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# CONGRUENCES FOR $_3F_2$ HYPERGEOMETRIC FUNCTIONS OVER FINITE FIELDS

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Abstract. We present congruences for Greene's $_3F_2$ hypergeometric functions over finite fields, which relate values of these functions to a simple polynomial in the characteristic of the field.

In the 1980's, J. Greene [G1][G2] initiated a study of finite field hypergeometric functions, and he found that they satisfy a variety of properties analogous to those of their classical counterparts. Recent works by S. Ahlgren and K. Ono [A-O][O] and M. Koike [K] have illustrated that certain special values of these functions are congruent to Apéry type numbers modulo the characteristic of the finite field. Here we present congruences of a different type which relate the values of these functions to a simple polynomial in the characteristic.

We begin by recalling Greene's definition. As usual, let $GF(p)$ denote the finite field with $p$ elements. We extend all characters $\chi$ of $GF(p)^*$ to $GF(p)$ by setting $\chi(0) := 0$. If $A$ and $B$ are two characters of $GF(p)$, then we denote the normalized Jacobi sum by

$$\binom{A}{B} := \frac{B(-1)}{p} J(A, \bar{B}) = \frac{B(-1)}{p} \sum_{x \in GF(p)} A(x)\bar{B}(1-x). \tag{1}$$

Definition 1. If $A_0, A_1, \dots A_n$ and $B_1, B_2, \dots B_n$ are characters of $GF(p)$, then Greene's hypergeometric function ${}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{matrix} \,\middle|\, x\right)_p$ is defined by

$${}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{matrix} \,\middle|\, x\right)_p := \frac{p}{p-1} \sum_{\chi} \binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi} \cdots \binom{A_n\chi}{B_n\chi} \chi(x),$$

where the summation is over all characters $\chi$ of $GF(p)$.

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We restrict our attention to the ${}_{n+1}F_n\left(\begin{matrix}\phi_p, & \phi_p, & \ldots, & \phi_p \\ & \epsilon_p, & \ldots, & \epsilon_p\end{matrix} \,\middle|\, \lambda\right)_p$ functions, where $\phi_p$ denotes the Legendre symbol modulo $p$ and $\epsilon_p$ is the trivial character; for convenience we denote these functions by ${}_{n+1}F_n(\lambda)_p$. S. Ahlgren and the first author [A-O] proved one of F. Beukers' Apéry number supercongruences by explicitly evaluating all of the ${}_4F_3(1)_p$ and relating these values to the zeta function of a specific Calabi-Yau manifold. They also showed that if $p$ is an odd prime, then [A-O, Theorem 3]

$$ {}_4F_3(1)_p \equiv -1 - p^{13} - p^{14} \pmod{32}. $$

Here we obtain many such congruences for ${}_3F_2(\lambda)_p$. For example, if $p \neq 2, 3, 5, 11$, then

$$ (2) \qquad {}_3F_2\left(\frac{2673}{2048}\right)_p \equiv \phi_p(-2)(1+p^{-1}+p^{-2}) \equiv \phi_p(-2)(1+p^2+p^3) \pmod{20}. $$

We begin by defining groups $G_i$, functions $\lambda_i(s)$ and sets $S_i$ as in Table 1 below.

If $p$ is prime and $n$ is a nonzero integer, then $\mathrm{ord}_p(n)$ shall denote the power of $p$ dividing $n$; we extend $\mathrm{ord}_p$ to $\mathbb{Q}$ in the obvious way.

Theorem 1. *For each $i$ in Table 1, define $N_i$ by*

$$ N_i := \begin{cases} 2|G_i| & \text{if } 4 \nmid |G_i|, \\ 4|G_i| & \text{otherwise.} \end{cases} $$

*If $s \in \mathbb{Q} - S_i$ and $p \geq 5$ is a prime for which*

$$ \mathrm{ord}_p(\lambda_i(s)(\lambda_i(s)-4)) = \mathrm{ord}_p(N_i) = 0, $$

*then*

$$ {}_3F_2\left(\frac{4}{4-\lambda_i(s)}\right)_p \equiv \phi_p(\lambda_i^2(s) - 4\lambda_i(s))\left(1 + p^{-1} + p^{-2}\right) \pmod{N_i}. $$

Remark. Example (2) is obtained by letting $s = 3/4$ when $i = 14$ in Theorem 1.

## 2. Proof of Theorem 1

If $\lambda \in \mathbb{Q} - \{0, 4\}$, define the elliptic curve $E(\lambda)/\mathbb{Q}$ by the equation

$$ (3) \qquad E(\lambda): \quad y^2 = (x-1)\left(x^2 + \frac{4-\lambda}{\lambda}\right). $$

The point $(1, 0)$ is a point of order 2 on $E(\lambda)$. (Note that this curve is isomorphic over $\mathbb{Q}$ to the curve (16) in [O].) This elliptic curve has discriminant

$$ (4) \qquad \Delta(E(\lambda)) := 1024\lambda^{-3}(\lambda - 4) $$

TABLE 1

| $i$ | $G_i$ | $\lambda_i(s)$ | $S_i$ |
|---|---|---|---|
| 1 | $\mathbb{Z}2$ | $s$ | $0, 4, \frac{9}{2}$ |
| 2 | $\mathbb{Z}2 \times \mathbb{Z}2$ | $\frac{4}{1-s^2}$ | $0, \pm 1, \pm\frac{1}{3}$ |
| 3 | $\mathbb{Z}4$ | $\frac{(8s+1)^2}{16s^2}$ | $0, -\frac{1}{8}, -\frac{1}{16}$ |
| 4 | $\mathbb{Z}2 \times \mathbb{Z}4$ | $\frac{4(16s^2+1)^2}{(4s+1)^2(4s-1)^2}$ | $0, \pm\frac{1}{4}$ |
| 5 | $\mathbb{Z}2 \times \mathbb{Z}4$ | $-\frac{(16s^2-24s+1)^2}{16s(4s-1)^2}$ | $0, \frac{1}{4}$ |
| 6 | $\mathbb{Z}2 \times \mathbb{Z}8$ | $\frac{(4096s^8+8192s^7+6144s^6+2048s^5+512s^4+256s^3+96s^2+16s+1)^2}{256(4s+1)^4(2s+1)^4s^4}$ | $0, -\frac{1}{4}, -\frac{1}{2}$ |
| 7 | $\mathbb{Z}2 \times \mathbb{Z}8$ | $-\frac{4(2048s^8+4096s^7+3072s^6+1024s^5-128s^4-256s^3-96s^2-16s-1)^2}{(8s^2+8s+1)(8s^2-1)(8s^2+4s+1)^2(4s+1)^4}$ | $0, -\frac{1}{4}, -\frac{1}{2}$ |
| 8 | $\mathbb{Z}2 \times \mathbb{Z}8$ | $\frac{(8192s^8+16384s^7+12288s^6+4096s^5+256s^4-256s^3-96s^2-16s-1)^2}{256s^4(8s^2+8s+1)(8s^2-1)(8s^2+4s+1)^2(2s+1)^4}$ | $0, -\frac{1}{4}, -\frac{1}{2}$ |
| 9 | $\mathbb{Z}8$ | $\frac{(8s^4-16s^3+16s^2-8s+1)^2}{16(s-1)^4s^4}$ | $0, \frac{1}{2}, 1$ |
| 10 | $\mathbb{Z}6$ | $-\frac{(3s^2-6s-1)^2}{16s^3}$ | $0, -1, -\frac{1}{9}$ |
| 11 | $\mathbb{Z}2 \times \mathbb{Z}6$ | $\frac{(s^4-12s^3+30s^2+228s-759)^2}{128(s-5)^3(s-3)(s+3)}$ | $1, \pm 3, 5, 9$ |
| 12 | $\mathbb{Z}2 \times \mathbb{Z}6$ | $-\frac{(s^4-12s^3+30s^2-156s+393)^2}{128(s-1)^3(s-9)(s-3)}$ | $1, \pm 3, 5, 9$ |
| 13 | $\mathbb{Z}2 \times \mathbb{Z}6$ | $\frac{4(s^4-12s^3+30s^2+36s-183)^2}{(s-5)^3(s-1)^3(s-9)(s+3)}$ | $1, \pm 3, 5, 9$ |
| 14 | $\mathbb{Z}10$ | $\frac{(2s^2-2s+1)^2(4s^4-12s^3+6s^2+2s-1)^2}{16(s-1)^5(s^2-3s+1)s^5}$ | $0, \frac{1}{2}, 1$ |
| 15 | $\mathbb{Z}12$ | $\frac{(24s^8-96s^7+216s^6-312s^5+288s^4-168s^3+60s^2-12s+1)^2}{16(s-1)^6(3s^2-3s+1)^2s^6}$ | $0, \frac{1}{2}, 1$ |

and $j$-invariant

$$j(E(\lambda)) := \frac{256(\lambda-3)^3}{\lambda-4}. \tag{5}$$

If $p$ is a prime such that $\mathrm{ord}_p(\Delta(E(\lambda))) = 0$, then $E(\lambda)$ is an elliptic curve when considered over $GF(p)$; in this case we say that $E(\lambda)$ has *good reduction at* $p$, and define ${}_3a_2(p;\lambda)$ by

$${}_3a_2(p;\lambda) = p + 1 - |E(\lambda)_p|, \tag{6}$$

where $|E(\lambda)_p|$ denotes the order of the Mordell-Weil group of $E(\lambda)$ over $GF(p)$.

Extending a result of J. Greene and D. Stanton [G-S], the first author proved the following theorem in [O].

Theorem 2. *If* $\lambda \in \mathbb{Q} - \{0, 4\}$ *and* $p \geq 5$ *is a prime for which* $\mathrm{ord}_p(\lambda(\lambda - 4)) = 0$, *then*

$${}_3F_2\left(\frac{4}{4-\lambda}\right)_p = \frac{\phi_p(\lambda^2 - 4\lambda)({}_3a_2(p;\lambda)^2 - p)}{p^2}.$$

Theorem 1 follows from Theorem 2 and the following elementary proposition regarding the numbers ${}_3a_2(p;\lambda) \pmod{N}$ when $E(\lambda)$ is the twist of an elliptic curve over $\mathbb{Q}$ whose Mordell-Weil group has a torsion subgroup of order $N$.

Proposition 3. *Suppose that* $E/\mathbb{Q}$ *is an elliptic curve for which* $j(E) = j(E(\lambda)) \neq 1728$, *and assume the torsion subgroup of* $E/\mathbb{Q}$ *has even order* $N$. *Let* $N' = 2N$ *if* $4 \nmid N$, *and* $N' = 4N$ *if* $4 \mid N$. *If* $p \geq 5$ *is a prime for which* $E$ *has good reduction and*

$$\mathrm{ord}_p(\lambda(\lambda-4)) = \mathrm{ord}_p(N) = 0,$$

*then*

$${}_3F_2\left(\frac{4}{4-\lambda}\right)_p \equiv \phi_p(\lambda^2 - 4\lambda)(1 + p^{-1} + p^{-2}) \pmod{N'}.$$

*Proof.* By (4), the condition that the odd prime $p$ satisfies $\mathrm{ord}_p(\lambda(\lambda-4)) = 0$ implies that $E(\lambda)$ is an elliptic curve over $GF(p)$. Since $j(E) = j(E(\lambda))$, $E$ is a twist of $E(\lambda)$ (see Proposition 1.4 (b) in Chapter III of [Si]).

We claim that in fact $E$ is a quadratic twist of $E(\lambda)$. To see this, note first that if $j(E) \neq 0, 1728$, then our claim is given by Proposition 5.4(i) in Chapter X of [Si]. In case $j(E) = 0$, we know that $E$ is given by an equation of the form $E: y^2 = x^3 - b$, $b \in \mathbb{Q} - \{0\}$. Since $N$ is even, $b = c^3$ for some $c \in \mathbb{Q} - \{0\}$ (so the point $(c, 0)$ has order 2). The same argument holds for $E(\lambda)$; indeed, one can check (since $\lambda = 3$ in this case) that $E(\lambda)$ is given by the equation $y^2 = x^3 - \frac{8}{27}$. Therefore $E$ is the $\frac{3c}{2}$-quadratic twist of $E(\lambda)$.

If $E_p$ denotes the curve $E$ considered over $GF(p)$, this implies that

$$ {}_3a_2(p;\lambda) = \pm(p+1-|E_p|). \tag{7}$$

Now, the fact that $\text{ord}_p(N) = 0$ gives that the reduction map $E \to E_p$ is injective on the torsion points of $E/\mathbb{Q}$ (see Proposition 3.1 (b) in Chapter VII of [Si]), and hence

$$|E_p| \equiv 0 \pmod{N}.$$

Therefore, by (7) we find that

$${}_3a_2(p;\lambda) \equiv \pm(p+1) \pmod{N}.$$

The claim now follows easily from Theorem 2. □

*Proof of Theorem 1.* We prove Theorem 1 in the case when $i = 10$ (i.e., $G_{10} = \mathbb{Z}6$); the remaining cases follow *mutatis mutandis.* Notice that the groups $G_i$ are exactly those which occur as torsion subgroups of elliptic curves over $\mathbb{Q}$ and have an element of order 2 (see, for example, Theorem 7.5 in Chapter VIII of [Si]).

Kubert [Ku] has shown that any elliptic curve $E/\mathbb{Q}$ having a rational point of order 6 can be given by an equation of the form

$$E : y^2 + (1-s)xy - (s^2+s)y = x^3 - (s^2+s)x^2, \tag{8}$$

where $s \in \mathbb{Q} - \{0, -1, -1/9\}$. This curve has discriminant

$$\Delta(E) = s^6(s+1)^3(9s+1), \tag{9}$$

and its $j$-invariant is

$$j(E) = \frac{(9s^4+12s^3+30s^2+12s+1)^3}{s^6(s+1)^3(9s+1)}.$$

Setting this equal to $j(E(\lambda))$, we obtain one rational solution $\lambda = \lambda_{10}(s)$, which is given in the table. By Proposition 3, for every $s \in \mathbb{Q} - S_{10}$ and every prime $p \geq 5$ for which $\text{ord}_p(\lambda_{10}(s)(\lambda_{10}(s)-4)) = 0$ and $E$ has good reduction at $p$, we have

$${}_3F_2\left(\frac{4}{4-\lambda_{10}(s)}\right) \equiv \phi_p(\lambda_{10}(s)^2 - 4\lambda_{10}(s))(1+p^{-1}+p^{-2}) \pmod{12}.$$

Since

$$\lambda_{10}(s)(\lambda_{10}(s)-4) = \frac{(3s^2-6s-1)^2(9s+1)(s+1)^3}{256s^6},$$

by (9) we find that all the potential odd primes of bad reduction for $E$ already have the property that $\text{ord}_p(\lambda_{10}(s)(\lambda_{10}(s)-4)) \neq 0$. This completes the proof when $i = 10$.

Theorem 1 is obtained by arguing as above for all the possible torsion subgroups of elliptic curves $E/\mathbb{Q}$ containing a rational point of order 2. All such curves have a convenient parametrization as in (8), and they are listed in Table 3 of [Ku]. □

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