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SOME PROPERTIES OF m-ISOMETRIES AND m-INVERTIBLE OPERATORS ON BANACH SPACES*

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Abstract This article has two purposes: the first is to give some structure results for the class of m-isometries, and the second purpose is to extend the notions of left and right inverses to m-left and m-right inverses respectively.

Key words m-isometry; m-left inverse; m-right inverse; approximate spectrum; point spectrum

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1 Introduction and Terminologies

Let X be a complex Banach space and let $\mathcal{L}(X)$ denote the set of bounded linear operators on X. X^* denotes the dual space of X. For an arbitrary operator $T \in \mathcal{L}(X)$, we use $N(T)$ to denote its kernel, $R(T)$ its range, and T^* its adjoint.

An operator $T \in \mathcal{L}(X)$ is said to be left invertible if there is an operator $S \in \mathcal{L}(X)$ such that $ST = I_X$, where I_X denotes the identity operator. The operator S is called a left inverse of T. By $L(X)$ we denote the set of all left invertible operators in $\mathcal{L}(X)$. For $T \in L(X)$, we denote by $\mathfrak{L}_T$ the set of all its left inverses, that is, $ST = I_X$ whenever $S \in \mathfrak{L}_T$. If $S \in \mathfrak{L}_T$ is a given left inverse of $T \in L(X)$, the family $\mathfrak{L}_T$ is characterized by

$$\mathfrak{L}_T = \{S + B(I_X - TS) : \ B \in \mathcal{L}(X)\}.$$

An operator $T \in \mathcal{L}(X)$ is said to be right invertible if there is an operator $R \in \mathcal{L}(X)$ such that $TR = I_X$. The operator R is called a right inverse of T. By $R(X)$ we denote the set of all right invertible operators in $\mathcal{L}(X)$. For $T \in R(X)$, we denote by $\mathfrak{R}_T$ the set of all its right inverses, that is, $TR = I_X$ whenever $R \in \mathfrak{R}_T$. If $R \in \mathfrak{R}_T$ is a given right inverse of $T \in R(X)$, the family $\mathfrak{R}_T$ is characterized by

$$\mathfrak{R}_T = \{R + (I_X - RT)A : \ A \in \mathcal{L}(X)\}.$$

The theory of right invertible operators started with works of D. Przeworska-Rolewicz, and then developed through many mathematicians (see [7]). The algebraic theory of generalized

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invertible operators was studied by Caradus, Israel, Anselone, Nashed, and also by many other authors (see [8, 10, 11]). In 1997, Nguyen Van Mau and Nguyen Minh Tuan introduced the class of right invertible operators of degree k, $k \in \mathbb{N}$, and investigated the algebraic characterizations of right invertible operators of degree $k = 1$.

An operator $T \in \mathcal{L}(X)$ when X is a Hilbert space is called m-isometry if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^{k} = 0. \tag{1.1}$$

Moreover, T is called a strict m-isometry if T an m-isometry but not an $(m-1)$-isometry. This class of operators was studied by several authors, see for example [1–5] and [9].

$\sigma_{ap}(T)$ and $\sigma_p(T)$ respectively represent the approximate point spectrum and the point spectrum of $T \in \mathcal{L}(X)$.

The reduced minimum modulus of a map $T \in \mathcal{L}(X)$ is defined by

$$\gamma(T) := \begin{cases} \inf\{\|Tx\| : d(x, N(T)) = 1\} & \text{if } T \neq 0 \\ \infty & \text{if } T = 0. \end{cases} \tag{1.2}$$

The reduced minimum modulus measures the closedness of the range of operators in the sense that $\gamma(T)$ is positive precisely when T has a closed range. Recall that the minimum modulus of T is defined by

$$\mu(T) := \inf\{\|Tx\|, x \in X : \|x\| = 1\} \tag{1.3}$$

Note that $\mu(T) > 0$ if and only if T is injective and has closed range.

The contents of this article are the following. Introduction and terminologies are described in the first part of this article. The second part is devoted to the study of some basic properties of the class of m-isometries. In the third section, we present some results concerning the structure of m-left inverses and m-right inverses operators.

2 Properties of m-Isometry

In this section, we will prove some basic properties of m-isometries.

Definition 2.1 [4] Let X be a Banach space, $m \in \mathbb{N}$ and $p \in [1, \infty)$. An operator $T \in \mathcal{L}(X)$ is called an (m, p)-isometry if, for any $x \in X$,

$$\sum_{k=0}^{m}(-1)^{k}\binom{m}{k}\|T^{m-k}x\|^{p} = 0. \tag{2.1}$$

It is called an m-isometry if it is an (m, p)-isometry for some $p \geq 1$.

Remark 2.1 A 1-isometry is an isometry. When X is a Hilbert space, Agler-Stankus definition of an m-isometry corresponds here to an (m, 2)-isometry.

Remark 2.2 If T is an m-isometry, then, T is bounded below and hence $\gamma(T) > 0$.

Proposition 2.1 Let $T \in \mathcal{L}(X)$ be an isometry and $S \in \mathcal{L}(X)$ with $ST = TS$, then ST is an m-isometry if and only if S is m-isometry.

Proof Let $x \in X$, then we have

$$\|T^jS^jx\| = \|S^jx\|, \quad j = 0, 1, \cdots, m.$$

We deduce that

$$\begin{aligned}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\|(TS)^{m-j}x\|^p &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}\|T^{m-j}S^{m-j}x\|^p\\ &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}\|S^{m-j}x\|^p,\end{aligned}$$

which completes the proof.

Corollary 2.1 Let $T \in \mathcal{L}(X)$ such that there exists $n_0 \in \mathbb{N}$ for which T^{n_0} is an m-isometry and T^{n-n_0} is an isometry, then, T^n is an m-isometry.

Proof It suffice to write $T^n = T^{n_0}T^{n-n_0}$ and apply Proposition 2.1.

Proposition 2.2 Let $T \in L(X)$ such that there exists $S \in \mathfrak{L}_T$ satisfing

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}\|S^{m-j}x\|^p = 0, \quad \text{for all } x \in R(T^m). \tag{2.2}$$

Then, T is an m-isometry.

Proof Let $x \in X$. As

$$\begin{aligned}0 &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}\|S^{m-j}(T^m x)\|^p = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\|T^j x\|^p\\ &= (-1)^m\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{m-j}\|T^j x\|^p\\ &= (-1)^m\sum_{j=0}^{m}(-1)^j\binom{m}{j}\|T^{m-j}x\|^p,\end{aligned}$$

it means that T is an m-isometry.

Theorem 2.1 Let $T \in \mathcal{L}(X)$ and assume that there exists a continuous projection P for which $R(T) = R(P)$. Then, the following conditions are equivalent.

1. T is an m-isometry.
2. T is left invertible and there exists an operator $S \in \mathfrak{L}_T$ satisfies

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}\|S^{m-j}x\|^p = 0, \quad \text{for all } x \in R(T^m). \tag{2.3}$$

Proof $1 \Longrightarrow 2$. If T is an $(m.p)$-isometry, then T is bounded below, hence T is one-to-one and $R(T)$ is closed. From the hypothesis, we have $X = R(P)\oplus N(P) = R(T)\oplus N(P)$ and hence the canonical projection $Q : X \longrightarrow R(T)$ is continuous.We consider the operator $S = T^{-1}Q$, where $T^{-1} : R(T) \longrightarrow X$, we have $S \in \mathcal{L}(X)$ and $ST = I_X$.

To prove that the operator S satisfies the required equation, let $y = T^m x \in R(T^m)$:

$$\begin{aligned}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\|S^{m-j}T^m x\|^p &= \sum_{j=0}^{m}(-1)^j\binom{m}{m-j}\|T^j x\|^p\\ &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^{m-k}x\|^p = 0.\end{aligned} \tag{2.4}$$

$2 \Longrightarrow 1$. Use Proposition 2.2.

Theorem 2.2 Let $T_1, T_2 \in \mathcal{L}(X)$ are two left invertible such that $T_1T_2 = T_2T_1$. If $A \in \mathfrak{L}_{T_1}$ and $B \in \mathfrak{L}_{T_2}$, then T_1T_2 is an m-isometry if and only if AB is an m-isometry on $R((T_1T_2)^m)$.

Proof As $AT_1 = I_X$ and $BT_2 = I_X$, we have $ABT_1T_2 = I_X$ and $ABT_1T_2 = BAT_1T_2$, which implies that $AB = BA$ on $R(T_1T_2)$.

Let $y = (T_1T_2)^m x \in R((T_1T_2)^m) \subset R(T_1T_2)$,

$$\begin{aligned}\sum_{j=0}^{m}(-1)^j \binom{m}{j} \|(AB)^{m-j}y\|^p &= \sum_{j=0}^{m}(-1)^j \binom{m}{j} \|A^{m-j}B^{m-j}(T_1T_2)^m x\|^p \\ &= \sum_{j=0}^{m}(-1)^j \binom{m}{j} \|(T_1T_2)^j x\|^p. \end{aligned} \tag{2.5}$$

This ends the proof

Theorem 2.3 Let $T, S \in \mathcal{L}(X)$, such that $TS = ST$. If T is a 2-isometry and S is an m-isometry, then, ST is an $(m+1)$-isometry,

Proof Assume that T is a 2-isometry and S is an $(m+1)$-isometry. We have

$$\|T^k x\|^p = k\|Tx\|^p - (k-1)\|x\|^p \quad \text{for } k = 1, 2, 3, \cdots.$$

Then,

$$\begin{aligned}&\sum_{j=0}^{m+1}(-1)^j \binom{m+1}{j} \|(TS)^{m+1-j}x\|^p \\ &= \Bigg\{ \sum_{j=0}^{m+1}(-1)^j \binom{m+1}{j} (m+1-j)\|TS^{m+1-j}x\|^p \\ &\quad - \sum_{j=0}^{m+1}(-1)^j \binom{m+1}{j} (m-j)\|S^{m+1-j}x\|^p \Bigg\} \\ &= (m+1)\sum_{j=0}^{m+1}(-1)^j \binom{m+1}{j} \|(S)^{m+1-j}Tx\|^p - \sum_{j=1}^{m+1}(-1)^j \binom{m+1}{j} j\|S^{m+1-j}Tx\|^p \\ &\quad -m\sum_{j=0}^{m+1}(-1)^j \binom{m+1}{j} \|S^{m+1-j}x\|^p - (m+1)\sum_{j=0}^{m}(-1)^j \binom{m}{j} \|S^{m-j}x\|^p \\ &= 0.\end{aligned}$$

Hence, the result follows.

Theorem 2.4 ([4], Proposition 2.4) Let $T \in \mathcal{L}(X)$ be an invertible m-isometry. Then,

1. T^{-1} is an m-isometry.
2. If m is even, then, T is an $(m-1)$-isometry.
3. $T(N(\beta_{m-1}^{\frac{1}{p}})) = N(\beta_{m-1}^{\frac{1}{p}})$, where $\beta_l(T,x) = \frac{1}{l!}\sum_{k=0}^{l}(-1)^{l-k}\binom{l}{k}\|T^k x\|^p$.

3 m-Left and m-Right Inverses

Now, we extend the notion of left inverse and right inverse to m-left inverse and m-right inverse respectively and present some properties of those classes denoted respectively by $\mathfrak{L}_T^m$

and $\mathfrak{R}_T^m$.

Definition 3.1 For $m \in \mathbb{N}$, an operator $T \in \mathcal{L}(X)$ is called

1. m-left invertible if there exists an operator $S \in \mathcal{L}(X)$ for which

$$S^mT^m - \binom{m}{1} S^{m-1}T^{m-1} + \cdots + (-1)^{m-1} \binom{m}{m-1} ST + (-1)^m I_X = 0. \tag{3.1}$$

In this case, S is called an m-left inverse of T.

2. m-right invertible if there exists an operator $R \in \mathcal{L}(X)$ for which

$$T^mR^m - \binom{m}{1} T^{m-1}R^{m-1} + \cdots + (-1)^{m-1} \binom{m}{m-1} TR + (-1)^m I_X = 0. \tag{3.2}$$

In this case, R is called an m-right inverse of T.

Definition 3.2 For $m \in \mathbb{N}$, an operator $T \in \mathcal{L}(X)$ is called m-invertible if T is both m-left and m-right invertible.

Remark 3.1 A 1-left inverse is a left inverse and a 1-right inverse is a right inverse.

The set of all m-left invertible operators in $\mathcal{L}(X)$ will be denoted by $L^m(X)$. For a $T \in L^m(X)$, we denote by $\mathfrak{L}_T^m$ the set of all m-left inverses of T, that is,

$$\mathfrak{L}_T^m = \left\{ S \in \mathcal{L}(X) : \sum_{j=0}^{m} (-1)^j \binom{m}{j} S^{m-j}T^{m-j} = 0 \right\}.$$

The set of all m-right invertible operators in $\mathcal{L}(X)$ will be denoted by $R^m(X)$. For a $T \in R^m(X)$, we denote by $\mathfrak{R}_T^m$ the set of all m-right inverses of T, that is,

$$\mathfrak{R}_T^m = \left\{ R \in \mathcal{L}(X) : \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{m-j}R^{m-j} = 0 \right\}.$$

Remark 3.2 If S is an m-left (resp. m-right) inverse of T, then, T is an m-right (resp. m-left) inverse of S.

Remark 3.3 Let $T \in L^2(X)$ and $S \in \mathfrak{L}_T^2$ be a 2-left inverse of it. We have

$$S(ST - I_X)T = ST - I_X, \tag{3.3}$$

and so,

$$\|S\| \geq \frac{1}{\|T\|}. \tag{3.4}$$

Lemma 3.1 Let $T \in L^2(X)$ and $S \in \mathfrak{L}_T^2$ be a 2-left inverse of it, we have

$$S^kT^k = kST - (k-1)I_X, \quad k = 0, 1, 2, \cdots. \tag{3.5}$$

Lemma 3.2 Let $T \in R^2(X)$ and $R \in \mathfrak{R}_T^2$ be a 2-right inverse of it, we have

$$T^kR^k = kTR - (k-1)I_X, \quad k = 0.1, 2, 3, \cdots. \tag{3.6}$$

S.Patel showed in [9, Theorem 2.1] that a power of 2-isometry is again a 2-isometry. We generalize this theorem to the class of 2-left (resp. 2-right) invertible.

Proposition 3.1 Let $m \geq 1$. If T is 2-left (resp. 2-right) invertible, then, T^n is m-left (resp. m-right) invertible for all n.

Proof Let $S \in \mathfrak{L}_T^2$, from Lemma 3.1, we have

$$S^{nk}T^{nk} = nkST - (nk-1)I_X, \quad k = 1, 2, 3, \cdots, n \in \mathbb{N}.$$

Hence,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}S^{nk}T^{nk} = 0.$$

Proposition 3.2 Let $T \in \mathcal{L}(X)$. The following properties hold:

1. If $T \in L^m(X)$, then $\gamma(S) > 0$ for all $S \in \mathfrak{L}_T^m$;
2. If $T \in R^m(X)$, then $\mu(R) > 0$ for all $R \in \mathfrak{R}_T^m$.

Proof

1. Let $u \in X^*$, we have

$$\begin{aligned}\|u\| &\leq \|T^*\|\Big(\|T^*\|^{m-1}\|S^*\|^{m-1} + \binom{m}{1}\|T^*\|^{m-2}\|S^*\|^{m-1} + \cdots \binom{m}{m-1}\Big)\|S^*u\| \\ &\leq m\|T^*\|\Big(\sum_{j=0}^{m-1}\binom{m-1}{j}\|T^*\|^{m-1-j}\|S^*\|^{m-1-j}\Big)\|S^*u\| \\ &\leq m\|T^*\|\Big(1 + \|S^*\|\|T^*\|\Big)^{m-1}\|S^*u\|.\end{aligned}$$

Using the fact that $\gamma(S^*) = \gamma(S)$ (see [6]) and $\|T^*\| = \|T\|$, we deduce that

$$\gamma(S) \geq \frac{1}{m\|T\|\Big(1 + \|S\|\|T\|\Big)^{m-1}}.$$

2. The argument is similar to part 1.

Proposition 3.3 We have the following inclusions:

1. If $T \in L^m(X)$ then, $\mathfrak{L}_T^m \subset \mathfrak{L}_T^{m+k}, k \in \mathbb{N}$.
2. If $T \in R^m(X)$, then, $\mathfrak{R}_T^m \subset \mathfrak{R}_T^{m+k}, k \in \mathbb{N}$.

Proof

1. It suffice to prove the result for $k = 1$. Let $S \in \mathfrak{L}_T^m$, we have

$$\begin{aligned}&S^mT^m - \binom{m}{1}S^{m-1}T^{m-1} + \binom{m}{2}S^{m-2}T^{m-2} + \cdots \\ &+(-1)^{m-1}\binom{m}{m-1}ST + (-1)^m I_X = 0.\end{aligned} \tag{3.7}$$

Multiplying this equation on the left by S and on the right by T, we obtain

$$\begin{aligned}&S^{m+1}T^{m+1} - \binom{m}{1}S^mT^m + \binom{m}{2}S^{m-1}T^{m-1} + \cdots \\ &+(-1)^{m-1}\binom{m}{m-1}S^2T^2 + (-1)^m ST = 0.\end{aligned} \tag{3.8}$$

We deduce by subtracting these two equations that

$$S^{m+1}T^{m+1} - \binom{m+1}{1} S^m T^m + \binom{m+1}{2} S^{m-1}T^{m-1} + \cdots$$
$$+(-1)^m \binom{m+1}{m} ST + (-1)^{m+1} I_X = 0. \tag{3.9}$$

2. The argument is similar to the one given in 1.

Remark 3.4 An m-left inverse in not necessarily an $(m-1)$-left inverse.

Example 3.1 We consider the Dirichlet space $\mathbb{D}$ consisting of all analytic functions $f : \{z \in \mathbb{C} : |z| < 1\} \longrightarrow \mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} \widehat{f}(k) z^k \quad \text{and} \quad \sum_{k=0}^{\infty} (k+1)|\widehat{f}(k)|^2 < \infty. \tag{3.10}$$

This space, equipped with the inner product

$$\langle f|g \rangle = \sum_{k=0}^{\infty} (k+1) \widehat{f}(k) \overline{\widehat{g}(k)} \tag{3.11}$$

is a Hilbert space.

Consider the operators T and S defined on $\mathbb{D}$, respectively, by

$$T(f)(z) = \sum_{k=0}^{\infty} \widehat{f}(k) z^{k+1} \quad \text{and} \quad S(f)(z) = \sum_{k=1}^{\infty} \frac{k+1}{k} \widehat{f}(k) z^{k-1}. \tag{3.12}$$

We have

$$ST(f)(z) = \sum_{k=1}^{\infty} \frac{k+1}{k} \widehat{f}(k-1) z^{k-1} \tag{3.13}$$

and

$$S^2T^2(f)(z) = \sum_{k=1}^{\infty} \frac{k+2}{k} \widehat{f}(k-1) z^{k-1}. \tag{3.14}$$

Therefore,

$$(S^2T^2 - 2ST + I)(f)(z) = \sum_{k=1}^{\infty} \left(\frac{k+2}{k} - 2\frac{k+1}{k} + 1 \right) \widehat{f}(k-1) z^{k-1} = 0. \tag{3.15}$$

Thus, S is a 2-left inverse of T, but not a 1-left inverse.

Lemma 3.3 Let $T \in R^m(X)$. Then, for $R \in \mathfrak{R}_T^m$,

$$T^{m+1}R^{m+1} = \sum_{k=0}^{m-1} (-1)^{m-k+1} \binom{m+1}{k} (m-k) T^k R^k. \tag{3.16}$$

Proof As

$$T^m R^m = -\sum_{k=0}^{m-1} (-1)^{m-k} \binom{m}{k} T^k R^k, \tag{3.17}$$

then,

$$T^{m+1}R^{m+1} = m(-1)^{m+1}I - \sum_{k=0}^{m-1}(-1)^{m-k}\binom{m+1}{k}T^kR^k$$
$$= \sum_{k=0}^{m-1}(-1)^{m-k+1}\binom{m+1}{k}(m-k)T^kR^k \tag{3.18}$$

Proposition 3.4 Let $T \in R^2(X)$. If $R_1 \in \mathfrak{R}_T^2$ and $R_2 \in \mathfrak{R}_T^{m-1}$ for $m \geq 3$ such that $R_1R_2 = R_2R_1$. Then, $R_1R_2 \in \mathfrak{R}_{T^2}^m$.

Proof By Lemma 3.2, we have

$$T^kT^kR_1^kR_2^k = T^k(kTR_1 - (k-1)I_X)R_2^k, \quad k = 0, 1, 2, \cdots. \tag{3.19}$$

From part (2) of Proposition 3.3, we deduce that

$$\begin{aligned}
&\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(T^2)^k(R_1R_2)^k \\
&= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}kT^kTR_1R_2^k - \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(k-1)T^kR_2^k \\
&= T\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^kR_2^kR_1 - \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}kT^kR_2^k \\
&= mT^2\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}T^kR_2^kR_2R_1 - mT\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}T^kR_2^kR_2 \\
&= 0,
\end{aligned} \tag{3.20}$$

which ends the proof.

Proposition 3.5 Let $T \in L^2(X)$. If $S_1 \in \mathfrak{L}_T^2$ and $S_2 \in \mathfrak{L}_T^{m-1}$ for $m \geq 3$ such that $S_1S_2 = S_2S_1$. Then, $S_1S_2 \in \mathfrak{L}_{T^2}^m$.

Consider a family of right invertible operators $T_i \in R(X)$ and a corresponding family of their right inverses $R_i \in \mathfrak{R}_{T_i}$ for $i = 1, 2, \cdots, n$ and some $n \in \mathbb{N}$. Then, the composition $T = T_1T_2\cdots T_n$ is right invertible, that is, $T \in R(X)$ and one of the right inverse $R \in R_T$ is given by $R = R_n \cdots R_1$.

In the following propositions, we give some conditions for which the composition of m-right (resp. m-lift) inverse is m-right (resp. m-left) inverse.

Proposition 3.6 Let $T \in R^2(X)$ and $R_1 \in \mathfrak{R}_T^2$. If $R_2 \in \mathcal{L}(X)$ such that

$$R_1R_2 = R_2R_1 \quad \text{and} \quad R\big((R_2 - I_X)^{m-1}\big) \subseteq N(TR_1 - I_X)\bigcap N(R_2 - I_X).$$

Then,

$$R_1R_2 \in \mathfrak{R}_T^m.$$

Proof Using Lemma 3.2, we have

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^k(R_1R_2)^k$$

$$= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}kTR_1R_2^k - \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(k-1)R_2^k$$
$$= (TR_1 - I_X)\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}kR_2^k + \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}R_2^k$$
$$= m(TR_1 - I_X)R_2(R_2 - I_X)^{m-1} + (-I_X + R_2)^m$$
$$= \{m(TR_1 - I_X)R_2 + (R_2 - I_X)\}(R_2 - I_X)^{m-1}. \tag{3.21}$$

This implies that $R_1R_2 \in \mathfrak{R}_T^m$.

Proposition 3.7 Let $T \in L^2(X)$. If $S_1 \in \mathfrak{L}_T^2$ and $S_2 \in \mathcal{L}(X)$ such that

$$S_1S_2 = S_2S_1, R(S_1T - I_X) \subseteq N(S_2 - I_X), \quad \text{and} \quad R\big((S_2 - I_X)^{m-1}\big) \subseteq N(S_2 - I_X).$$

Then,

$$S_1S_2 \in \mathfrak{L}_T^m.$$

The proof of the following theorem is inspired from [5]

Theorem 3.1 Let $T \in \mathcal{L}(X)$, $T_1 \in L^2(X)$, and $T_2 \in R^m(X)$. We have

1. The operator $\pounds_1 : \mathcal{L}(X) \longrightarrow \mathcal{L}(X)$ defined by $\pounds_1(T) = T_1TT_2$ is $(m+1)$-left invertible.

2. The operator $\pounds_2 : \mathcal{L}(X) \longrightarrow \mathcal{L}(X)$ defined by $\pounds_2(T) = T_2TT_1$ is $(m+1)$-right invertible.

Proof Let $S \in \mathfrak{L}_{T_1}^2$ and $R \in \mathfrak{R}_{T_2}^m$.

1. We prove that $\pounds_l(T) = STR$ is an $(m+1)$-left inverse of $\pounds_1$. In fact, we have

$$\pounds_l^k\pounds_1^k(T) = S^kT_1^kTT_2^kR^k$$

and

$$\sum_{k=0}^{m+1}(-1)^{m+1-k}\binom{m+1}{k}\pounds_l^k\pounds_1^k(T)$$
$$= \sum_{k=0}^{m+1}(-1)^{m+1-k}\binom{m+1}{k}S^kT_1^kTT_2^kR^k$$
$$= \sum_{k=0}^{1}(-1)^{m+1-k}\binom{m+1}{k}S^kT_1^kTT_2^kR^k + \sum_{k=2}^{m-1}(-1)^{m+1-k}\binom{m+1}{k}S^kT_1^kTT_2^kR^k$$
$$+ \sum_{k=m}^{m+1}(-1)^{m+1-k}\binom{m+1}{k}S^kT_1^kTT_2^kR^k, \tag{3.22}$$

and using Lemma 3.1 and Lemma 3.3 we obtain

$$\sum_{k=0}^{m+1}(-1)^{m+1-k}\binom{m+1}{k}\pounds_l^k\pounds_1^k(T)$$
$$= (-1)^{m+1}T + (-1)^m(m-1)ST_1TT_2R + \sum_{k=0}^{m-1}(-1)^{m+1-k}\binom{m+1}{k}kST_1TT_2^kR^k$$
$$- \sum_{k=0}^{m-1}(-1)^{m+1-k}\binom{m+1}{k}(k-1)TT_2^kR^k + (m^2+m)\sum_{k=0}^{m-1}(-1)^{m-k}\binom{m}{k}ST_1TT_2^kR^k$$

$$-(m^2-1)\sum_{k=0}^{m-1}(-1)^{m-k}\binom{m}{k}TT_2^kR^k$$
$$+[(m+1)ST_1-mI_X]T\Big(\sum_{k=0}^{m-1}(-1)^{m-k+1}(m-k)\binom{m+1}{k}T_2^kR^k\Big)$$
$$=\sum_{k=0}^{m-1}\alpha_{(0,k)}TT_2^kR^k+\sum_{0}^{m-1}\alpha_{(1.k)}\binom{m+1}{k}ST_1TT_2^kR^k \tag{3.23}$$

with

$$\alpha_{(0,0)}=(-1)^{m+1}-(m^2-1)(-1)^m+m^2(-1)^m=0.$$
$$\alpha_{(0,1)}=-(m^2-1)(-1)^{m-1}\binom{m}{1}-m(-1)^m\binom{m+1}{1}(m-1)=0.$$
$$\alpha_{(0,k)}=(-1)^{m-k}\Big[(k-1)\binom{m+1}{k}-(m^2-1)\binom{m}{k}+m\binom{m+1}{k}(m-k)\Big]$$
$$=0,\quad k\geq 2.$$
$$\alpha_{(1,0)}=(-1)^m\big[(m^2+m)-(m+1)m\big]=0.$$
$$\alpha_{(1,1)}=(-1)^m\Big[(m+1)-(m^2+m)m+(m+1)\binom{m+1}{1}(m-1)\Big]=0.$$
$$\alpha_{(1,k)}=(-1)^{m-k+1}\Big[\binom{m+1}{k}k-m(m+1)\binom{m}{k}+(m+1)\binom{m+1}{k}(m-k)\Big]$$
$$=0,\quad k\geq 2.$$

This complete the proof.

2. We prove by a similar argument that $\mathcal{L}_r(T)=RTS$ is an $(m+1)$-right inverse of $\mathcal{L}_2$.

With the notation of Theorem 3.1, we can prove the following result.

Theorem 3.2 Let $T_1\in L^m(X)$ and $T_2\in R^2(X)$. Then, $\mathcal{L}_1$ is $(m+1)$-right invertible and $\mathcal{L}_2$ is $(m+1)$-left invertible.

In the following Lemma, we give some spectral properties of an m-right inverse, for an integer $m\geq 1$

Lemma 3.4 Let $T\in R^m(X)$ and $R\in\mathfrak{R}_T^m$. Then, we have the following properties.

1. $0\notin\sigma_{ap}(R)$.
2. $\{\frac{1}{\lambda},\lambda\in\sigma_{ap}(R)\}\subset\sigma_{ap}(T)$.
3. $\{\frac{1}{\lambda},\lambda\in\sigma_{p}(R)\}\subset\sigma_{p}(T)$.

Proof

1. Suppose, contrary to our claim, that is $0\in\sigma_{ap}(R)$. Then, there exists a sequence $(x_n)_n\in X$ with $\|x_n\|=1$ and $Rx_n\longrightarrow 0$ as $n\longrightarrow+\infty$. Then, $T^{m-j}R^{m-j}x_n\longrightarrow 0$ for $j=0,1,\cdots,m-1$. By the assumption, we obtain

$$x_n=-(-1)^m\sum_{j=0}^{m-1}(-1)^j\binom{m}{j}T^{m-j}R^{m-j}x_n\longrightarrow 0,$$

which is impossible.

2. Let $\lambda\in\sigma_{ap}(R)$, then there exists a sequence $(x_n)_n\in X$ with $\|x_n\|=1$ and $(R-\lambda I_X)x_n\longrightarrow 0$ as $n\longrightarrow+\infty$. Then, $(T^{m-j}R^{m-j}-\lambda^{m-j}T^{m-j})(x_n)\longrightarrow 0$. By the assumption,

we obtain

$$\begin{aligned}&\sum_{j=0}^{m}(-1)^j\binom{m}{j}(T^{m-j}R^{m-j}-\lambda^{m-j}T^{m-j})(x_n)\\=&-\sum_{j=0}^{m}(-1)^j\binom{m}{j}(\lambda^{m-j}R^{m-j})(x_n)\longrightarrow 0.\end{aligned}$$

We find that $(T-\frac{1}{\lambda})^m(x_n)\longrightarrow 0$ as $n\longrightarrow\infty$. From this, it follows that $\frac{1}{\lambda}\in\sigma_{ap}(T)$.

3. The argument is similar to Part 2.

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