BOUNDS FOR THE SECOND HANKEL DETERMINANT OF CERTAIN BI-UNIVALENT FUNCTIONS

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Abstract. In the present work, we propose to investigate the second Hankel determinant inequalities for certain class of analytic and bi-univalent functions. Some interesting applications of the results presented here are also discussed.
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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we will show the family of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\beta)$ of starlike functions of order β in $\mathbb{U}$ and the class $\mathcal{K}(\beta)$ of convex functions of order β in $\mathbb{U}$. By definition, we have

$$\mathcal{S}^*(\beta) := \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta;\ z \in \mathbb{U};\ 0 \leq \beta < 1 \right\}$$

and

$$\mathcal{K}(\beta) := \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta;\ z \in \mathbb{U};\ 0 \leq \beta < 1 \right\}.$$

The arithmetic means of some functions and expressions is very frequently used in mathematics, specially in geometric function theory. Making use of the arithmetic means Mocanu [22] introduced the class of $\alpha-$convex $(0 \leqq \alpha \leqq 1)$ functions (later called as Mocanu-convex functions) as follows:

$$\mathcal{M}(\alpha) := \left\{ f : f \in \mathcal{S} \text{ and } \Re\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0;\ z \in \mathbb{U} \right\}.$$

In [21], it was shown that if the above analytical criteria holds for $z \in \mathbb{U}$, then f is in the class of starlike functions $\mathcal{S}^*(0)$ for α real and is in the class of convex functions $\mathcal{K}(0)$ for $\alpha \geq 1$. In general, the class of $\alpha-$convex functions determines the arithmetic bridge between starlikeness and convexity.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

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and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

For $0 \leq \beta < 1$, a function $f \in \sigma$ is in the class $S^*_\sigma(\beta)$ of bi-starlike function of order β, or $\mathcal{K}_{\sigma,\beta}$ of bi-convex function of order β if both f and f^{-1} are respectively starlike or convex functions of order β. Also, a function f is in the class $\mathcal{M}^\alpha_\Sigma(\beta)$ of bi-Mocanu convex function of order β if both f and f^{-1} are respectively Mocanu convex function of order β. For a brief history and interesting examples of functions which are in (or which are not in) the class σ, together with various other properties of the bi-univalent function class σ one can refer the work of Srivastava et al. [26] and references therein. Various subclasses of the bi-univalent function class σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were found in several recent investigations (see, for example, [2, 5, 7, 10, 16, 19, 23]). However, the problem to find the coefficient bounds on $|a_n|$ $(n = 3, 4, \dots)$ for functions $f \in \sigma$ is still an open problem.

For integers $n \geq 1$ and $q \geq 1$, the $q-$th Hankel determinant, defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1).$$

The Hankel determinant plays an important role in the study of singularities (see [8]). This is also an important in the study of power series with integral coefficients [4, 8]. The properties of the Hankel determinants can be found in [27]. The Hankel determinants $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2a_4 - a_2^3$ are well-known as Fekete-Szegö and second Hankel determinant functionals respectively. Further Fekete and Szegö [9] introduced the generalized functional $a_3 - \delta a_2^2$, where δ is some real number. In 1969, Keogh and Merkes [14] discussed the Fekete-Szegö problem for the classes $\mathcal{S}^*$ and $\mathcal{K}$. Recently, several authors have investigated upper bounds for the Hankel determinant of functions belonging to various subclasses of univalent functions [1, 6, 13, 15, 17, 18] and the references therein. On the other hand, Zaprawa [28, 29] extended the study on Fekete-Szegö problem for certain subclasses of bi-univalent function class σ. Following Zaprawa [28, 29], the Fekete-Szegö problem for functions belonging to various other subclasses of bi-univalent functions were considered in [3, 12, 20]. Very recently, the upper bounds of $H_2(2)$ for the classes $S^*_\sigma(\beta)$ and $K_\sigma(\beta)$ were discussed by Deniz et al. [7].

Next we state the following lemmas we shall use to establish the desired bounds in our study.

Lemma 1.1. [24] *If the function $p \in \mathcal{P}$ is given by the series*

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots, \tag{1.3}$$

then the following sharp estimate holds:

$$|c_k| \le 2, \qquad k = 1, 2, \cdots. \tag{1.4}$$

Lemma 1.2. [11] *If the function $p \in \mathcal{P}$ is given by the series (1.3), then*

$$\begin{aligned} 2c_2 &= c_1^2 + x(4 - c_1^2) \\ 4c_3 &= c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \end{aligned}$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

Inspired by the works of [7, 28] we consider the following subclass of the function class σ.

For $0 \le \alpha \le 1$ and $0 \le \beta < 1$, a function $f \in \sigma$ given by (1.1) is said to be in the class $\mathcal{M}_\sigma^\alpha(\beta)$ if the following conditions are satisfied:

$$\Re\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) \ge \beta \qquad (z \in \mathbb{U})$$

and for $g = f^{-1}$

$$\Re\left((1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right)\right) \ge \beta \qquad (w \in \mathbb{U}).$$

The class was introduced and studied by Li and Wang [16], further the study was extended by Ali et al. [2]. In this paper we shall obtain the functional $H_2(2)$ for functions f belongs to the class $\mathcal{M}_\sigma^\alpha(\beta)$ and its special classes.

2. Bounds for the second Hankel determinant

We begin this section with the following theorem:

Theorem 2.1. *Let f of the form (1.1) be in $\mathcal{M}_\sigma^\alpha(\beta)$. Then*

$$|a_2a_4 - a_3^2| \le \begin{cases} \frac{4(1-\beta)^2}{3(1+\alpha)^3(1+3\alpha)}\left[4(1-\beta)^2 + (1+\alpha)^2\right]\ ; \\ \qquad \beta \in \left[0, 1 - \frac{(1+\alpha)[3(1+3\alpha)+\sqrt{9(1+3\alpha)^2-48(1+\alpha)(1+3\alpha)+128(1+2\alpha)^2}]}{16(1+2\alpha)}\right] \\ \frac{(1-\beta)^2}{(1+\alpha)(1+3\alpha)}\frac{[(1-\beta)^2(1+3\alpha)(13+7\alpha)-12(1-\beta)(1+\alpha)(1+2\alpha)(1+3\alpha)-4(1+\alpha)^2(9\alpha^2+8\alpha+2)]}{[16(1-\beta)^2(1+2\alpha)-6(1-\beta)(1+\alpha)(1+3\alpha)](1+2\alpha)+(1+\alpha)^2[3(1+\alpha)(1+3\alpha)-8(1+2\alpha)^2]}\ ; \\ \qquad \beta \in \left(1 - \frac{(1+\alpha)[3(1+3\alpha)+\sqrt{9(1+3\alpha)^2+128(1+2\alpha)^2}]}{32(1+2\alpha)}, 1\right). \end{cases}$$

Proof. Let $f \in \mathcal{M}_\sigma^\alpha(\beta)$. Then

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \beta + (1-\beta)p(z) \tag{2.1}$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) = \beta + (1-\beta)q(w), \tag{2.2}$$

where $p, q \in \mathcal{P}$ and defined by

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots \tag{2.3}$$

and

$$q(z) = 1 + d_1w + d_2w^2 + d_3w^3 + \dots \tag{2.4}$$

It follows from (2.1), (2.2), (2.3) and (2.4) that

$$(1+\alpha)a_2 = (1-\beta)c_1 \tag{2.5}$$
$$2(1+2\alpha)a_3-(1+3\alpha)a_2^2 = (1-\beta)c_2 \tag{2.6}$$
$$3(1+3\alpha)a_4-3(1+5\alpha)a_2a_3+(1+7\alpha)a_2^3 = (1-\beta)c_3 \tag{2.7}$$

and

$$-(1+\alpha)a_2 = (1-\beta)d_1 \tag{2.8}$$
$$(3+5\alpha)a_2^2-(2+4\alpha)a_3 = (1-\beta)d_2 \tag{2.9}$$
$$(12+30\alpha)a_2a_3-(10+22\alpha)a_2^3-(3+9\alpha)a_4 = (1-\beta)d_3. \tag{2.10}$$

From (2.5) and (2.8), we find that

$$c_1=-d_1 \tag{2.11}$$

and

$$a_2=\frac{1-\beta}{1+\alpha}c_1. \tag{2.12}$$

Now, from (2.6), (2.9) and (2.12), we have

$$a_3=\frac{(1-\beta)^2}{(1+\alpha)^2}c_1^2+\frac{1-\beta}{4+8\alpha}(c_2-d_2). \tag{2.13}$$

Also, from (2.7) and (2.10), we find that

$$a_4=\frac{(2+8\alpha)(1-\beta)^3}{(3+9\alpha)(1+\alpha)^3}c_1^3+\frac{5(1-\beta)^2}{8(1+\alpha)(1+2\alpha)}c_1(c_2-d_2)+\frac{1-\beta}{6(1+3\alpha)}(c_3-d_3). \tag{2.14}$$

Then, we can establish that

$$\begin{aligned}|a_2a_4-a_3^2| = \quad & \left|\frac{-1}{3}\frac{(1-\beta)^4}{(1+\alpha)^3(1+3\alpha)}c_1^4+\frac{(1-\beta)^3}{8(1+\alpha)^2(1+2\alpha)}c_1^2(c_2-d_2)\right. \\ & \left.+\frac{(1-\beta)^2}{6(1+\alpha)(1+3\alpha)}c_1(c_3-d_3)-\frac{(1-\beta)^2}{16(1+2\alpha)^2}(c_2-d_2)^2\right|.\end{aligned} \tag{2.15}$$

According to Lemma 1.2 and (2.11), we write

$$c_2-d_2=\frac{(4-c_1^2)}{2}(x-y) \tag{2.16}$$

and

(2.17)

$$c_3-d_3=\frac{c_1^3}{2}+\frac{c_1(4-c_1^2)(x+y)}{2}-\frac{c_1(4-c_1^2)(x^2+y^2)}{4}+\frac{(4-c_1^2)[(1-|x|^2)z-(1-|y|^2)w]}{2}$$

for some x, y, z and w with $|x|\le 1$, $|y|\le 1$, $|z|\le 1$ and $|w|\le 1$. Using (2.16) and (2.17) in 2.15, we have

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \Bigg| \frac{-(1-\beta)^4 c_1^4}{3(1+\alpha)^3(1+3\alpha)} + \frac{(1-\beta)^3 c_1^2(4-c_1^2)(x-y)}{16(1+\alpha)^2(1+2\alpha)} + \frac{(1-\beta)^2 c_1}{6(1+\alpha)(1+3\alpha)} \\
&\quad \times \Bigg[\frac{c_1^3}{2} + \frac{c_1(4-c_1^2)(x+y)}{2} - \frac{c_1(4-c_1^2)(x^2+y^2)}{4} \\
&\quad + \frac{(4-c_1^2)[(1-|x|^2)z-(1-|y|^2)w]}{2} \Bigg] - \frac{(1-\beta)^2(4-c_1^2)^2}{64(1+2\alpha)^2}(x-y)^2 \Bigg| \\
&\le \frac{(1-\beta)^4}{3(1+\alpha)^3(1+3\alpha)} c_1^4 + \frac{(1-\beta)^2 c_1^4}{12(1+\alpha)(1+3\alpha)} + \frac{(1-\beta)^2 c_1(4-c_1^2)}{6(1+\alpha)(1+3\alpha)} \\
&\quad + \left[\frac{(1-\beta)^3 c_1^2(4-c_1^2)}{16(1+\alpha)^2(1+2\alpha)} + \frac{(1-\beta)^2 c_1^2(4-c_1^2)}{12(1+\alpha)(1+3\alpha)} \right](|x|+|y|) \\
&\quad + \left[\frac{(1-\beta)^2 c_1^2(4-c_1^2)}{24(1+\alpha)(1+3\alpha)} - \frac{(1-\beta)^2 c_1(4-c_1^2)}{12(1+\alpha)(1+3\alpha)} \right](|x|^2+|y|^2) \\
&\quad + \frac{(1-\beta)^2(4-c_1^2)^2}{64(1+2\alpha)^2}(|x|+|y|)^2.
\end{aligned}$$

Since $p \in \mathcal{P}$, so $|c_1| \le 2$. Letting $c_1 = c$, we may assume without restriction that $c \in [0,2]$. Thus, for $\gamma_1 = |x| \le 1$ and $\gamma_2 = |y| \le 1$, we obtain

$$|a_2a_4 - a_3^2| \le T_1 + T_2(\gamma_1+\gamma_2) + T_3(\gamma_1^2+\gamma_2^2) + T_4(\gamma_1+\gamma_2)^2 = F(\gamma_1,\gamma_2),$$

$$\begin{aligned}
T_1 &= T_1(c) = \frac{(1-\beta)^4}{3(1+\alpha)^3(1+3\alpha)} c^4 + \frac{(1-\beta)^2 c^4}{12(1+\alpha)(1+3\alpha)} + \frac{(1-\beta)^2 c(4-c^2)}{6(1+\alpha)(1+3\alpha)} \ge 0 \\
T_2 &= T_2(c) = \frac{(1-\beta)^3 c^2(4-c^2)}{16(1+\alpha)^2(1+2\alpha)} + \frac{(1-\beta)^2 c^2(4-c^2)}{12(1+\alpha)(1+3\alpha)} \ge 0 \\
T_3 &= T_3(c) = \frac{(1-\beta)^2 c^2(4-c^2)}{24(1+\alpha)(1+3\alpha)} - \frac{(1-\beta)^2 c(4-c^2)}{12(1+\alpha)(1+3\alpha)} \le 0 \\
T_4 &= T_4(c) = \frac{(1-\beta)^2(4-c^2)^2}{64(1+2\alpha)^2} \ge 0.
\end{aligned}$$

Now we need to maximize $F(\gamma_1,\gamma_2)$ in the closed square $\mathbb{S} := \{(\gamma_1,\gamma_2) : 0 \le \gamma_1 \le 1, 0 \le \gamma_2 \le 1\}$ for $c \in [0,2]$. We must investigate the maximum of $F(\gamma_1,\gamma_2)$ according to $c \in (0,2)$, $c = 0$ and $c = 2$ taking into account the sign of $F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2$.

Firstly, let $c \in (0,2)$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$ for $c \in (0,2)$, we conclude that

$$F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2 < 0.$$

Thus, the function F cannot have a local maximum in the interior of the square $\mathbb{S}$. Now, we investigate the maximum of F on the boundary of the square $\mathbb{S}$.

For $\gamma_1 = 0$ and $0 \le \gamma_2 \le 1$ (similarly $\gamma_2 = 0$ and $0 \le \gamma_1 \le 1$) we obtain

$$F(0,\gamma_2) = G(\gamma_2) = T_1 + T_2\gamma_2 + (T_3+T_4)\gamma_2^2$$

(i) The case $T_3 + T_4 \ge 0$: In this case for $0 < \gamma_2 < 1$ and any fixed c with $0 < c < 2$, it is clear that $G'(\gamma_2) = 2(T_3+T_4)\gamma_2 + T_2 > 0$, that is, $G(\gamma_2)$ is an increasing function. Hence, for fixed $c \in (0,2)$, the maximum of $G(\gamma_2)$ occurs at $\gamma_2 = 1$ and

$$\max G(\gamma_2) = G(1) = T_1 + T_2 + T_3 + T_4.$$

(ii) The case $T_3+T_4<0$: Since $T_2+2(T_3+T_4)\geq 0$ for $0<\gamma_2<1$ and any fixed c with $0<c<2$, it is clear that $T_2+2(T_3+T_4)<2(T_3+T_4)\gamma_2+T_2<T_2$ and so $G'(\gamma_2)>0$. Hence for fixed $c\in(0,2)$, the maximum of $G(\gamma_2)$ occurs at $\gamma_2=1$ and

Also for $c=2$ we obtain

$$F(\gamma_1,\gamma_2)=\frac{4(1-\beta)^2}{3(1+\alpha)^3(1+3\alpha)}\left[4(1-\beta)^2+(1+\alpha)^2\right]. \tag{2.18}$$

Taking into account the value (2.18) and the cases i and ii, for $0\leq\gamma_2<1$ and any fixed c with $0\leq c\leq 2$,

$$\max G(\gamma_2)=G(1)=T_1+T_2+T_3+T_4.$$

For $\gamma_1=1$ and $0\leq\gamma_2\leq 1$ (similarly $\gamma_2=1$ and $0\leq\gamma_1\leq 1$), we obtain

$$F(1,\gamma_2)=H(\gamma_2)=(T_3+T_4)\gamma_2^2+(T_2+2T_4)\gamma_2+T_1+T_2+T_3+T_4.$$

Similarly, to the above cases of T_3+T_4, we get that

$$\max H(\gamma_2)=H(1)=T_1+2T_2+2T_3+4T_4.$$

Since $G(1)\leq H(1)$ for $c\in(0,2)$, $\max F(\gamma_1,\gamma_2)=F(1,1)$ on the boundary of the square $\mathbb{S}$. Thus the maximum of F occurs at $\gamma_1=1$ and $\gamma_2=1$ in the closed square $\mathbb{S}$.

Let $K:(0,2)\to\mathbb{R}$

$$K(c)=\max F(\gamma_1,\gamma_2)=F(1,1)=T_1+2T_2+2T_3+4T_4. \tag{2.19}$$

Substituting the values of T_1, T_2, T_3 and T_4 in the function K defined by (2.19), yields

$$\begin{aligned}K(c)=&\frac{(1-\beta)^2}{48(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}\left\{\left[16(1-\beta)^2(1+2\alpha)^2\right.\right.\\&-6(1-\beta)(1+\alpha)(1+2\alpha)(1+3\alpha)-8(1+\alpha)^2(1+2\alpha)^2+3(1+\alpha)^3(1+3\alpha)\big]\,c^4\\&+24(1+\alpha)\left[(1-\beta)(1+2\alpha)(1+3\alpha)+2(1+\alpha)(1+2\alpha)^2-(1+\alpha)^2(1+3\alpha)\right]c^2\\&\left.+48(1+\alpha)^3(1+3\alpha)\right\}.\end{aligned}$$

Assume that $K(c)$ has a maximum value in an interior of $c\in(0,2)$, by elementary calculation, we find

$$\begin{aligned}K'(c)=&\frac{(1-\beta)^2}{12(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}\left\{\left[16(1-\beta)^2(1+2\alpha)^2\right.\right.\\&-6(1-\beta)(1+\alpha)(1+2\alpha)(1+3\alpha)-8(1+\alpha)^2(1+2\alpha)^2+3(1+\alpha)^3(1+3\alpha)\big]\,c^3\\&\left.+12(1+\alpha)\left[(1-\beta)(1+2\alpha)(1+3\alpha)+2(1+\alpha)(1+2\alpha)^2-(1+\alpha)^2(1+3\alpha)\right]c\right\}.\end{aligned}$$

After some calculations we concluded following cases:

Case 1. Let

$$[16(1-\beta)^2(1+2\alpha)-6(1-\beta)(1+\alpha)(1+3\alpha)](1+2\alpha)+(1+\alpha)^2[3(1+\alpha)(1+3\alpha)-8(1+2\alpha)^2]\geq 0,$$

that is,

$$\beta\in\left[0,1-\frac{(1+\alpha)[3(1+3\alpha)+\sqrt{9(1+3\alpha)^2-48(1+\alpha)(1+3\alpha)+128(1+2\alpha)^2}]}{16(1+2\alpha)}\right].$$

Therefore $K'(c) > 0$ for $c \in (0,2)$. Since K is an increasing function in the interval $(0,2)$, maximum point of K must be on the boundary of $c \in (0,2)$, that is, $c = 2$. Thus, we have

$$\max_{0<c<2} K(c) = K(2) = \frac{4(1-\beta)^2}{3(1+\alpha)^3(1+3\alpha)} \left[4(1-\beta)^2 + (1+\alpha)^2\right].$$

Case 2. Let

$$[16(1-\beta)^2(1+2\alpha)-6(1-\beta)(1+\alpha)(1+3\alpha)](1+2\alpha)+(1+\alpha)^2[3(1+\alpha)(1+3\alpha)-8(1+2\alpha)^2] < 0,$$

that is,

$$\beta \in \left[1 - \frac{(1+\alpha)[3(1+3\alpha) + \sqrt{9(1+3\alpha)^2 - 48(1+\alpha)(1+3\alpha) + 128(1+2\alpha)^2}]}{16(1+2\alpha)}, 1\right].$$

Then $K'(c) = 0$ implies the real critical point $c_{0_1} = 0$ or

$$c_{0_2} = \sqrt{\frac{-12(1+\alpha)[(1-\beta)(1+2\alpha)(1+3\alpha)+2(1+\alpha)(1+2\alpha)^2-(1+\alpha)^2(1+3\alpha)]}{[16(1-\beta)^2(1+2\alpha)-6(1-\beta)(1+\alpha)(1+3\alpha)](1+2\alpha)+(1+\alpha)^2[3(1+\alpha)(1+3\alpha)-8(1+2\alpha)^2]}}.$$

When

$$\beta \in \left(1 - \frac{(1+\alpha)[3(1+3\alpha)+\sqrt{9(1+3\alpha)^2-48(1+\alpha)(1+3\alpha)+128(1+2\alpha)^2}]}{16(1+2\alpha)}, 1 - \frac{(1+\alpha)[3(1+3\alpha)+\sqrt{9(1+3\alpha)^2+128(1+2\alpha)^2}]}{32(1+2\alpha)}\right],$$

we observe that $c_{0_2} \geq 2$, that is, c_{0_2} is out of the interval $(0,2)$. Therefore, the maximum value of $K(c)$ occurs at $c_{0_1} = 0$ or $c = c_{0_2}$ which contradicts our assumption of having the maximum value at the interior point of $c \in [0,2]$.

When $\beta \in \left(1 - \frac{(1+\alpha)[3(1+3\alpha)+\sqrt{9(1+3\alpha)^2+128(1+2\alpha)^2}]}{32(1+2\alpha)}, 1\right)$, we observe that $c_{0_2} < 2$, that is, c_{0_2} is an interior of the interval $[0,2]$. Since $K''(c_{0_2}) < 0$, the maximum value of $K(c)$ occurs at $c = c_{0_2}$. Thus, we have

$$\begin{aligned}\max_{0\leq c\leq 2} K(c) &= K(c_{0_2}) \\ &= \frac{(1-\beta)^2}{(1+\alpha)(1+3\alpha)} \frac{[(1-\beta)^2(1+3\alpha)(13+7\alpha)-12(1-\beta)(1+\alpha)(1+2\alpha)(1+3\alpha)-4(1+\alpha)^2(9\alpha^2+8\alpha+2)]}{[16(1-\beta)^2(1+2\alpha)-6(1-\beta)(1+\alpha)(1+3\alpha)](1+2\alpha)+(1+\alpha)^2[3(1+\alpha)(1+3\alpha)-8(1+2\alpha)^2]}.\end{aligned}$$

This completes the proof. □

Remark 2.2. For $\alpha = 0$ and $\alpha = 1$, Theorem 2.1 would reduce to a known results in [7, Theorem 2.1, Theorem 2.3].

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