INITIAL COEFFICIENT BOUNDS FOR CERTAIN CLASSES OF MEROMORPHIC BI-UNIVALENT FUNCTIONS

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ABSTRACT. In this paper we extend the concept of bi-univalent to the class of meromorphic functions. We propose to investigate the coefficient estimates for two classes of meromorphic bi-univalent functions. Also, we find estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in these new classes. Some interesting remarks and applications of the results presented here are also discussed.
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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

It is well known that every function $h \in \mathcal{S}$ has an inverse h^{-1}, defined by

$$h^{-1}(h(z)) = z, \ \ (z \in \mathbb{U})$$

and

$$h(h^{-1}(w)) = w, \ \ (|w| < r_0(h); \ r_0(h) \geq \frac{1}{4}),$$

where

$$h^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots . \tag{1.2}$$

A function $h \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if both $h(z)$ and $h^{-1}(z)$ are univalent in $\mathbb{U}$. Let $\Sigma_{\mathcal{B}}$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

In 1967, Lewin [14] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. On the other hand, Brannan and Clunie [2] (see also [3, 4, 23]) and Netanyahu [15] made an attempt to introduce various subclasses of the bi-univalent function class $\Sigma_{\mathcal{B}}$ and obtained non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1.1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2\}; \ \ \mathbb{N} := \{1, 2, 3, \cdots\})$ is still an open problem. Following Brannan and Taha [4], many researchers (see[1, 5, 7, 8, 10, 16, 20, 22, 24, 26]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class $\Sigma_{\mathcal{B}}$ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Let Σ denote the class of functions f of the form

$$f(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}, \tag{1.3}$$

which are mermorphic univalent functions defined in

$$\mathcal{V} := \{z : z \in \mathbb{C} \text{ and } 1 < |z| < \infty\}.$$

It is well known that every function $f \in \Sigma$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathcal{V})$$

and

$$f^{-1}(f(w)) = w \qquad (M < |w| < \infty,\ M > 0).$$

Furthermore, the inverse function f^{-1} has a series expansion of the form

$$f^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n}, \tag{1.4}$$

where $M < |w| < \infty$.

The coefficient problem was investigated for various interesting subclasses of the meromorphic univalent functions (see, for example [6, 12, 18]). In 1951, Springer [21] conjectured on the coefficient of the inverse of meromorphic univalent functions, latter the problem was investigated by many researchers for various subclasses (see, for details [11, 12, 13, 19, 25]).

Analogous to the bi-univalent analytic functions, a function $f \in \Sigma$ is said to be meromorphic bi-univalent if both f and f^{-1} are meromorphic univalent in $\mathcal{V}$. We denote by $\Sigma_{\mathcal{M}}$ the class of all meromorphic bi-univalent functions in $\mathcal{V}$ given by (1.3).

A function f in the class Σ is said to be meromorphic bi-univalent starlike of order $\alpha (0 \leq \alpha < 1)$ if it satisfies the following inequalities

$$f \in \Sigma_{\mathcal{M}},\ \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha\ (z \in \mathcal{V}) \text{ and } \Re\left(\frac{wg'(w)}{g(w)}\right) > \alpha\ (w \in \mathcal{V}),$$

where $g(w) = f^{-1}(w)$ is the inverse of $f(z)$ whose series expansion is given by (1.4), a simple calculation shows that

$$g(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2}{w^3} + \dots. \tag{1.5}$$

We denote by $\Sigma^*_{\mathcal{M}}(\alpha)$ the class of all meromorphic bi-univalent starlike functions of order α. Similarly, a function f in the class Σ is said to be meromorphic bi-univalent strongly starlike of order $\alpha (0 < \alpha \leq 1)$ if it satisfies the following conditions

$$f \in \Sigma_{\mathcal{M}},\ \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}\ (z \in \mathcal{V}) \text{ and } \left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2}\ (w \in \mathcal{V}),$$

where $g(w)$ is given by (1.5). We denote by $\widetilde{\Sigma}^*_{\mathcal{M}}(\alpha)$ the class of all meromorphic bi-univalent strongly starlike functions of order α. The classes $\Sigma^*_{\mathcal{M}}(\alpha)$ and $\widetilde{\Sigma}^*_{\mathcal{M}}(\alpha)$ were introduced and studied by Halim et al. [9].

Motivated by the works of Halim et al. [9] we define the following general subclasses $\Sigma^*_{\mathcal{M}}(\alpha, \mu, \lambda)$ and $\widetilde{\Sigma}^*_{\mathcal{M}}(\alpha, \mu, \lambda)$ of the function class Σ.

Definition 1.1. A function f given by (1.3) is said to be in the class $\Sigma^*_{\mathcal{M}}(\alpha,\mu,\lambda)$ if the following conditions are satisfied:

$$f\in\Sigma_{\mathcal{M}},\ \Re\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)>\alpha\ (\mu\geq 0,\ \lambda\geq 1,\ \lambda>\mu; z\in\mathcal{V}) \tag{1.6}$$

and

$$\Re\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)>\alpha\ (\mu\geq 0,\ \lambda\geq 1,\ \lambda>\mu; w\in\mathcal{V}) \tag{1.7}$$

for some $\alpha(0\leq\alpha<1)$, where g is given by (1.5).

Definition 1.2. A function f given by (1.3) is said to be in the class $\widetilde{\Sigma}^*_{\mathcal{M}}(\alpha,\mu,\lambda)$ if the following conditions are satisfied:

$$f\in\Sigma_{\mathcal{M}},\ \left|\arg\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right|<\frac{\alpha\pi}{2}\ (\mu\geq 0,\ \lambda\geq 1,\ \lambda>\mu; z\in\mathcal{V}) \tag{1.8}$$

and

$$\left|\arg\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)\right|<\frac{\alpha\pi}{2}\ (\mu\geq 0,\ \lambda\geq 1,\ \lambda>\mu; w\in\mathcal{V}) \tag{1.9}$$

for some $\alpha(0<\alpha\leq 1)$, where g is given by (1.5).

It is interesting to note that, for $\lambda=1$ and $\mu=0$ the classes $\Sigma^*_{\mathcal{M}}(\alpha,\mu,\lambda)$ and $\widetilde{\Sigma}^*_{\mathcal{M}}(\alpha,\mu,\lambda)$ respectively, reduces to the classes $\Sigma^*_{\mathcal{M}}(\alpha)$ and $\widetilde{\Sigma}^*_{\mathcal{M}}(\alpha)$ introduced and studied by Halim et al. [9].

The object of the present paper is to extend the concept of bi-univalent to the class of meromorphic functions defined on $\mathcal{V}$ and find estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in the above-defined classes $\Sigma^*_{\mathcal{M}}(\alpha,\mu,\lambda)$ and $\widetilde{\Sigma}^*_{\mathcal{M}}(\alpha,\mu,\lambda)$ of the function class $\Sigma_{\mathcal{M}}$ by employing the techniques used earlier by Halim et al. [9].

In order to derive our main results, we shall need the following lemma.

Lemma 1.3. *(see [17]) If $\varphi\in\mathcal{P}$, then $|c_k|\leqq 2$ for each k, where $\mathcal{P}$ is the family of all functions φ, analytic in $\mathbb{U}$, for which*

$$\Re\{\varphi(z)\}>0 \qquad (z\in\mathbb{U}),$$

where

$$\varphi(z)=1+c_1z+c_2z^2+\cdots \qquad (z\in\mathbb{U}).$$

2. Coefficient Bounds for the Function Classes $\Sigma^*_{\mathcal{M}}(\alpha,\mu,\lambda)$ and $\widetilde{\Sigma}^*_{\mathcal{M}}(\alpha,\mu,\lambda)$

We begin this section by finding the estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in the class $\Sigma^*_{\mathcal{M}}(\alpha,\mu,\lambda)$.

Theorem 2.1. *Let the function $f(z)$ given by (1.3) be in the following class:*

$$\Sigma^*_{\mathcal{M}}(\alpha,\mu,\lambda) \qquad (0\leq\alpha<1;\ \lambda\geq 1;\ \mu\geq 0;\ \ \lambda>\mu).$$

Then

$$|b_0| \leq \frac{2(1-\alpha)}{\lambda-\mu} \tag{2.1}$$

and

$$|b_1| \leq 2(1-\alpha)\sqrt{\frac{(1-\mu)^2(1-\alpha)^2}{(\lambda-\mu)^4}+\frac{1}{(2\lambda-\mu)^2}}\ . \tag{2.2}$$

Proof. It follows from (1.6) and (1.7) that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}=\alpha+(1-\alpha)p(z) \tag{2.3}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}=\alpha+(1-\alpha)q(w), \tag{2.4}$$

where $p(z)$ and $q(w)$ are functions with positive real part in $\mathcal{V}$ and have the following forms:

$$p(z)=1+\frac{p_1}{z}+\frac{p_2}{z^2}+\cdots \tag{2.5}$$

and

$$q(z)=1+\frac{q_1}{w}+\frac{q_2}{w^2}+\cdots, \tag{2.6}$$

respectively. Now, equating coefficients in (2.3) and (2.4), we get

$$(\mu-\lambda)b_0=(1-\alpha)p_1, \tag{2.7}$$

$$(\mu-2\lambda)(b_1+(\mu-1)\frac{b_0^2}{2})=(1-\alpha)p_2, \tag{2.8}$$

$$(\lambda-\mu)b_0=(1-\alpha)q_1 \tag{2.9}$$

and

$$(2\lambda-\mu)(b_1-(\mu-1)\frac{b_0^2}{2})=(1-\alpha)q_2. \tag{2.10}$$

From (2.7) and (2.9), we get

$$p_1=-q_1 \tag{2.11}$$

and

$$b_0^2=\frac{(1-\alpha)^2(p_1^2+q_1^2)}{2(\lambda-\mu)^2}. \tag{2.12}$$

Since $\Re\{p(z)\}>0$ in $\mathcal{V}$, the function $p(1/z)\in\mathcal{P}$ and hence the coefficients p_n and similarly the coefficients q_n of the function q satisfy the inequality in Lemma 1.3, we get

$$|b_0|\leq\frac{2-2\alpha}{\lambda-\mu}.$$

This gives the bound on $|b_0|$ as asserted in (2.1).

Next, in order to find the bound on $|b_1|$, we use (2.8) and (2.10), which yields,

$$(1-\mu)^2(2\lambda-\mu)^2b_0^4-4(1-\alpha)^2p_2q_2=4(2\lambda-\mu)^2b_1^2. \tag{2.13}$$

It follows from (2.13) that

$$b_1^2=\frac{(1-\mu)^2b_0^4}{4}-\frac{(1-\alpha)^2}{(2\lambda-\mu)^2}p_2q_2.$$

Substituting the estimate obtained (2.12), and applying Lemma 1.3 once again for the coefficients p_2 and q_2, we readily get

$$|b_1| \leq 2(1-\alpha)\sqrt{\frac{(1-\mu)^2(1-\alpha)^2}{(\lambda-\mu)^4} + \frac{1}{(2\lambda-\mu)^2}}\ .$$

This completes the proof of Theorem 2.1. □

Next we the estimate the coefficients $|b_0|$ and $|b_1|$ for functions in the class $\widetilde{\Sigma}^*_{\mathcal{M}}(\alpha,\mu,\lambda)$.

Theorem 2.2. *Let the function $f(z)$ given by (1.1) be in the following class:*

$$\widetilde{\Sigma}^*_{\mathcal{M}}(\alpha,\mu,\lambda) \qquad (0<\alpha\leq 1;\ \ \lambda\geq 1;\ \ \mu\geq 0;\ \ \lambda>\mu).$$

Then

$$|b_0| \leq \frac{2\alpha}{\lambda-\mu} \tag{2.14}$$

and

$$|b_1| \leq 2\alpha^2\sqrt{\frac{1}{(2\lambda-\mu)^2} + \frac{(1-\mu)^2}{(\lambda-\mu)^4}}. \tag{2.15}$$

Proof. It follows from (1.8) and (1.9) that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = [p(z)]^{\alpha} \tag{2.16}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = [q(w)]^{\alpha}, \tag{2.17}$$

where $p(z)$ and $q(w)$ have the forms (2.5) and (2.6), respectively. Now, equating the coefficients in (2.16) and (2.17), we get

$$(\mu-\lambda)b_0 = \alpha p_1, \tag{2.18}$$

$$(\mu-2\lambda)(b_1+(\mu-1)\frac{b_0^2}{2}) = \frac{1}{2}\left[\alpha(\alpha-1)p_1^2+2\alpha p_2\right], \tag{2.19}$$

$$-(\lambda-\mu)b_0 = \alpha q_1 \tag{2.20}$$

and

$$(2\lambda-\mu)(b_1-(\mu-1)\frac{b_0^2}{2}) = \frac{1}{2}\left[\alpha(\alpha-1)q_1^2+2\alpha q_2\right]. \tag{2.21}$$

From (2.18) and (2.20), we find that

$$p_1 = -q_1 \tag{2.22}$$

and

$$b_0^2 = \frac{\alpha^2(p_1^2+q_1^2)}{2(\lambda-\mu)^2}. \tag{2.23}$$

As discussed in the proof of Theorem 2.1, applying Lemma 1.3 for the coefficients p_2 and q_2, we immediately have

$$|b_0| \leq \frac{2\alpha}{\lambda-\mu}.$$

This gives the bound on $|b_0|$ as asserted in (2.14).

Next, in order to find the bound on $|b_1|$, by using (2.19) and (2.21), we get

$$2(2\lambda-\mu)^2b_1^2+(2\lambda-\mu)^2(1-\mu)^2\frac{b_0^4}{2}=\frac{\alpha^2(\alpha-1)^2(p_1^4+q_1^4)}{4}+\alpha^2(p_2^2+q_2^2)+\alpha^2(\alpha-1)(p_1^2p_2+q_1^2q_2). \tag{2.24}$$

It follows from (2.24) and (2.23) that

$$\begin{aligned}2(2\lambda-\mu)^2b_1^2=&\frac{\alpha^2(\alpha-1)^2(p_1^4+q_1^4)}{4}+\alpha^2(p_2^2+q_2^2)+\alpha^2(\alpha-1)(p_1^2p_2+q_1^2q_2)\\&-\frac{(2\lambda-\mu)^2(1-\mu)^2\alpha^4}{8(\mu-\lambda)^4}(p_1^2+q_1^2)^2.\end{aligned}$$

Applying Lemma 1.3 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|b_1|\le 2\alpha^2\sqrt{\frac{1}{(2\lambda-\mu)^2}+\frac{(1-\mu)^2}{(\lambda-\mu)^4}}.$$

This completes the proof of Theorem 2.2. □

Remark 2.3. For $\lambda=1$ and $\mu=0$ the bounds obtained in Theorems 2.1 and 2.2 are coincidence with outcome of [9, Theorem 1 and Theorem 2]. Similarly, various interesting corollaries and consequences could be derived from our results, the details involved may be left to the reader.

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