On Triangular Norm-Based Propositional Fuzzy Logics

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Abstract

Fuzzy logics based on triangular norms and their corresponding conorms are investigated. An affirmative answer to the question whether in such logics a specific level of satisfiability of a set of formulas can be characterized by the same level of satisfiability of its finite subsets is given. Tautologies, contradictions and contingencies with respect to such fuzzy logics are studied, in particular for the important cases of min-max and Lukasiewicz logics. Finally, fundamental t-norm-based fuzzy logics are shown to provide a gradual transition between min-max and Lukasiewicz logics.

Key words: Fuzzy Logics, Min-max logic, Lukasiewicz Logic, Triangular Norms, Satisfiability.

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0 Introduction

Fuzzy logic emerged as a necessary extension of Zadeh's [13] theory of fuzzy sets. Its development during the last two decades was stimulated by its applications in fields like artificial intelligence and heuristic decision making (see Zadeh [16]). Fuzzy logical systems formalize the rules of reasoning with vague statements in various settings (see, e.g., Zadeh [15] and Dubois and Prade [3]). The variety of fuzzy logical systems discussed in the literature reflects the diversity of ways in which vagueness can be approached from a logical point of view as well as the heterogeneous nature of the vagueness concepts appearing in real world problems (see R. L. De Mantaras [6]).

In this work, by the term "fuzzy logic" we mean a specific $[0,1]$-valued propositional logic whose connectives are interpreted via triangular norms. Recall (see, for instance, Schweizer and Sklar [12], Butnariu and Klement [1]) that a *triangular norm* (*t-norm* for short) is a function $T : [0,1]\times[0,1] \to [0,1]$ which is symmetric ($T(x,y) = T(y,x)$), associative ($T(x,T(y,z)) = T(T(x,y),z)$), has 1 as unit element ($T(x,1) = x$), and is monotone ($x_1 \leq x_2$ implies $T(x_1,y) \leq T(x_2,y)$). The function $S : [0,1]\times[0,1] \to [0,1]$ defined by $S(x,y) = 1 - T(1-x, 1-y)$ is called the *triangular conorm* (*t-conorm*) *corresponding to* T. In these terms, a fuzzy propositional logic can be described as a $[0,1]$-valued logic in which the disjunction symbol $\vee$, the conjunction symbol $\wedge$, and the negation symbol $\neg$ are interpreted respectively as follows

$$\begin{aligned} \mathbf{t}(\varphi \wedge \psi) &= T(\mathbf{t}(\varphi), \mathbf{t}(\psi)), \\ \mathbf{t}(\varphi \vee \psi) &= S(\mathbf{t}(\varphi), \mathbf{t}(\psi)), \\ \mathbf{t}(\neg\varphi) &= 1 - \mathbf{t}(\varphi) \end{aligned}$$

for every truth assignment $\mathbf{t}$.

This work is aimed to analyze properties of and interrelations among fuzzy logics. The formal definition of the concept of fuzzy logic we are dealing with is presented in details in Section 2. This definition is an attempt to interrelate the syntax and the semantics of the logics based on t-norms which were studied separately in different contexts, and under various aspects by many people (see, for instance, Novak [9], Pavelka [10], Dubois and Prade [3], and the references therein). Section 3 is devoted to satifiability problems in fuzzy logics. The question there is whether a specific level of satifiability (see Definition 2.4) of a set of formulas in a fuzzy logic can be characterized by the same level of satifiability of its finite subsets. The affirmative answer is given by Theorem 2.3 and Corollary 2.5. These results may provide a way of defining and studying a concept of consistency that admits fuzzy degrees (in the interval $[0,1]$) like the truth concept in fuzzy logics. This new concept of consistency may be more suitable to handle collections of sentences involved in fuzzy logical modelling of real life phenomena. The higher the consistency level of such a description, the higher its practical reliability. This is in contrast to bivalued logical represantations, which are either consistent or inconsistent, with no intermediate alternatives. We feel that in some practical situations we can restrict ourselves to logical descriptions with a sufficiently high level of consistency, even if they do not have the ideal 100% degree of consistency. In Section 4 we examine the traditional classes of tautologies, contradictions, and contingencies with respect to fuzzy logic. Theorem 3.4 gives a semantical characterization of these classes. In Section 5 we discuss tautologies, contradictions, and contingencies in the context of the so called min-max logics. Theorem 4.1 and its subsequent corollaries make precise their semantical characterization in this more specific context. Section 6 deals with a class of fuzzy logics defined via infinitary disjunctions. We show that whenever a fuzzy logic has a fundamental t-norm-based interpretation, it automatically incorporates the Lukasiewicz $[0,1]$-valued logic (see Rose and Rosser [11]) as well as the min-max fuzzy logic (see Definition 1.1). Precisely, Proposition 5.4 says that, given a fundamental t-norm-based fuzzy logic $\mathcal{P}$, any well-formed formula φ in Łukasiewicz logic or in min-max fuzzy logic is semantically equivalent to a well-formed formula φ^* of $\mathcal{P}$ where φ^* can be computed from φ by a specific recursive procedure. As a consequence we obtain Theorems 5.4 and 5.5 emphasizing the interconnectivity among min-max and Łukasiewicz fuzzy logics, on one hand, and the so-called s-fuzzy logics (Definition 5.3), on the other hand. These results point out a class of fuzzy logics in which "fuzzy reasoning" can be done by simultanuously exploiting the flexibility of the low exclusiveness of the conjunction of min-max fuzzy logic and the high exclusiveness of the conjunction of the Łukasiewicz fuzzy logic, where the level of exclusiveness is measured via the maximal truth value of the proposition $p \wedge \neg p$ for an arbitrary p. Taking advantage of this particularity of s-fuzzy logics should lead to better classification and pattern recognition procedures.

1 Preliminaries

In this section we make precise some notions which will be used along this work. We start by recalling that a *propositional logic* $\mathcal{P} = (\mathcal{L}, \mathcal{S})$ consists of a *language* $\mathcal{L}$ and a *structure* $\mathcal{S}$ described as follows:

(i) The language $\mathcal{L} = (\mathbf{P}, \mathbf{C})$ consists of a set $\mathbf{P}$ of *atomic symbols* and a finite set $\mathbf{C}$ of *connectives*. Each connective $\diamond \in \mathbf{C}$ is assigned a fixed *arity* which is either a natural number n or ∞.

(ii) The *structure* of $\mathcal{P}$ is an ordered pair $\mathcal{S} = (V, \mathcal{O})$, where V is a set whose elements are called *truth values*, and $\mathcal{O}$ is a mapping that assigns to each connective $\diamond \in \mathbf{C}$ a function $F_\diamond : \prod_{i=1}^{\ell} V_i \to V$, where $\ell \in \mathbb{N} \cup \{\infty\}$ is the arity of $\diamond$ and for each i, $V_i = V$. $F_\diamond$ is called the *interpretation* of the connective $\diamond$ in $\mathcal{P}$.

Note that different propositional logics may have the same language and differ by their structure only. If the set V of truth values is exactly $\{0,1\}$, then the logic $\mathcal{P}$ is called *bivalued*. Given a propositional logic $\mathcal{P}$, *well-formed formulas in* $\mathcal{P}$ (*$\mathcal{P}$-formulas*, for short) are defined recursively as follows:

1. Every atomic symbol $p \in \mathbf{P}$ is a formula.

2. If $\diamond$ is an n-ary connective, where $n \in \mathbb{N}$, and φ_1, φ_2, ..., φ_n are n $\mathcal{P}$-formulas then $\diamond(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is a $\mathcal{P}$-formula.

3. If $\diamond$ is an infinitary connective and if $\{\varphi_n\}_{n=1}^{\infty}$ is an infinite sequence of $\mathcal{P}$-formulas, then $\diamond(\varphi_1, \varphi_2, \ldots)$ is an $\mathcal{P}$-formula.

The set of all $\mathcal{P}$-formulas will be denoted by $\mathcal{F}(\mathcal{P})$ or, simply, $\mathcal{F}$, if no confusion is possible, and we write $\varphi \diamond \psi$ instead of $\diamond(\varphi, \psi)$, whenever $\diamond$ is a binary connective, $\diamond \varphi$ instead of $\diamond(\varphi)$, whenever $\diamond$ is a unary connective, and $\overset{\infty}{\underset{n=1}{\diamond}} \varphi_n$ instead of the syntactically infinitely long expression $\diamond(\varphi_1, \varphi_2, \ldots)$, whenever $\diamond$ is an infinitary connective.

A function $\mathbf{t} : \mathbf{P} \to V$ is usually called a *truth assignment*. Clearly, $\mathbf{t}$ depends on the set of truth values V only and hence, it may be shared by different propositional logics that have V in common. For a specific propositional logic $\mathcal{P}$, $\mathbf{t}$ has a unique extension $\overline{\mathbf{t}}_{\mathcal{P}} : \mathcal{F}(\mathcal{P}) \to V$ defined via the following recursion:

1. If $\varphi = p$, where p is an atomic symbol, then $\overline{\mathbf{t}}_{\mathcal{P}}(p) = \mathbf{t}(p)$;

2. If $\varphi = \diamond(\varphi_1, \varphi_2, \ldots, \varphi_n)$ then $\overline{\mathbf{t}}_{\mathcal{P}}(\varphi) = F_{\diamond}(\overline{\mathbf{t}}_{\mathcal{P}}(\varphi_1), \overline{\mathbf{t}}_{\mathcal{P}}(\varphi_2), \ldots, \overline{\mathbf{t}}_{\mathcal{P}}(\varphi_n))$;

3. If $\varphi = \overset{\infty}{\underset{n=1}{\diamond}} \varphi_n$, then $\overline{\mathbf{t}}_{\mathcal{P}}(\varphi) = F_{\diamond}(\{\overline{\mathbf{t}}(\varphi_n)\}_{n=1}^{\infty})$.

We call $\overline{\mathbf{t}}_{\mathcal{P}}$ *the natural extension of* $\mathbf{t}$ *to* $\mathcal{F}(\mathcal{P})$ and denote it $\overline{\mathbf{t}}$ whenever confusions are not possible.

This paper concerns t-norm-based *fuzzy propositional logics* (*fuzzy logics* for short), i.e., propositional logics $\mathcal{P}$ such that

(i) The language $\mathcal{L}$ of $\mathcal{P}$ consists of a non-empty set of atoms $\mathbf{P}$, and a collection $\mathbf{C}$ of connectives that contains at least the *negation* $\neg$ and a connective $\diamond$ called *disjunction* which is either binary or infinitary.

(ii) The structure $\mathcal{S}$ of $\mathcal{P}$ consists of the set $V = [0,1]$ and the mapping $\mathcal{O}$ such that

$$F_{\neg}(x) = 1 - x \quad \text{and} \quad F_{\diamond}(\{x_i\}_{i=1}^{\ell}) = \mathop{\mathbf{S}}_{i=1}^{\ell} x_i, \tag{1.1}$$

where $\ell \in \{2, \infty\}$ and $\mathop{\mathbf{S}}_{i=1}^{\ell} x_i$ is the ℓ-ary extension of a t-conorm S as defined in the sequel.

Let T denote an arbitrary t-norm. For every natural number $n \geq 2$ the *n-ary extension of T*, $\mathop{\mathbf{T}}_{i=1}^{n} : [0,1]^n \to [0,1]$, is recursively defined by

$$\mathop{\mathbf{T}}_{i=1}^{n+1}(x_1, x_2, \ldots, x_{n+1}) = \begin{cases} T(x_1, x_2) & \text{if } n = 1, \\ T\Big(\mathop{\mathbf{T}}_{i=1}^{n}(x_1, x_2, \ldots, x_n), x_{n+1}\Big) & \text{if } n > 1. \end{cases}$$

We write $\mathop{\mathbf{T}}_{i=1}^{n} x_i$ instead of $\mathop{\mathbf{T}}_{i=1}^{n}(x_1, \ldots, x_n)$. *The infinitary extension of T*, $\mathop{\mathbf{T}}_{n=1}^{\infty} : [0,1]^{\mathbb{N}} \to [0,1]$, is defined by

$$\mathop{\mathbf{T}}_{n=1}^{\infty} x_n = \lim_{n\to\infty} \mathop{\mathbf{T}}_{i=1}^{n} x_i \,.$$

The limit exists since the sequence $\left\{\mathop{\mathbf{T}}_{i=1}^{n} x_i\right\}_{n=2}^{\infty}$ is non-increasing in $[0,1]$. The t-conorm S corresponding to T is extended analogously via

$$\mathop{\mathbf{S}}_{i=1}^{n} x_i = 1 - \mathop{\mathbf{T}}_{i=1}^{n}(1 - x_i) \quad \text{and} \quad \mathop{\mathbf{S}}_{n=1}^{\infty} x_n = 1 - \mathop{\mathbf{T}}_{n=1}^{\infty}(1 - x_n).$$

The t-norm T, whose corresponding t-conorm S is involved in (1.1), is said to be the t-norm on which $\mathcal{P}$ is based, and we call $\mathcal{P}$ a *t-norm-based fuzzy* *logic.* Fuzzy logics frequently encountered in practical applications are based on the *fundamental t-norms* $\{T_s \mid 0 \le s \le \infty\}$, where

$$T_s(x,y) = \begin{cases} \log_s \left[1 + \dfrac{(s^x - 1)\cdot(s^y - 1)}{s - 1} \right] & \text{if } 0 < s < \infty,\ s \neq 1, \\ \min(x,y) & \text{if } s = 0, \\ x\cdot y & \text{if } s = 1, \\ \max(0, x + y - 1) & \text{if } s = \infty. \end{cases} \tag{1.2}$$

Definition 1.1 A fuzzy logic that is based on T_0 is called *min-max logic*, a logic based on T_∞ is called a *Lukasiewicz logic.*

2 Compactness in Triangular Norm-Based Fuzzy Logics

In this section we are concerned with the compactness problem in fuzzy logics that are based on continuous t-norms. Throughout this section $\mathcal{P}$ represents a fuzzy logic whose language is $\mathcal{L} = (\mathbf{P}, \{\neg, \vee\})$, where $\vee$ is a binary connective called *disjunction.* We assume that the t-norm T on which $\mathcal{P}$ is based is continuous and, therefore, its corresponding t-conorm S is continuous too.

The fuzzy logic $\mathcal{P}$ is called *compact* if "satisfiability" of any collection of formulas $\Gamma \subseteq \mathcal{F}$ can be decided by proving satisfiability of all finite subsets of Γ. The following definition makes this statement precise.

Definition 2.1 Let $\Gamma \subseteq \mathcal{F}$ be a set of formulas and let $K \subseteq [0,1]$. We say that Γ is *K-satisfialble* if there exists a truth assignment $\mathbf{t}$ such that $\overline{\mathbf{t}}(\varphi) \in K$ for every $\varphi \in \Gamma$. We say that Γ is *finitely K-satisfiable* if every finite subset of Γ is K-satisfiable.

The question is now whether, or under which conditions, K-satisfiability and finite K-satisfiability are equivalent. To answer that, for every formula φ define $H_\varphi : [0,1]^{\mathbf{P}} \to [0,1]$ by

$$H_\varphi(\mathbf{t}) = \overline{\mathbf{t}}(\varphi). \tag{2.3}$$

We use the following lemma.

Lemma 2.2 *If $\varphi \in \mathcal{F}$, then the function H_φ is continuous.*

Proof: By induction on the length of φ. If $\varphi \in \mathbf{P}$, then H_φ is a projection from $[0,1]^{\mathbf{P}}$ on one of its coordinates and thus it is continuous. If $\varphi = \neg\psi$ where H_ψ is continuous then, for every $\mathbf{t} \in [0,1]^{\mathbf{P}}$, $H_\varphi(\mathbf{t}) = 1 - H_\psi(\mathbf{t})$, and H_φ is continuous. If $\varphi = \psi \vee \chi$ where H_ψ and H_χ are continuous, then $H_\varphi(\mathbf{t}) = S(H_\psi(\mathbf{t}), H_\chi(\mathbf{t}))$ and, therefore, H_φ is continuous too. ∎

The following result shows that fuzzy logics based on continuous t-norms have a compactness property similar to that in bivalued logics.

Theorem 2.3 (Compactness Property) *If $\Gamma \subseteq \mathcal{F}$ and if K is a closed subset of $[0,1]$, then Γ is K-satisfiable if and only if it is finitely K-satisfiable.*

Proof: The only thing to prove is that Γ is K-satisfiable whenever it is finitely K-satisfiable. If Γ is finite the result is obvious. Assume that Γ is not finite. For every $\varphi \in \Gamma$, let

$$C_\varphi = \{\mathbf{t} \in [0,1]^{\mathbf{P}} \mid \overline{\mathbf{t}}(\varphi) \in K\}$$

This is a closed subset of $[0,1]^{\mathbf{P}}$ since $C_\varphi = H_\varphi^{-1}[K]$ and H_φ is continuous (cf. Lemma 2.2). By hypothesis, for every finite subset $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ of Γ, there is a truth assignment $\mathbf{t}$ such that, for each $i = 1, 2, \ldots, n$, we have $\overline{\mathbf{t}}(\varphi_i) \in K$, i.e., $\bigcap_{i=1}^{n} C_{\varphi_i} \neq \emptyset$. Therefore, $\{C_\varphi\}_{\varphi \in \Gamma}$ is a collection of closed sets having the finite intersection property. Tychonoff's Theorem (see Munkres [8], page 229) ensures that $[0,1]^{\mathbf{P}}$ is a compact topological space. Hence from the finite intersection property we deduce that $\bigcap_{\varphi\in\Gamma} C_\varphi \neq \emptyset$. Let $\mathbf{t}$ be a truth assignment in $\bigcap_{\varphi\in\Gamma} C_\varphi$. By definition of C_φ, $\overline{\mathbf{t}}(\varphi) \in K$ for every $\varphi \in \Gamma$, and therefore Γ is K-satisfiable. ∎

The compactness property of bivalued propositional logic (see Enderton [4], page 59) is a special case of Theorem 2.3 with $K = \{1\}$ (the choice of S is irrelevant in this case since every t-conorm coincides with the truth table of $\vee$ in bivalued logic). The case when K is a singleton is interesting in a more general context.

Definition 2.4 Let $r \in [0,1]$ and let Γ be a set of $\mathcal{P}$-formulas. We call Γ *r-satisfiable* if it is K-satisfiable for $K = \{r\}$. We call Γ *finitely r-satisfiable* if each finite subset of Γ is r-satisfiable. The maximal $r \in [0,1]$ for which Γ is r-satisfiable (if it exists) is called *the consistency degree* of Γ.

Corollary 2.5 *Let $r \in [0,1]$ and $\Gamma \subseteq \mathcal{F}$. Then Γ is r-satisfiable if and only if it is finitely r-satisfiable.*

Corollary 2.5 ensures that a special "level" of satisfiability of a collection Γ of $\mathcal{P}$-formulas is guaranteed whenever this level of satisfiability is guaranteed for every finite subset of Γ. The next result says that a set of formulas has a well-defined consistency degree whenever it is r-satisfiable for some $r \in [0,1]$.

Proposition 2.6 *If $\Gamma \subseteq \mathcal{F}$ is r-satisfiable for some $r \in [0,1]$, then there exists a number $r^* = r(\Gamma)$ such that*

$$r^* = \max\{r \mid \Gamma \text{ is } r\text{-satisfiable } \}$$

Proof: Let $C = \{r \in [0,1] \mid \Gamma \text{ is } r\text{-satisfiable }\}$. Then, by hypothesis, C is not empty. We show that C is closed and, therefore, it has a maximal element r^*. By Lemma 2.2 the function H_φ is continuous and, therefore, it is a closed subset of the product topological space $[0,1]^{\mathbf{P}} \times [0,1]$. Let

$$K = \bigcap_{\varphi \in \Gamma} H_\varphi$$

Then K is a closed subset of the topological space $[0,1]^{\mathbf{P}} \times [0,1]$. By Tychonoff's Theorem, $[0,1]^{\mathbf{P}} \times [0,1]$ is compact and, therefore, K is compact (see Munkres [8], page 165). Now let $\pi : [0,1]^{\mathbf{P}} \times [0,1] \to [0,1]$ be the natural projection map $\pi(\mathbf{t}, r) = r$. Clearly, π is a continuous function and, therefore, the image $\pi[K]$, is a compact subset of $[0,1]$ (see Munkres [8], page 167). If $r \in C$ then, for some $\mathbf{t} \in [0,1]^{\mathbf{P}}$, $\mathbf{t}(\varphi) = r$ for every $\varphi \in \Gamma$, i.e., $H_\varphi(\mathbf{t}) = r$ for every $\varphi \in \Gamma$. Therefore, $(\mathbf{t}, r) \in K$ and thus $r \in \pi[K]$. Hence $C \subseteq \pi[K]$. Suppose now that $r \in \pi[K]$. Then, for some $\mathbf{t} \in [0,1]^{\mathbf{P}}$, $(\mathbf{t}, r) \in K$ and consequently $r = H_\varphi(\mathbf{t})$ for every $\varphi \in \Gamma$. By the definition of H_φ, it follows that $r = \mathbf{t}(\varphi)$ for every $\varphi \in \Gamma$, i.e., $r \in C$. Hence $C = \pi[K]$. ∎

3 Validation sets of formulas in fuzzy logic

In this section we study the truth values that a given $\mathcal{P}$-formula φ may have, i.e., we would like to determine the nature of the validation set

$$V_\varphi = \{\overline{\mathbf{t}}(\varphi) \mid \mathbf{t} \in [0,1]^{\mathbf{P}}\}.$$

Proposition 3.1 *If $\varphi \in \mathcal{F}$, then V_φ is a closed subinterval of* $[0,1]$.

Proof: Observe that the space $[0,1]^{\mathbf{P}}$ with the product topology is connected and compact (see Munkres [8], pages 146-187). Therefore, the image of $[0,1]^{\mathbf{P}}$ via H_φ is a closed subinterval of $[0,1]$. Since

$$V_\varphi = \{\overline{\mathbf{t}}(\varphi) \mid \mathbf{t} \in [0,1]^{\mathbf{P}}\} = \{H_\varphi(\mathbf{t}) \mid \mathbf{t} \in [0,1]^{\mathbf{P}}\} = H_\varphi\left([0,1]^{\mathbf{P}}\right),$$

the proof of the proposition is complete. ∎

We can be more specific about the nature of V_φ. To this end we recall the following definition (see Monk [7]). Let us abbreviate the formula $\neg\alpha \vee \beta$ by $\alpha \to \beta$.

Definition 3.2 (see Monk [7]) A set $\Gamma \subseteq \mathcal{F}$ is said to be *closed under Modus Ponens*, if $\psi \in \Gamma$ whenever $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$. We say that Γ is *the closure of the set* $\Delta \subseteq \mathcal{F}$ *under Modus Ponens* if

(a) $\Delta \subseteq \Gamma$;

(b) Γ is closed under Modus Ponens;

(c) If Γ' is closed under Modus Ponens, and $\Delta \subseteq \Gamma'$, then $\Gamma \subseteq \Gamma'$.

Obviously, the closure of any set $\Delta \subseteq \mathcal{F}$ under Modus Ponens is exactly the set

$$\mathrm{MP}(\Delta) = \bigcap \left\{ \Gamma' \;\middle|\; \Delta \subseteq \Gamma' \text{ and } \Gamma' \text{ is closed under Modus Ponens} \right\}.$$

By analogy to what is usually done in bivalued logic, we use the concept of closure under Modus Ponens in conjunction with that of a logical axiom in order to classify $\mathcal{P}$-formulas. Recall the following definition:

Definition 3.3 A formula $\varphi \in \mathcal{F}$ is called a *logical axiom* if, for some $\alpha, \beta, \gamma \in \mathcal{F}$, it has one of the following forms:

(A1) $\alpha \to (\beta \to \alpha)$

(A2) $[\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$

(A3) $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$

The set of all the logical axioms is denoted by $\mathcal{A}$. We denote $\mathcal{T} := \mathrm{MP}(\mathcal{A})$ and the elements of $\mathcal{T}$ are called *tautologies*. If $\neg\varphi \in \mathcal{T}$, then φ is called a *contradiction*. If φ is neither a tautology nor a contradiction, then φ is called a *contingency*. A basic fact of bivalued propositional logic is (see Monk [7], pages 115-140): φ is a tautology if and only if $\overline{\mathbf{t}}(\varphi) = 1$ for every truth assignment $\mathbf{t} : \mathbf{P} \to \{0, 1\}$. Consequently, in bivalued propositional logic, φ is a contradiction if and only if $\overline{\mathbf{t}}(\varphi) = 0$, for every truth assignment $\mathbf{t} \in [0, 1]^{\mathbf{P}}$, and φ is a contingency if and only if $\overline{\mathbf{t}}(\varphi) = 1$ and $\overline{\mathbf{s}}(\varphi) = 0$ for some bivalued truth assignments $\mathbf{s}$ and $\mathbf{t}$.

The following result describes V_φ for formulas φ of a large class of fuzzy logics as, for instance, those based on fundamental t-norms.

Theorem 3.4 *Let $\mathcal{P}$ be a fuzzy logic based on a t-norm T which has no zero divisors, i.e.,*

$$T(x, y) = 0 \Longrightarrow (x = 0 \text{ or } y = 0).$$

For every $\mathcal{P}$-formula φ we have:

(a) *φ is a tautology if and only if, for some $a > 0$, $V_\varphi = [a, 1]$.*

(b) *φ is a contradiction if and only if, for some $b < 1$, $V_\varphi = [0, b]$.*

(c) *φ is a contingency if and only if $V_\varphi = [0, 1]$.*

Proof:

(a) To show sufficiency, suppose that, for some $a > 0$, $V_\varphi = [a, 1]$, and assume that φ is not a tautology. By the completeness theorem of bivalued propositional logic (see Monk [7], page 123) there is a truth assignment $\mathbf{t}$, such that $\overline{\mathbf{t}}(\varphi) = 0$, i.e., $0 \in V_\varphi$, which contradicts our assumption.

To show necessity, suppose that φ is a tautology. Then $1 \in V_\varphi$ and, therefore, by Proposition 3.1, $V_\varphi = [a, 1]$ for some $a \in [0, 1]$. It remains to show that $a > 0$. Let $\mathbf{t} : \mathbf{P} \to [0, 1]$ be a truth assignment, and let $p_0 \in \mathbf{P}$. A truth assignment $\mathbf{s}$ is called a *p_0-variant of* $\mathbf{t}$ if $\mathbf{s}(p) = \mathbf{t}(p)$, for every $p \neq p_0$. We need the following:

Claim 1: *Let $\psi \in \mathcal{F}$, $\mathbf{t}$ a truth assignment, and $p_0 \in \mathbf{P}$. If $\overline{\mathbf{t}}(\psi) \in \{0,1\}$ and if $0 < \mathbf{t}(p_0) < 1$, then $\overline{\mathbf{s}}(\psi) = \overline{\mathbf{t}}(\psi)$ for every p_0-variant truth assignment $\mathbf{s}$ of $\mathbf{t}$.*

Proof of claim 1: It is obviously true for every atomic formula. Suppose that it is also true for α with $\overline{\mathbf{t}}(\neg\alpha) \in \{0,1\}$. Then clearly $\overline{\mathbf{t}}(\alpha) \in \{0,1\}$. By hypothesis $\overline{\mathbf{s}}(\alpha) = \overline{\mathbf{t}}(\alpha)$, for every p_0-variant truth assignment s of $\mathbf{t}$ and, consequently, $\overline{\mathbf{s}}(\neg\alpha) = \overline{\mathbf{t}}(\neg\alpha)$. Now suppose that $\psi = \alpha \vee \beta$ and that Claim 1 is true for α and β. We distinguish two cases:

Case 1: $\overline{\mathbf{t}}(\psi) = 0$. Then $\overline{\mathbf{t}}(\alpha) = \overline{\mathbf{t}}(\beta) = 0$ and, by hypothesis, $\overline{\mathbf{s}}(\alpha) = \overline{\mathbf{s}}(\beta) = 0$, and therefore, $\overline{\mathbf{s}}(\alpha \vee \beta) = 0$ for every p_0-variant $\mathbf{s}$ of $\mathbf{t}$.

Case 2: $\overline{\mathbf{t}}(\psi) = 1$. Since T has no zero divisors, it follows that $\overline{\mathbf{t}}(\alpha) = 1$ or $\overline{\mathbf{t}}(\beta) = 1$. Without loss of generality, assume that $\overline{\mathbf{t}}(\alpha) = 1$. By hypothesis $\overline{\mathbf{s}}(\alpha) = 1$, for every p_0-variant $\mathbf{s}$ of $\mathbf{t}$ and, therefore, $\overline{\mathbf{s}}(\alpha \vee \beta) = 1$. This completes the proof of Claim 1.

Returning to the proof of the proposition, suppose that $a = 0$. Then there exists some truth assignment $\mathbf{t} : \mathbf{P} \to [0,1]$ such that $\overline{\mathbf{t}}(\varphi) = 0$. Let $Q = \{p_1, p_2, \ldots, p_n\}$ be the set of all atoms that appear in φ such that $0 < \mathbf{t}(p_i) < 1$. We may assume that for every $p \notin Q$, $\mathbf{t}(p) \in \{0,1\}$, since if p does not appear in φ then it does not affect $\overline{\mathbf{t}}(\varphi)$. Define $\mathbf{s}_1 : \mathbf{P} \to [0,1]$ by $\mathbf{s}_1(p_1) = 0$ and $\mathbf{s}_1(p) = \mathbf{t}(p)$, for every $p \neq p_1$. By Claim 1, $\overline{\mathbf{s}}_1(\varphi) = 0$. Similarly, define $\mathbf{s}_2 : \mathbf{P} \to [0,1]$ by $\mathbf{s}_2(p_2) = 0$ and $\mathbf{s}_2(p) = \mathbf{s}_1(p)$, for every $p \neq p_2$. Again, by Claim 1, we have $\overline{\mathbf{s}}_2(\varphi) = 0$. Repeating this procedure n times, a bivalued truth assignment $\mathbf{s}_n : \mathbf{P} \to \{0,1\}$ is obtained such that $\overline{\mathbf{s}}_n(\varphi) = 0$. This contradicts the assumption that φ is a tautology. Hence, $a > 0$.

(b) If φ is a contradiction, then $\neg\varphi$ is a tautology and, therefore, it follows from (a) that $V_{\neg\varphi} = [a, 1]$, for some $a > 0$. By consequence, $V_\varphi = [0, 1-a]$ where $1 - a < 1$.

(c) If φ is a contingency, then there are bivalued truth assignments $\mathbf{s}$ and $\mathbf{t}$ such that $\overline{\mathbf{s}}(\varphi) = 0$ and $\overline{\mathbf{t}}(\varphi) = 1$, i.e., both 0 and 1 are elements of V_φ and, therefore, by Theorem 3.4, $V_\varphi = [0, 1]$. ∎

The next thing we are interested in is the nature of the numbers a and b that appear in Theorem 3.4. Experiments with appropriate computer software (developed by the third author) show that the number a depends on the tautology φ and on the type of the logic $\mathcal{P}$, in general, in a way which is not yet clarified. However, in the case of the min-max fuzzy logics, this numbers are specified in the next section.

4 Satisfiability in min-max fuzzy logic

In this section we restrict our attention to the satisfiability problem in min-max fuzzy logics. Its importance stems from the fact that, among all fuzzy logics, min-max logics are the most used in practice. In the context of a min-max logic $\mathcal{P}_0$, Proposition 3.1 has a more precise form:

Theorem 4.1 *A formula φ in a min-max logic $\mathcal{P}_0$ is a tautology if and only if $\overline{\mathbf{t}}(\varphi) \geq 0.5$ for every $\mathbf{t} \in [0, 1]^{\mathbf{P}}$.*

Proof: We show that every tautology φ belongs to the set

$$\Gamma = \{\psi \in \mathcal{F} \mid \overline{\mathbf{t}}(\psi) \geq 0.5, \ \forall\, \mathbf{t} \in [0, 1]^{\mathbf{P}}\}.$$

First, we show that the set $\mathcal{A}$ of logical axioms of $\mathcal{P}_0$ is contained in Γ. Let φ be in $\mathcal{A}$.

(I) If φ is of form (A1), then

$$\begin{aligned} \overline{\mathbf{t}}(\varphi) &= \overline{\mathbf{t}}(\alpha \to (\beta \to \alpha)) = \overline{\mathbf{t}}(\neg\alpha \vee \neg\beta \vee \alpha) \\ &= \max\Big(1 - \overline{\mathbf{t}}(\alpha),\ 1 - \overline{\mathbf{t}}(\beta),\ \overline{\mathbf{t}}(\alpha)\Big) \geq 0.5. \end{aligned}$$

(II) If φ is of form (A2), then

$$\begin{aligned}
\overline{\mathbf{t}}(\varphi) &= \overline{\mathbf{t}}\Big([\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]\Big) \\
&= \overline{\mathbf{t}}\Big(\neg[\neg\alpha \vee (\neg\beta \vee \gamma)] \vee [\neg(\neg\alpha \vee \beta) \vee (\neg\alpha \vee \gamma)]\Big) \\
&= \overline{\mathbf{t}}\Big(\neg(\neg\alpha \vee \neg\beta \vee \gamma) \vee \neg(\neg\alpha \vee \beta) \vee \neg\alpha \vee \gamma\Big) \\
&= \max\Big(1 - \overline{\mathbf{t}}(\neg\alpha \vee \neg\beta \vee \gamma), 1 - \overline{\mathbf{t}}(\neg\alpha \vee \beta), 1 - \overline{\mathbf{t}}(\alpha), \overline{\mathbf{t}}(\gamma)\Big).
\end{aligned}$$

If $\overline{\mathbf{t}}(\alpha) \leq 0.5$ or $\overline{\mathbf{t}}(\gamma) \geq 0.5$, then clearly $\overline{\mathbf{t}}(\varphi) \geq 0.5$. If $\overline{\mathbf{t}}(\beta) \leq 0.5$ then $1 - \overline{\mathbf{t}}(\neg\alpha \vee \beta) \geq 0.5$. In the only remaining case, namely $\overline{\mathbf{t}}(\alpha) > 0.5$, $\overline{\mathbf{t}}(\beta) > 0.5$, and $\overline{\mathbf{t}}(\gamma) < 0.5$, we get $\overline{\mathbf{t}}(\neg\alpha \vee \neg\beta \vee \gamma) < 0.5$ and, therefore, $1 - \overline{\mathbf{t}}(\neg\alpha \vee \neg\beta \vee \gamma) > 0.5$ which implies $\overline{\mathbf{t}}(\varphi) \geq 0.5$.
(III) If φ is of form (A3) then

$$\begin{aligned}
\overline{\mathbf{t}}(\varphi) &= \overline{\mathbf{t}}\Big((\neg\alpha \to \neg\beta) \to (\beta \to \alpha)\Big) \\
&= \overline{\mathbf{t}}\Big(\neg(\alpha \vee \neg\beta) \vee (\neg\beta \vee \alpha)\Big) \\
&= \max\Big(1 - \overline{\mathbf{t}}(\alpha \vee \neg\beta),\ \overline{\mathbf{t}}(\alpha \vee \neg\beta)\Big) \geq 0.5
\end{aligned}$$

Now for showing that $\mathfrak{T}$ is a subset of Γ it is sufficient to prove that Γ is closed under Modus Ponens. In order to prove that, let $\mathbf{t} \in [0,1]^{\mathbf{P}}$ and let $\varphi, \psi \in \mathfrak{F}$ such that $\varphi, \varphi \to \psi \in \Gamma$. We distinguish the following three cases:

Case 1: $\overline{\mathbf{t}}(\varphi \to \psi) > 0.5$ and $\overline{\mathbf{t}}(\varphi) \geq 0.5$. Then from $\max\Big(1 - \overline{\mathbf{t}}(\varphi), \overline{\mathbf{t}}(\psi)\Big) > 0.5$ and $\overline{\mathbf{t}}(\varphi) \geq 0.5$ it follows that $\overline{\mathbf{t}}(\psi) > 0.5$.

Case 2: $\overline{\mathbf{t}}(\varphi \to \psi) \geq 0.5$ and $\overline{\mathbf{t}}(\varphi) > 0.5$. In this case $1 - \overline{\mathbf{t}}(\varphi) < 0.5$ and $\max\Big(1 - \overline{\mathbf{t}}(\varphi),\ \overline{\mathbf{t}}(\psi)\Big) \geq 0.5$ and, therefore, $\overline{\mathbf{t}}(\psi) \geq 0.5$.

Case 3: $\overline{\mathbf{t}}(\varphi \to \psi) = 0.5$ and $\overline{\mathbf{t}}(\varphi) = 0.5$. Assume, by contradiction, that $\overline{\mathbf{t}}(\psi) < 0.5$ in this case.

Claim 2: *If $\chi \in \mathcal{F}$, if $\mathbf{s} \in [0,1]^{\mathbf{P}}$, and if $\overline{\mathbf{s}}(\chi) = 0.5$, then there exists at least one atom $p \in \mathbf{P}$ such that p appears in χ and $\mathbf{s}(p) = 0.5$.*

For proving this claim, suppose that $\overline{\mathbf{s}}(\chi) = 0.5$ for some truth assignment $\mathbf{s} \in [0,1]^{\mathbf{P}}$. Let Q be the set of all $p \in \mathbf{P}$ that appear in χ and let $\mathcal{F}_Q$ be the set of all formulas that are obtainable from atomic formulas in Q only. Suppose that $\mathbf{t}(p) \neq 0.5$ for every $p \in Q$. Then, it follows by induction on $\mathcal{F}_Q$ that $\overline{\mathbf{t}}(\alpha) \neq 0.5$, for every $\alpha \in \mathcal{F}_Q$. This contradicts the fact that $\overline{\mathbf{s}}(\chi) = 0.5$ and $\chi \in \mathcal{F}_Q$. Hence, claim 2 is proved.

Let P_0 be the set of all $p \in \mathbf{P}$ that appear in φ and for which $\mathbf{t}(p) = 0.5$. By Claim 2 $P_0 \neq \emptyset$ and, obviously, P_0 is finite. For every $\delta > 0$, let

$$W(\mathbf{t}, \delta) = \left\{ \mathbf{s} \in [0,1]^{\mathbf{P}} \,\middle|\, \forall p \in P_0,\ 0.5 - \delta < \mathbf{s}(p) < 0.5 + \delta \right\}.$$

From the definition of the product topology it is clear that the family

$$\left\{ W(\mathbf{t}, \delta) \,\middle|\, \delta > 0 \right\}$$

is a basis for the neighborhood system of $\mathbf{t}$ in the product topological space $[0,1]^{\mathbf{P}}$ (see Munkres [8], pages 112 and 190). Suppose $r = \overline{\mathbf{t}}(\psi) < 0.5$ and let $\epsilon > 0$ be small enough such that $r + \epsilon < 0.5$. Let $U = H_{\psi}^{-1}\Big([0, r+\epsilon)\Big)$. By Lemma 2.2 H_ψ is continuous and, therefore, U is an open subset of $[0,1]^{\mathbf{P}}$. Clearly $\mathbf{t} \in U$. Hence, there exists $\delta > 0$ such that $W_\delta \subseteq U$.

Define a truth assignment $\mathbf{s}$ by $\mathbf{s}(p) = \mathbf{t}(p)$ if $p \in \mathbf{P} - P_0$, and $\mathbf{s}(p) = 0.5 + \frac{\delta}{2}$ if $p \in P_0$. Obviously, $\mathbf{s} \in W(\mathbf{t}, \delta)$ and $\mathbf{s}(p) \neq 0.5$, for every atom p that appears in φ. Claim 2 implies that $\overline{\mathbf{s}}(\varphi) \neq 0.5$ and thus $\overline{\mathbf{s}}(\varphi) > 0.5$. Since $\mathbf{s} \in W(\mathbf{t}, \delta) \subseteq U$, it results that $\overline{\mathbf{s}}(\psi) < r + \epsilon < 0.5$. Therefore, $\max(1 - \overline{\mathbf{s}}(\varphi),\ \overline{\mathbf{s}}(\psi)) < 0.5$ and this contradicts the assumption that $\varphi \to \psi \in \Gamma$. Hence, $\overline{\mathbf{t}}(\psi) \geq 0.5$ in this case too. This shows that the condition of the Theorem is necessary.

For proving sufficiency, suppose that φ is not a tautology. Then, there is a truth assignment $\mathbf{t} : \mathbf{P} \to \{0,1\}$ such that $\overline{\mathbf{t}}(\varphi) = 0$, i.e., $\overline{\mathbf{t}}(\varphi) < 0.5$. ∎

From Theorem 4.1 we deduce the following:

Corollary 4.2 *A formula* φ *in a min-max logic is a contradiction if and only if* $\overline{\mathbf{t}}(\varphi) \leq 0.5$ *for every* $\mathbf{t} \in [0,1]^{\mathbf{P}}$. *A formula* φ *is a contingency if and only if there are* $\mathbf{s}, \mathbf{t} \in [0,1]^{\mathbf{P}}$ *for which* $\overline{\mathbf{s}}(\varphi) < 0.5$ *and* $\overline{\mathbf{t}}(\varphi) > 0.5$.

The next corollary specifies the nature of the set V_φ in min-max fuzzy logics.

Corollary 4.3 *Let* φ *be a formula in a min-max logic.*
(a) φ *is a tautology if and only if* $V_\varphi = [0.5, 1]$;
(b) φ *is a contradiction if and only if* $V_\varphi = [0, 0.5]$;
(c) φ *is a contingency if and only if* $V_\varphi = [0, 1]$.

5 Comparing various fuzzy logics

In this section we deal with a class of fuzzy logics in which an infinitary disjunction is interpreted via fundamental t-conorms. We aim at showing that min-max fuzzy logics are "weaker" than Łukasiewicz fuzzy logics and, implicitly, "weaker" than infinitary fuzzy logics that are based on fundamental t-norms. The meaning of "weaker" in this context is that any reasoning which can be done in the "weaker" logic can be translated into a semantically equivalent reasoning in the "stronger" logic. From a philosophical point of view, this implies that Łukasiewicz logics as well as the infinitary fundamental t-norm-based logics provide a richer environment for approximate reasoning than min-max fuzzy logics in the same context. To make precise this idea we introduce the following notion:

Definition 5.1 Let $\mathcal{P}_1 = (\mathcal{P}_1, \mathcal{S}_1)$ and $\mathcal{P}_2 = (\mathcal{P}_2, \mathcal{S}_2)$ be two fuzzy logics such that $\mathcal{P}_1$ and $\mathcal{P}_2$ have the same set of atomic symbols $\mathbf{P}$. We say that $\mathcal{P}_1$ is *weaker* than $\mathcal{P}_2$ if there exists a function $f : \mathcal{F}(\mathcal{P}_1) \to \mathcal{F}(\mathcal{P}_2)$ such that, for every formula $\varphi \in \mathcal{F}(\mathcal{P}_1)$ and for every $\mathbf{t} \in [0,1]^{\mathbf{P}}$,

$$\overline{\mathbf{t}}_{\mathcal{P}_1}(\varphi) = \overline{\mathbf{t}}_{\mathcal{P}_2}(f(\varphi))$$

where $\overline{\mathbf{t}}_{\mathcal{P}_i}$ denotes the natural extension of $\mathbf{t}$ to $\mathcal{F}(\mathcal{P}_i)$, $(i = 1, 2)$.

A min-max fuzzy logic is weaker than any Łukasiewicz fuzzy logic having the same set of atoms. This results from:

Proposition 5.2 *Let $\mathcal{P}_0$ be a min-max fuzzy logic and let $\mathcal{P}_\infty = (\mathcal{P}_\infty, \mathcal{S}_\infty)$ be a Lukasiewicz fuzzy logic. If the logics $\mathcal{P}_0$ and $\mathcal{P}_\infty$ have the same set of atoms, then $\mathcal{P}_0$ is weaker than $\mathcal{P}_\infty$.*

Proof: Let $\mathcal{P}_i = (\mathbf{P}, \mathbf{C}_i)$ be the language of $\mathcal{P}_i$, $i \in \{0, \infty\}$, and suppose that $\mathbf{C}_0 = \{\neg, \vee\}$ and $\mathbf{C}_\infty = \{\backsim, \triangledown\}$. Define the function $f : \mathcal{F}(\mathcal{P}_0) \to \mathcal{F}(\mathcal{P}_\infty)$ by the following recursive procedure:

(i) if $\varphi \in \mathbf{P}$, then $f(\varphi) = \varphi$;

(ii) if $\varphi \in \mathcal{F}(\mathcal{P}_0)$ and if $f(\varphi)$ is defined, then $f(\neg\varphi) = \backsim f(\varphi)$;

(iii) if $\varphi, \psi \in \mathcal{F}(\mathcal{P}_0)$ and $f(\varphi)$, $f(\psi)$ are already defined then

$$f(\varphi \vee \psi) = \backsim(\backsim f(\varphi) \triangledown f(\psi)) \triangledown f(\psi) \tag{5.4}$$

For every truth assignment $\mathbf{t} \in [0,1]^{\mathbf{P}}$ denote by $\overline{\mathbf{t}}_i$, $i \in \{0, \infty\}$, the natural extension of $\mathbf{t}$ to $\mathcal{F}(\mathcal{P}_i)$. We claim that, if $\varphi \in \mathcal{F}(\mathcal{P}_0)$, then

$$\overline{\mathbf{t}}_0(\varphi) = \overline{\mathbf{t}}_\infty(f(\varphi)) \tag{5.5}$$

If $\varphi \in \mathbf{P}$, then (5.5) clearly holds by Definition 5.1. If $\varphi = \neg\psi$ where $\overline{\mathbf{t}}_0(\psi) = \overline{\mathbf{t}}_\infty(f(\psi))$, then

$$\begin{aligned}\overline{\mathbf{t}}_\infty(f(\varphi)) &= \overline{\mathbf{t}}_\infty(f(\neg\psi)) = \overline{\mathbf{t}}_\infty(\backsim f(\psi)) = 1 - \overline{\mathbf{t}}_\infty(f(\psi)) \\ &= 1 - \overline{\mathbf{t}}_0(\psi) = \overline{\mathbf{t}}_0(\neg\psi) = \overline{\mathbf{t}}_0(\varphi)\end{aligned}$$

i.e., (5.5) holds in this case. If $\varphi = \alpha \vee \beta$ such that (5.5) holds for α and β, then

$$\begin{aligned}
\overline{\mathfrak{t}}_\infty(f(\alpha \vee \beta)) &= \overline{\mathfrak{t}}_\infty\Big(\backsim(\backsim f(\alpha) \triangledown f(\beta)) \triangledown f(\beta)\Big) \\
&= \min\Big(1, \overline{\mathfrak{t}}_\infty(\backsim(\backsim f(\alpha) \triangledown f(\beta))) + \overline{\mathfrak{t}}_\infty(f(\beta))\Big) \\
&= \min\Big(1, 1 - \overline{\mathfrak{t}}_\infty(\backsim f(\alpha) \triangledown f(\beta)) + \overline{\mathfrak{t}}_\infty(f(\beta))\Big) \\
&= \min\Big(1, 1 - \min(1, 1 - \overline{\mathfrak{t}}_0(\alpha) + \overline{\mathfrak{t}}_0(\beta)) + \overline{\mathfrak{t}}_0(\beta)\Big) \\
&= \max\Big(\overline{\mathfrak{t}}_0(\alpha), \overline{\mathfrak{t}}_0(\beta)\Big) \\
&= \overline{\mathfrak{t}}_0(\alpha \vee \beta) = \overline{\mathfrak{t}}_0(\varphi) \quad \blacksquare
\end{aligned}$$

We are going to show that any Łukasiewicz (and implicitly, any min-max) fuzzy logic is weaker than the infinitary fuzzy logic based on fundamental t-norms and having the same atoms. To this end, we introduce the following notion.

Definition 5.3 Let $s \in [0, \infty]$. A fuzzy logic $\mathcal{P}^* = (\mathcal{L}, \mathcal{S})$ is called an *s-fuzzy logic* if it is based on the fundamental t-norm T_s and its set of connectives consists of the symbols $\eqsim$ (negation), $\bigsqcup_{n=1}^{\infty}$ (infinitary disjunction), and $\Rightarrow$ (binary implication) whose interpretation in $\mathcal{S}$ is given by

$$F_{\Rightarrow}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

The following result shows how min-max and Łukasiewicz fuzzy logics compare to s-fuzzy logics with $s \in (0, \infty)$:

Theorem 5.4 *Let $s \in (0, \infty)$. Suppose that $\mathcal{P}_0$ is a min-max fuzzy logic and $\mathcal{P}_\infty$ is a Łukasiewicz fuzzy logic having the same set of atoms $\mathbf{P}$ as $\mathcal{P}_0$. If*

$\mathcal{P}_s^* = (\mathcal{L}_s, \mathcal{S}_s)$ *is an s-fuzzy logic having the set of atoms* $\mathbf{P}$*, then* $\mathcal{P}_0$ *and* $\mathcal{P}_\infty$ *are weaker than* $\mathcal{P}_s^*$*.*

Proof: For any $\mathbf{t} \in [0,1]^{\mathbf{P}}$ we denote by $\overline{\mathbf{t}}_s$ the natural extension of $\mathbf{t}$ to $\mathcal{F}(\mathcal{P}_s^*)$. According to Proposition 5.2, there exists a function $f : \mathcal{F}(\mathcal{P}_0) \to \mathcal{F}(\mathcal{P}_\infty)$ such that $\overline{\mathbf{t}}_\infty(f(\varphi)) = \overline{\mathbf{t}}_0(\varphi)$, for every $\mathbf{t} \in [0,1]^{\mathbf{P}}$ and $\varphi \in \mathcal{F}(\mathcal{P}_0)$. It is sufficient to prove that $\mathcal{P}_\infty$ is weaker than $\mathcal{P}_s^*$ since, in this case, there exists a function $g : \mathcal{F}(\mathcal{P}_\infty) \to \mathcal{F}(\mathcal{P}_s^*)$ such that

$$\overline{\mathbf{t}}_s(g(\varphi)) = \overline{\mathbf{t}}_\infty(\varphi) \tag{5.6}$$

whenever $\mathbf{t} \in [0,1]^{\mathbf{P}}$ and $\varphi \in \mathcal{F}(\mathcal{P}_\infty)$. This implies that the function $h : \mathcal{F}(\mathcal{P}_0) \to \mathcal{F}(\mathcal{P}_s^*)$ defined by $h = g \circ f$ has the property $\overline{\mathbf{t}} \circ h = \overline{\mathbf{t}}_0$, for any $\mathbf{t} \in [0,1]^{\mathbf{P}}$, i.e., $\mathcal{P}_0$ is weaker than $\mathcal{P}_s^*$. It remains to show that there exists a function $g : \mathcal{F}(\mathcal{P}_\infty) \to \mathcal{F}(\mathcal{P}_s^*)$ satisfying (5.6) for all $\mathbf{t} \in [0,1]^{\mathbf{P}}$ and $\varphi \in \mathcal{F}(\mathcal{P}_\infty)$. To this end we introduce some notations. We presume that the set of connectives of $\mathcal{P}_\infty$ is $\{\backsim, \nabla\}$ and the set of connectives of $\mathcal{P}_s^*$ is $\{\neg, \bigsqcup_{n=1}^{\infty}, \Rightarrow\}$. For a sequence $\{\varphi_n\}_{n=1}^{\infty} \subseteq \mathcal{F}(\mathcal{P}_s^*)$, denote

$$\prod_{n=1}^{\infty} \varphi_n := \neg \left(\bigsqcup_{n=1}^{\infty} \neg \varphi_n \right).$$

For every finite sequence $\varphi_1, \varphi_2, \ldots, \varphi_n \in \mathcal{F}(\mathcal{P}_\infty)$, we let

$$\bigsqcup_{i=1}^{n} \varphi_i := \bigsqcup_{i=1}^{\infty} \psi_i,$$

where

$$\psi_i := \begin{cases} \varphi_i & \text{if } 1 \le i \le n, \\ \neg(\varphi_i \Rightarrow \varphi_i) & \text{if } i > n, \end{cases}$$

Also,

$$\prod_{i=1}^{n} \varphi_i := \neg \left(\bigsqcup_{i=1}^{n} \neg \varphi_i \right).$$

We sometimes write

$$\varphi_1 \sqcup \varphi_2 := \bigsqcup_{i=1}^{2} \varphi_i \quad \text{and} \quad \varphi_1 \sqcap \varphi_2 := \bigsqcap_{i=1}^{2} \varphi_i .$$

With these notations in mind, define the function $g : \mathcal{F}(\mathcal{P}_\infty) \to \mathcal{F}(\mathcal{P}_s^*)$ by the following recursive procedure:

(i) if $p \in \mathbf{P}$, then $g(p) = p$;

(ii) if $\varphi = \backsim\psi$ where $\psi \in \mathcal{F}(\mathcal{P}_\infty)$ and $g(\psi)$ is already defined, then $g(\varphi) = \neg g(\psi)$;

(iii) if $\varphi = \alpha \triangledown \beta$ where $\alpha, \beta \in \mathcal{F}(\mathcal{P}_\infty)$ and $g(\alpha)$ and $g(\beta)$ are already defined, then put

$$\begin{aligned} \alpha_1 &:= g(\alpha), & \beta_1 &:= g(\beta), \\ \alpha_{n+1} &:= \alpha_n \sqcup \beta_n, & \beta_{n+1} &:= \alpha_n \sqcap \beta_n \end{aligned} \tag{5.7}$$

and define

$$g(\varphi) = \bigsqcup_{n=1}^{\infty} \gamma_n$$

where

$$\gamma_n = \begin{cases} \alpha_1 & \text{if } n = 1, \\ \beta_{n-1} & \text{if } n > 1. \end{cases} \tag{5.8}$$

According to (i)-(iii) above g is well-defined. Now we aim at showing that it satisfies (5.6) for all $\mathbf{t} \in [0,1]^{\mathbf{P}}$ and $\varphi \in \mathcal{F}(\mathcal{P}_\infty)$. To this end we use the following:

Claim 3: *For every $s \in [0,\infty]$ and $x \in [0,1]$ let*

$$c(s,x) = \begin{cases} x & \textit{if } s \geq 1, \\ \frac{s^x - 1}{s-1} & \textit{if } s < 1. \end{cases}$$

Then

$$T_s(a, T_s(a,b)) \leq c(s,a) \cdot T_s(a,b), \tag{5.9}$$

for all $a, b \in [0, 1]$.

According to Butnariu and Klement [2] (Proposition 1.12), if $s \geq 1$, then

$$T_s(a, T_s(a, b)) \leq T_1(a, T_s(a, b)) = a \cdot T_s(a, b)$$

for all $a, b \in [0, 1]$, and (5.9) holds in this case. Suppose that $s < 1$. Denote $c = c(s, a)$ for some fixed $a, b \in [0, 1]$. Then (5.9) amounts to

$$\ln\left[1 + c^2(s^b - 1)\right] \geq c \cdot \ln\left[1 + c(s^b - 1)\right].$$

This is equivalent to

$$\sum_{n=1}^{\infty}(-1)^{n-1} \cdot \frac{c^{2n}(s^b - 1)^n}{n} \geq \sum_{n=1}^{\infty}(-1)^{n-1} \cdot \frac{c^{n+1}(s^b - 1)^n}{n} \tag{5.10}$$

which is clearly satisfied because of $c \in [0, 1]$, proving Claim 3. Returning to the proof of the Proposition, denote $G = \overline{\mathbf{t}}_s \circ g$ and observe that it is sufficient to show that

$$G(\varphi \triangledown \psi) = S_\infty(G(\varphi), G(\psi)) \tag{5.11}$$

for all $\varphi, \psi \in \mathcal{F}(\mathcal{P}_\infty)$ satisfying $G(\varphi) = \overline{\mathbf{t}}_\infty(\varphi)$ and $G(\psi) = \overline{\mathbf{t}}_\infty(\psi)$. In order to prove (5.11) observe that

$$G(\varphi \triangledown \psi) = \overline{\mathbf{t}}_s(g(\varphi \triangledown \psi)) = \overline{\mathbf{t}}_s\left(\bigsqcup_{n=1}^{\infty} \gamma_n\right) = \mathop{\mathbf{S}_s}_{n=1}^{\infty} \overline{\mathbf{t}}_s(\gamma_n), \tag{5.12}$$

where $\{\gamma_n\}_{n=1}^{\infty}$ is the sequence of formulas in $\mathcal{F}(\mathcal{P}_s^*)$ associated to φ and ψ by the rules (5.7) and (5.8). Denote $a = G(\varphi)$, $b = G(\psi)$, and

$$a_n = \overline{\mathbf{t}}_s(\alpha_n), \quad b_n = \overline{\mathbf{t}}_s(\beta_n)$$

From Frank [5] it follows that

$$T_s(x, y) + S_s(x, y) = x + y,$$

for every $x, y \in [0, 1]$. This implies

$$a_n + b_n = a + b, \tag{5.13}$$

by induction on n, because $\overline{\mathbf{t}}_s$ is the natural extension of $\mathbf{t}$ to $\mathcal{F}(\mathcal{P}_s^*)$. Let

$$q := S_\infty(a,b)$$

We distinguish two cases:

Case 1: $q < 1$. Then, by the definition of S_∞ we have $q = a + b$ and from (5.13) we get that $a_n \leq q < 1$. Applying the lemma inductively we obtain, for $n \geq 2$

$$b_n = \overline{\mathbf{t}}_s(\beta_n) = T_s(\overline{\mathbf{t}}_s(\alpha_{n-1}), \overline{\mathbf{t}}_s(\beta_{n-1})) \leq \begin{cases} q[c(s,q)]^{n-2} & \text{if } s < 1, \\ q^n & \text{if } s > 1. \end{cases} \tag{5.14}$$

This shows that

$$\lim_{n\to\infty} b_n = 0$$

since $q < 1$ and $c(s,q) < 1$ for $s < 1$. Consequently,

$$q = a + b = \lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n$$

because of (5.13). It can be shown by induction that $\mathop{\mathbf{S}_s}\limits_{i=1}^{n} \overline{\mathbf{t}}_s(\gamma_n) = a_n$ for each $n \in \mathbb{N}$. Therefore,

$$G(\varphi \bigtriangledown \psi) = \mathop{\mathbf{S}_s}\limits_{n=1}^{\infty} \overline{\mathbf{t}}_s(\gamma_n) = \lim_{n\to\infty} a_n = q = S_\infty(a,b), \tag{5.15}$$

and the proposition is proven in this case.

Case 2: $q = 1$. We show that (5.15) holds in this case too, i.e., $\lim\limits_{n\to\infty} a_n = q$. Suppose, by contradiction, that $\lim\limits_{n\to\infty} a_n \neq q$. Then, for some $d \in [0,1]$ we have

$$\lim_{n\to\infty} a_n \leq d < q = 1$$

because of (5.13) and the fact that the sequence $\{a_n\}_{n=1}^{\infty}$ is nondecreasing. Hence, for all $n \in \mathbb{N}$,

$$a_n \leq \lim_{n\to\infty} a_n \leq d < 1.$$

This implies (by induction) that (5.14) still holds with q replaced by d. Thus $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} a_n = q$, a contradiction. ∎

Theorem 5.4 allows to conclude that s-fuzzy logics provide a richer environment for fuzzy reasoning than the commonly used min-max and Łukasiewicz fuzzy logics. Heuristically speaking, this is not very surprising since, by contrast to min-max and Łukasiewicz fuzzy logics, the s-fuzzy logics use infinitary connectives. In fact even if we expand the min-max and/or Łukasiewicz fuzzy logics by making the disjunction infinitary, they are still weaker than s-fuzzy logics with $s \in (0, \infty)$. Precisely, we have the following:

Theorem 5.5 *Let* $\{\mathcal{P}_s^* \mid s \in [0, \infty]\}$ *be a family of* s*-fuzzy logics having the set of atoms* $\mathbf{P}$*. Then* $\mathcal{P}_0^*$ *and* $\mathcal{P}_\infty^*$ *are weaker than* $\mathcal{P}_s^*$ *for all* $s \in (0, \infty)$*.*

Proof: The fact that $\mathcal{P}_0^*$ is weaker than $\mathcal{P}_\infty^*$ is trivial. We prove that $\mathcal{P}_\infty^*$ is weaker than $\mathcal{P}_s^*$ for $s \in (0, \infty)$. To this end, $\{\backsim, \Rightarrow, \overset{\infty}{\underset{n=1}{\nabla}}\}$ denotes the set of connectives of $\mathcal{P}_\infty^*$ and $\{\neg, \overset{s}{\Rrightarrow}, \bigsqcup_{n=1}^{\infty}\}$ is the connectives set of $\mathcal{P}_s^*$. We define the function $g : \mathcal{F}(\mathcal{P}_\infty^*) \to \mathcal{F}(\mathcal{P}_s^*)$ by the following recursive procedure

(i) if $\delta \in \mathbf{P}$, then $g(\delta) = \delta$;

(ii) if $\delta = \backsim\psi$ and $g(\psi)$ is already defined, then $g(\delta) = \neg g(\psi)$;

(iii) if $\delta = (\alpha \Rightarrow \beta)$ and if $g(\alpha)$ and $g(\beta)$ are already defined, then $g(\delta) = [g(\alpha) \overset{s}{\Rrightarrow} g(\beta)]$;

(iv) if $\delta = \overset{\infty}{\underset{n=1}{\nabla}} \varphi_n$ where $g(\varphi_n)$ is already defined for each $n \in \mathbb{N}$, then

$$g(\delta) = \bigsqcup_{n=1}^{\infty} \psi_n,$$

where $\psi_1 = g(\varphi_1)$ and, for $n \geq 2$,

$$\psi_n = \bigsqcup_{k=1}^{\infty} \gamma_{k+1}$$

with γ_{k+1} given by (5.8) for $\varphi = \overset{k-1}{\underset{i=1}{\nabla}} \varphi_i$ and $\psi = \varphi_{k+1}$.

Let $G = \overline{\mathbf{t}}_s \circ g$. The only thing to prove is that

$$G\left(\mathop{\nabla}_{n=1}^{\infty} \varphi_n\right) = \mathop{\mathbf{S}_\infty}_{n=1}^{\infty} \overline{\mathbf{t}}_\infty(\varphi_n)$$

where $\varphi_n \in \mathcal{F}(\mathcal{P}_\infty^*)$, and $\overline{\mathbf{t}}_\infty(\varphi_n) = G(\varphi_n)$ for all $n \in \mathbb{N}$. Observe that

$$\overline{\mathbf{t}}_\infty\left(\mathop{\nabla}_{n=1}^{\infty} \varphi_n\right) = \mathop{\mathbf{S}_\infty}_{n=1}^{\infty} \overline{\mathbf{t}}_\infty(\varphi_n) = \lim_{n\to\infty} \mathop{\mathbf{S}_\infty}_{i=1}^{n} \overline{\mathbf{t}}_\infty(\varphi_i) = \lim_{n\to\infty} \overline{\mathbf{t}}_\infty\left(\mathop{\nabla}_{i=1}^{n} \varphi_i\right).$$

It can be shown by induction (using Theorem 5.4) that, for each $n \in \mathbb{N}$,

$$\overline{\mathbf{t}}_\infty\left(\mathop{\nabla}_{i=1}^{n} \varphi_i\right) = \overline{\mathbf{t}}_s\left(\bigsqcup_{i=1}^{n} \psi_i\right).$$

Thus

$$\begin{aligned}
\overline{\mathbf{t}}_\infty\left(\mathop{\nabla}_{n=1}^{\infty} \varphi_n\right) &= \lim_{n\to\infty} \overline{\mathbf{t}}_s\left(\bigsqcup_{i=1}^{n} \psi_i\right) = \lim_{n\to\infty} \mathop{\mathbf{S}_s}_{i=1}^{n} \overline{\mathbf{t}}_s(\psi_i) \\
&= \mathop{\mathbf{S}_\infty}_{n=1}^{\infty} \overline{\mathbf{t}}_s(\psi_n) = \overline{\mathbf{t}}_s\left(\bigsqcup_{n=1}^{\infty} \psi_n\right) \\
&= G\left(\mathop{\nabla}_{n=1}^{\infty} \varphi_n\right). \quad \blacksquare
\end{aligned}$$

Remark 5.6

(i) Proposition 5.2 states that $\mathcal{P}_0$ is weaker than $\mathcal{P}_\infty$. However, $\mathcal{P}_\infty$ is not weaker than $\mathcal{P}_0$. Suppose, by contradiction, that there is a function $h : \mathcal{F}(\mathcal{P}_\infty) \to \mathcal{F}(\mathcal{P}_0)$ such that for every $\varphi \in \mathcal{F}(\mathcal{P}_\infty)$ and for every $\mathbf{t} \in [0,1]^{\mathbf{P}}$, $\overline{\mathbf{t}}_\infty(\varphi) = \overline{\mathbf{t}}_0(h(\varphi))$. Put $\varphi = p_0 \vee \neg p_0$. Then, for every $\mathbf{t} \in [0,1]^{\mathbf{P}}$, $\overline{\mathbf{t}}_\infty(\varphi) = 1$. Let $\mathbf{s} \in [0,1]^{\mathbf{P}}$ be such that $\mathbf{s}(p) = 0.5$ for every $p \in \mathbf{P}$. Then, $\overline{\mathbf{s}}_0(h(\varphi)) = 0.5$, i.e., $\overline{\mathbf{s}}_\infty(\varphi) = 1 \neq 0.5 = \overline{\mathbf{s}}_0(\varphi)$.

(ii) Theorem 5.4 rises the question whether $\mathcal{P}_s$ and $\mathcal{P}_r$ comparable by the weakness relation. As far as we know this problem is open.

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