respect to a specific η, then the above theorem holds also on K.

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