Primes, Quantum Chaos, and Computers

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1. Introduction

The primes, which are the fundamental building blocks of the multiplicative structure of the integers, have fascinated mathematicians at least since the time of the ancient Greeks. Euclid already knew that there are infinitely many primes, and his elegant proof is still commonly used today. Further progress in studying primes was made over the succeeding centuries. However, many very basic questions remain unanswered, such as that of accurate determination of the number of primes below a given bound. This paper describes some of the computational work (both actual calculations and algorithmic improvements) that has been done in this area. It is an interesting story that involves many interactions over several centuries between number theory and numerical analysis, and in recent years has also included possible new connections to physics.

Over the last century it has become clear that the primes are intimately connected to the Riemann zeta function, which is defined for complex s with $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} . \tag{1.1}$$

This function is named after Riemann as a result of a short paper [21] he published in 1859 on number theory, his only published work in this area. This pathbreaking paper was the foundation of modern prime number theory.

While the zeta function is named after Riemann, its history can be traced back two centuries before Riemann [1,27]. In 1650 Mengoli, an Italian mathematician, published a book on summation of series. In it he proved that the harmonic series diverges; by a slight abuse of our modern notation, he showed that

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty .$$

He also considered several convergent infinite series. Some he was able to sum in closed form. However, he failed in his attempts to evaluate

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} , \tag{1.2}$$

and he declared that to find this sum ‘‘requires the help of a richer intellect [than mine]’’ [27].

Although Mengoli’s problem of evaluating $\zeta(2)$ was reasonably well known, and several mathematicians attempted to solve it, success was only achieved by Euler in the 1700’s [1,27]. In his first attempt, Euler invented a method that enabled him to obtain a very good numerical approximation to $\zeta(2)$. This method, which is now called the Euler-Maclaurin summation formula, relies on the fact that a sum of terms that vary smoothly as functions of the index of summation can be approximated well by the corresponding integral, so that, for example,

$$\sum_{n=M}^{N} \frac{1}{n^2} = \int_{M}^{N} \frac{dx}{x^2} + \text{error} . \tag{1.3}$$

This was a major advance over what had been done before. The Euler-Maclaurin formula has been used in innumerable other applications since then, and represents one of the first and most fruitful connections between number theory and numerical analysis.

Euler’s next step, in 1735, completely solved Mengoli’s problem, and showed that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} . \tag{1.4}$$

Euler’s proof was not rigorous by our standards, but it is remarkable that he was able to obtain it. In retrospect, given our knowledge that π is transcendental (this was only proved about a century ago), Euler’s evaluation (1.4) of $\zeta(2)$ explains why Mengoli failed with his attempts; since Mengoli’s techniques were based on telescoping series and related elementary methods, he could not obtain an answer such as (1.4).

So far we have discussed connections of the zeta function with summation of series and numerical analysis, and have not dealt at all with primes, which are the main topic of this paper. In 1737 Euler showed that $\zeta(s)$, which is defined by (1.1) for $\mathrm{Re}(s) > 1$, can also be written as

$$\zeta(s) \;=\; \prod_p \, (1 - p^{-s})^{-1} \qquad \mathrm{Re}(s) > 1 \,, \tag{1.5}$$

where p runs over the primes. This expansion, now called the Euler product of $\zeta(s)$, follows from the fact that every positive integer has a unique expansion as a product of prime powers, and so when one expands the product in (1.5) and rearranges the terms, the sum (1.1) results.

By considering the behavior of the product (1.5) for $s = 1 + \delta$, $\delta > 0$, $\delta \to 0$, Euler was able to show that

(i) there exist infinitely many primes,

(ii) $\sum p^{-1} = \infty$.

Result (i) was just another (and rather complicated) proof of Euclid's theorem. On the other hand, (ii) was already new, and showed the power of analysis in a number theoretic setting.

The next big step in the study of primes and their connection to the zeta function was Riemann's paper in 1859 [21]. He considered the zeta function for a complex s (unlike Euler, who only used real s). The definition (1.1) shows that $\zeta(s)$ is an analytic function of s for $\mathrm{Re}(s) > 1$, and the Euler product (1.5) shows that $\zeta(s)$ does not vanish in that region. Riemann showed that $\zeta(s)$ can be continued analytically to the entire complex plane with the exception of $s = 1$, where $\zeta(s)$ has a first order pole. He also showed that the behavior of the zeta function at s is related to its behavior at $1 - s$ by the following functional equation:

$$\xi(s) \;=\; \xi(1-s) \,, \tag{1.6}$$

where

$$\xi(s) \;=\; \pi^{s/2}\,\Gamma(s/2)\,\zeta(s)\;, \tag{1.7}$$

and $\Gamma(z)$ is the Euler gamma function.

Riemann observed that the zeros of the zeta function determine the distribution of primes. Euclid proved that there are infinitely many primes, and Euler that the sum of their reciprocals diverges. However, mathematicians were interested in much more precise information. Around 1800, Gauss and Legendre independently conjectured that if $\pi(x)$ is the number of primes $p \leq x$, then

$$\pi(x) \sim \frac{x}{\log x} \qquad \text{as} \qquad x \to \infty \tag{1.8}$$

(i.e., $\pi(x)(\log x)/x \to 1$ as $x \to \infty$). This conjecture was made on empirical grounds, after examining computations of primes in short intervals at large heights. The conjecture (1.8) was proved in 1896 (independently by Hadamard [8] and de la Vallée Poussin [26]), and is now known as the Prime Number Theorem (PNT). The proofs required the development of analytic methods sketched in Riemann's memoir of about 40 years earlier, and the development of the infrastructure of complex analysis. These proofs relied strongly on properties of the zeros of the zeta function.

2. Zeros of the zeta function

Where are the zeros of the zeta function? If we consider the complex s-plane, we have a first order pole of $\zeta(s)$ at $s = 1$, and the half-plane $\mathrm{Re}(s) > 1$ is free of zeros by the Euler product (1.5). The functional equation (1.6) then shows that in the half-plane $\mathrm{Re}(s) < 0$, $\zeta(s)$ has zeros only at the negative even integers, $s = -2, -4, \ldots$ (of multiplicity one in each case). These zeros are called the *trivial zeros*. All the remaining zeros, of which there are infinitely many, are inside the *critical strip*, $0 \leq \mathrm{Re}(s) \leq 1$. These zeros are called the *nontrivial zeros* of $\zeta(s)$, and determine the behavior of $\zeta(s)$ and therefore of the primes.

The famous *Riemann Hypothesis* states that all of the nontrivial zeros of $\zeta(s)$ lie on the *critical line* $\mathrm{Re}(s) = 1/2$, right in the middle of the critical strip. In spite of many attempts, nobody has been able to prove the Riemann Hypothesis. We do know that there are no zeros on the boundaries of the critical strip, $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, and we have bounds on how close (as a function of their height) a zero can come to these boundaries. Unfortunately these bounds do not even provide us with any fixed $\delta > 0$ such that all the zeros would have to lie in $\mathrm{Re}(s) \le 1 - \delta$.

The main reason for the intense interest in the location of the zeros is that they determine the remainder term in the Prime Number Theorem. We wrote it in the form (1.8). However, it turns out that one gets a better approximation to $\pi(x)$ if one replaces $x/(\log x)$ in (1.8) by

$$\mathrm{Li}(x) = \int_2^x \frac{dv}{\log v} . \tag{2.1}$$

It is easy to show that

$$\mathrm{Li}(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty , \tag{2.2}$$

and so to a first order approximation replacing $x/(\log x)$ by $\mathrm{Li}(x)$ does not change the quantitative meaning of (1.8). However, it was proved within a few years of the Hadamard and de la Vallée Poussin breakthrough that

$$|\pi(x) - \mathrm{Li}(x)| \le c_1 x \exp(-c_2 (\log x)^{1/2}) \tag{2.3}$$

for some constants $c_1, c_2 > 0$, which is considerably stronger than (1.8). Even stronger results have been proved since then. However, in all the rigorous results, the bound for $|\pi(x) - \mathrm{Li}(x)|$ is larger than $x^{1-\delta}$ for every $\delta > 0$ for x large. Thus the remainder term in the PNT is not all that much smaller than the main term.

If the Riemann Hypothesis is true, then

$$|\pi(x) - \mathrm{Li}(x)| \leq c_3 x^{1/2} \log x \tag{2.4}$$

for some $c_3 > 0$. Thus in this case the remainder term in the PNT would be only about the square root of the main term. Conversely, if (2.4) is valid, than the Riemann Hypothesis is also true. Thus the location of the nontrivial zeros is intimately related to the distribution of primes.

Since $\zeta(s)$ is real for $s \in \mathbb{R}$, we find that $\zeta(\bar{\rho}) = 0$ whenever $\zeta(\rho) = 0$. Therefore we can restrict to considering just those nontrivial zeros ρ with $\mathrm{Im}(\rho) \geq 0$. We will number these in order of increasing distance from the real axis;

$$0 \leq \mathrm{Im}(\rho_1) \leq \mathrm{Im}(\rho_2) \leq \cdots .$$

We will also write $\rho_n = 1/2 + i\gamma_n$. All the zeros that have ever been checked satisfy the Riemann Hypothesis, and so have $\gamma_n \in \mathbb{R}$. (They are also all of multiplicity one, but here again we have no proof that this holds in general.)

The first four nontrivial zeros are

$$\begin{array}{c} 1/2 + i \ 14.1347251417\ldots \\ 1/2 + i \ 21.0220396387\ldots \\ 1/2 + i \ 25.0108575801\ldots \\ 1/2 + i \ 30.4248761258\ldots . \end{array}$$

Nothing is known about the γ_n, but they are thought likely to be transcendental numbers, algebraically independent of any reasonable numbers that have ever been considered.

Given the importance of the Riemann Hypothesis for the distribution of primes, and the lack of success in attempts to prove it, it is not surprising that many computations have been carried out to check whether it is true or not. Table 1 presents a listing of the published verifications of the Riemann Hypothesis. On each line, next to the name of the investigator is the date of publication, and then an integer *n* that means that the first *n* zeros all lie on the critical line. Due to the special properties of the zeta function, these verifications are rigorous in the sense that (assuming all the calculations are performed correctly) the zeros are shown to lie right on the critical line, and not just within $\pm 10^{-10}$ of it, say.

Table 1 about here

There are several interesting features to these computations. The most noticeable is surely the huge increase in the number of zeros that were checked, which has gone up by a factor of 10^8, from 15 to 1.5×10^9. To a large extent this was due to advances in technology. Gram, Backlund, and Hutchinson used paper and pencil methods, with some help from trigonometric and logarithmic tables. Titchmarsh relied on large electromechanical tabulating machinery, and his work, foreshadowing the era of extensive reliance on research grants, was carried out ‘‘with the help of a Government Grant’’ [23]. (Most of the calculations were done under the supervision of Comrie by the operators of the machines.) Around the time of the Titchmarsh computations, Turing decided to investigate the Riemann Hypothesis. After failing to prove it theoretically, he decided to carry out some computations, and to do this obtained a grant for £40 from the Royal Society of London to build a special purpose computer to calculate the zeros [10]. (Unlike the purely theoretical Turing machines that are the basis of the theory computation, this one was going to be a very real machine built out of gears that Turing was making himself.) Unfortunately, World War II intervened, and the construction of the machine was never completed. During and after the war, electronic digital computers were developed, and around 1950 Turing programmed one of them to search for counterexamples to the Riemann Hypothesis [25]. Although this machine was one of the world’s most powerful at that time, it was extremely weak by modern standards, roughly equivalent to a Hewlett-Packard HP-15C pocket programmable calculator in speed, and far inferior to it in reliability and ease of use. However, technology soon improved, and further computations want far beyond Turing’s. The calculation of van de Lune, to Riele, and Winter [13] that verified the Riemann Hypothesis for the first 1.5×10^9 zeros was carried out on the Cyber 205, a modern supercomputer that is far faster (on the order of 10^6 times faster) than the machine used by Turing. It is about 10^{10} times faster than a human being that relies on paper and pencil. Moreover, van de Lune and his collaborators used their computers for about 1500 hours.

While the progress in technology was extremely rapid, and contributed greatly to the computation of the zeros, a very significant contribution was also made by advances in algorithms. This is a very common, and generally unappreciated, feature of many areas of computational science. For example, in numerical solutions of some systems of partial differential equations, improvements in algorithms have done at least as much to speeding up the computations as have hardware advances. In the case of the Riemann zeta function, Gram, Backlund, and Hutchinson all used a method based on the Euler-Maclaurin summation formula. While this method is very effective, and makes it possible to calculate $\zeta(s)$ to arbitrary accuracy anywhere, it is not very efficient. To compute $\zeta(1/2 + it)$ for large t, for example, requires on the order of t operations. Since the n-th zero is at height $\sim 2\pi n/(\log n)$ as $n \to \infty$ (with the 1.5×10^9-th zero at height approximately 5×10^8), the Euler-Maclaurin method would be impractical for computations near the 10^9-th zero, say.

Fortunately for all those interested in numerical evidence about the zeta function, a more efficient algorithm was discovered, and has been used in all the computations listed in Table 1 starting with that of Titchmarsh. This algorithm was ‘‘discovered’’ in Riemann’s unpublished notes by C. L. Siegel [22], and is based on what is now known as the Riemann-Siegel formula. The description of Table 1 says that it presents a listing of the published verifications of the Riemann Hypothesis. However, these are not the only computations of the zeros that have been carried out. Riemann had actually verified this conjecture numerically for some initial zeros (at least 3, and possibly as many as 20, see [9]), and had done so by means of a very efficient method that takes only on the order of $t^{1/2}$ steps to compute $\zeta(1/2+it)$, as opposed to t for the Euler-Maclaurin method. To obtain his formula, Riemann had to develop new analytic tools, especially the saddle-point method (which did not become widely known until almost 60 years later, when Debye published and developed it further). For computations near the 10^9-th zero, the Riemann-Siegel formula speeds up the calculations by factors of over 10^4.

Recently, an even faster method for computing large sets of zeros of the zeta function was invented by A. Schönhage and the author [17]. It enables one to compute all of the roughly $T^{1/2}$ zeros of $\zeta(1/2+it)$ for $T \le t \le T+T^{1/2}$ in time on the order of $T^{1/2}$, so that the expected number of steps per zero is essentially constant. (To be more precise, it takes on the order of $(\log T)^c$ steps per zero for a constant c.) This algorithm is not only of theoretical interest, but has actually been implemented [16], and used to compute the sets of zeros shown in Table 2. In this table, the entry for $N = 2 \times 10^{20}$ means that 101 million zeros starting with zero number $2 \times 10^{20} - 633{,}984$ have been computed. (All these zeros satisfy the Riemann Hypothesis.)

Table 2 about here

The new algorithm is based on the Riemann-Siegel formula, but involves several new ingredients. The main ones are

1. The Fast Fourier Transform.
2. Efficient and accurate band-limited function interpolation.
3. A method for the rapid evaluation of a rational function at many points simultaneously.

The use of all these ingredients illustrates the wide range of tools that are useful in studying prime numbers. The Fast Fourier Transform is a fairly recent and basic tool of the data analysis. Band-limited function interpolation is used commonly in signal processing. The rational function evaluation method of [17] is very similar to the one proposed by Greengard and Rokhlin [7] for astrophysical many-body simulations, and could be used in its place.

The new algorithm offers striking improvements in computing zeros at large heights. Near the 10^{20}-th zeros, the Euler-Maclaurin method requires about 10^{20} operations for a single zero. The Riemann-Siegel formula reduces this to about 10^{11} operations, and the new algorithm to about 10^6. (A further improvement down to about 10^5 is possible, but has not been implemented.) Hence one gains about 15 orders of magnitude through algorithmic ideas, as opposed to about 10 orders of magnitude improvement due to technology. However, it has to be acknowledged that technology is crucial to making the algorithmic improvements work. To use the new methods effectively, one needs large and fast machines.

3. Computers and the Riemann Hypothesis

So far we have been discussing methods for computing zeros of the zeta functions and actual computations. The obvious question is whether there is much point in such extensive efforts. No amount of computation can by itself prove the Riemann Hypothesis. If this conjecture is false, of course, then one could expect to find a counterexample. However, given that the first 1.5×10^9 zeros do lie on the critical line, is there any reason to doubt that all the other nontrivial zeros are also on that line?

Unfortunately (or perhaps fortunately, depending on how complicated one likes mathematics to be), number theory has many examples of conjectures that are plausible, are supported by seemingly overwhelming numerical evidence, and yet are false. One famous example of this is provided by another conjecture of Riemann's, namely that for $x \geq x_0$,

$$\pi(x) < \mathrm{Li}(x) . \tag{3.1}$$

This conjecture is known to be true for all $x \leq 10^{12}$, and there are good reasons for believing that it is true for all $x \leq 10^{30}$. However, Littlewood showed in 1914 that it is false. At this point we know there is a counterexample below 10^{400} [20], but we do not know where the smallest counterexample is located.

In view of situations such as the above one, where the smallest counterexample is quite large, it makes some sense to explore small segments very high up, which was done in the computations listed in Table 2. However, even those computations do not go up very high. Furthermore, if counterexamples to the Riemann Hypothesis are rare, then searching small intervals is not likely to find them.

The computations that have been carried out can be viewed as supporting the truth of the Riemann Hypothesis in ways that extend beyond just merely verifying its truth for a billion or two of its zeros. The behavior of the zeta function near the real axis is very regular, and this forces initial zeros to lie on the critical line. As one goes up much further, one can find very wild behavior, so that tiny changes in the zeta function would force some zeros off the critical line. The fact that this has not happened in any of the cases that have been examined is often taken as strong evidence that there must be a solid reason for it, and so the Riemann Hypothesis ought to be true [6,15,16]. Such thinking is of course very speculative, but many people find it persuasive.

The main justification for the extensive recent computations of zeros of the zeta function was the hope that they might shed some insight into the truth not only of the Riemann Hypothesis, but of some conjectures that are even stronger. Hilbert and Pólya conjectured that the zeros of $\zeta(s)$ correspond to eigenvalues of some positive linear operator. If this conjecture were true, the Riemann Hypothesis would follow. According to Pólya, this conjecture was based more on wishful thinking than on any evidence. The aim was to find some way to prove the Riemann Hypothesis, and Pólya reasoned by analogy. The way that mathematicians or physicists often force numbers to lie on a line is through the use of symmetry; e.g., the eigenvalues of a symmetric matrix are real. Pólya's hope was that a similar reason might lie behind the Riemann

Hypothesis. Neither Hilbert nor Pólya ever suggested a specific operator or even a specific space this operator might act on. Still, their conjecture is regarded as perhaps the most promising approach to proving the Riemann Hypothesis, and it was a stimulus to much of the recent computational work on zeros of the zeta function.

4. Quantum chaos

How can one obtain evidence for a conjecture as vague as that of Hilbert and Pólya? If we had a specific space and a specific operator, we could hope to compute some of its eigenvalues and check whether they agree with the values coming from the zeros of $\zeta(s)$. However, we have neither a space nor an operator to test. What we do instead is to assume that the operator associated to the zeta function behaves in some ways like a random operator. This obviously cannot be completely correct, since any operator associated to an object as special as the zeta function cannot be random. However, in the absence of other knowledge, we might hope that the local behavior of the zeros, say, might be like that of eigenvalues of random operators, just as the local behavior of primes is conjectured to be, and appears to be, fairly random.

Figures 1 and 2 about here

Mathematical physicists have developed an extensive theory of distribution of eigenvalues of random matrices [5,14,19], with the goal of modeling the energy level distributions in highly excited nuclei and other many-particle systems. There are several different spaces of random matrices, but the one that is most likely to be relevant to the study of the zeta function is the so-called GUE (Gaussian unitary ensemble). If we let

$$\delta_n = (\gamma_{n+1} - \gamma_n) \frac{\log(\gamma_n/(2\pi))}{2\pi}$$

be the normalized difference between two consecutive zeros, then δ_n has mean value 1, and can be compared to the difference between consecutive eigenvalues in the GUE. Figure 1 shows such a comparison using the first million zeros of $\zeta(s)$. (The scatterplot is the histogram of the δ_n, and the smooth curve is the probability density function for differences of consecutive eigenvalues in the GUE.) We can see that there is a reasonably good fit, but it is far from perfect. As we remarked before, the behavior of $\zeta(s)$ is very constrained at low heights. Therefore it is more instructive to consider Fig. 2, which shows a similar plot, but this one based on about a million zeros near zero number 10^{20}. This time the fit is almost perfect!

The excellent fit between zeta function data and the GUE does not prove the existence of the Hilbert-Pólya operator. However, it is remarkable that the chain of wild conjectures that were mentioned above leads to a prediction that appears to be remarkably correct. This certainly suggests that there is something to all the wild guesses.

The analogy with random matrices should not be carried too far, because the zeros of $\zeta(s)$ are after all very special. This can be seen in the data, where there are long-range dependencies between zeros. Figure 3 shows a graph of

$$2 \log \left| \sum_{i=N+1}^{N+40000} e^{i\gamma_n x} \right| ,$$

for N close to 10^{20}. If the γ_n were random, we would expect the sum to be small for x away from 0. Instead, we see spikes at values of $x = \log p^m$ for m a prime. From a number theorist's point of view this is quite well understood, and is a reflection of the connection between the zeta function and primes. From a physicist's point of view, such phenomena are interesting because

> ‘‘... exactly this behavior would be expected if the γ_n were eigenvalues not of a random matrix but of the Hamiltonian operator obtained by quantizing some still unknown dynamical system without time-reversal symmetry, whose phase-space trajectories are chaotic.’’ (Comment made by M. Berry [4], on a related observation about the zeros of the zeta

function.)

Various long range dependencies have been studied in recent years by mathematical physicists in their explorations of quantum chaos. This field is somewhat controversial. Some have characterized it as

> ‘‘... an emerging science that is leading to the discovery of unfamiliar regimes of behavior in microscopic systems’’ [2],

while other have called it

> ‘‘... a poorly characterized disease for which we have only identified some of the symptoms.’’ [12].

Whichever description is accurate, quantum chaos is concerned with the transition from quantum mechanical systems, which do not exhibit chaotic behavior (which is quite a different thing from the usual probabilistic nature of quantum mechanism) and classical Newtonian systems, which do. The study of quantum chaos has led to a variety of predictions about dependencies between energy levels that are very far from each other. Reasoning by analogy, one can apply some of these predictions to zeros of the zeta function, and can than show both numerically [4] and through mathematical proof [17] that these predictions are indeed correct. While this again proves nothing, it suggests again that a Hilbert-Pólya operator does exist.

Figure 3 about here

The analogies between zeta function zeros and physical systems are useful to physicists as well as number theorists. The main reason is that it is hard to compute energy levels in complicated systems, either experimentally or mathematically. By comparison, algorithms for computing zeros of the zeta function are quite efficient. Therefore if one assumes that the zeros of the zeta function do come from a Hilbert-Pólya operator, which furthermore behaves like a very general operator, then one can use the zeros to test predictions for physical systems. Thus we have a situation where mathematics and physics help each other to gain greater understanding.

5. Suggestions for further reading

This paper provides a very brief introduction to several different topics and their interactions. For further information about the history of the zeta function, the reader is referred to [1,6,27]. Computational aspects of the zeta function are discussed in [6,15,17,22]. The general theory of the zeta function is presented in [11,18,24]. For a discussion of the variety of conjectures about the zeta function, see [15,18]. Random matrix theories are treated in [5,14,19]. Quantum chaos is described in [3].

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Figure Captions

Fig. 1. Probability density of the normalized spacings δ_n. Solid line: GUE prediction. Scatterplot: empirical data based on initial 1,000,000 zeros.

Fig. 2. Probability density of the normalized spacings δ_n. Solid line: GUE prediction. Scatterplot: empirical data based on 1,041,600 zeros near zero number 2×10^{20}.

Fig. 3. Graph of $2\log|\Sigma\exp(i\gamma_n y)|$, where n runs over $10^{20}+1 \leq n \leq 10^{20}+40{,}000$.

Table 1. Numerical verifications of the Riemann Hypothesis for the first n zeros.

Investigator	n
Gram (1903)	10
Backlund (1914)	79
Hutchinson (1925)	138
Titchmarsh et al. (1936)	1,041
Turing (1953)	1,104
Lehmer (1956)	25,000
Meller (1958)	35,337
Lehman (1966)	250,000
Rosser et al. (1969)	3,500,000
Brent (1979)	81,000,001
van de Lune et al. (1986)	1,500,000,000

Table 2. Large computed sets of zeros of the Riemann zeta function.

N	number of zeros	index of first zero in set
10^{12}	1,592,196	$N - 6{,}032$
10^{14}	1,685,452	$N - 736$
10^{16}	16,480,973	$N - 5{,}946$
10^{18}	16,671,047	$N - 8{,}839$
10^{19}	16,749,725	$N - 13{,}607$
10^{20}	175,587,726	$N - 30{,}769{,}710$
2×10^{20}	101,305,325	$N - 633{,}984$

Primes, Quantum Chaos, and Computers

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ABSTRACT

There is a long history of close connections between number theory and numerical analysis. This paper illustrates some of the connections involving the Riemann zeta function. Numerical data, collected as a result of improvements in both hardware and in algorithms, is yielding new information about some of the basic conjectures of number theory. It is also used to explore some new connections that have been proposed between distribution of primes and new areas of physics.