# ORIENTABILITY AND COMPLETE INTERSECTIONS FOR RING SPECTRA (CASE FOR SUPPORT).

## 1. Track record

J. P. C. Greenlees has been employed by Sheffield University since 1990; he was appointed to a Personal Chair in 1995; also, he was awarded the 1995 Junior Berwick Prize of the London Mathematical Society, and was a Nuffield Science Research Fellow during 1996–97. His research is mainly inspired by equivariant algebraic topology and homotopy theory, but his work has produced numerous interactions with commutative algebra and representation theory. Greenlees has had several visiting appointments at the University of Chicago, and makes regular short term visits. He was the Principal Organizer of the four month programme "New contexts for stable homotopy" at the Isaac Newton Institute, Cambridge in Autumn 2002, at which ring spectra were one of the four main themes.

Recent EPSRC supported projects include two VFs: Lyubeznik (under GR/L83486, investigating the commutative algebra consequences of the structures arising in topology [15]), and Bruner (VF funded under GR/M27845, investigating the connective $K$-theory of classifying spaces [2, 14]; the sequel, investigated the real case VF GR/S01139/01 [3]). A very successful project on connections between local cohomology and algebraic topology (GR/M71350) finished in 2002 [6, 7, 8, 9]. This work has aroused considerable interest, and been the subject of many working seminars (for example at MSRI, in Bochum-Bonn-Dusseldorf-Wuppertal, Barcelona, Lausanne, and Nice) The project Higher Structures on elliptic cohomology (EP/C52084X/1) began in 2005. The PI was a named participant in the Chicago NSF grants in 1995-98 and 1998-2001.

## 2. Background

There are now many examples [6, 7, 8, 9, 1, 19] showing that by formulating properties of commutative rings in a homotopy invariant way, one can extend them to 'rings up to homotopy'. The rings up to homotopy might be differential graded rings, or highly structured ring spectra. Amongst the examples this covers is the ring $C^*(X; k)$ of cochains on a space $X$, and various chromatic localizations of the sphere. Applying methods from commutative algebra to these new examples is illuminating in itself, but such rings up to homotopy have homology which is a conventional graded ring, commutative or non-commutative, and many interesting rings arise this way. We concentrate here particulary on properties highlighted by Morita theory, for instance, the duality properties of the ring up to homotopy, which have implications for the homology ring.

The paper [7] provided a conceptual framework, which allows us to view the following dualities

- Poincaré duality for manifolds
- Gorenstein duality for commutative rings
- Benson-Carlson duality for cohomology rings of finite groups
- Poincaré duality for groups
- Gross-Hopkins duality in chromatic stable homotopy

as instances of a single phenomenon.

In terms of concrete benefits, this gives a local cohomology theorem for the homology of homotopically Gorenstein rings. In particular it gives a new proof of the local cohomology theorem for cohomology rings of finite $p$-groups [13], and an extension to cohomology rings of classifying spaces of $p$-compact groups. It also gives new insight into the local cohomology theorem for other equivariant cohomology theories as in [12, 16].

## 3. The context, and two examples.

In this section we set the scene by describing results from [7].

With notation suggested by commutative local rings, the context is set by taking a map $R \longrightarrow k$ of ring spectra, and we work in a derived category $D(R)$ of $R$-modules. By taking $k$ as an $R$-module, we obtain a derived Morita theory relating left $R$-modules to right modules over the endomorphism ring $\mathcal{E} = \mathrm{Hom}_R(k, k)$. Since $\mathcal{E}$ only sees the behaviour through the module $k$ we consider the subcategory $\mathrm{Cell}_k$ of $D(R)$ generated by using arbitrary coproducts and triangles.

We are interested in understanding homotopy invariant versions of various well-known conditions on the ring $R$. One of the themes is the way that regularity conditions on $R$ are reflected in growth conditions on $\mathcal{E}$.

To give some context to the discussion, we introduce two examples.

**Example 3.1.** *(Local algebra)* We take $R$ to be a classical commutative Noetherian local ring in degree 0, with maximal ideal $I$ and residue field $k$. The subcategory $\mathrm{Cell}_k$ consists of objects whose homology is $I$-power torsion ([6]).

The endomorphism ring spectrum $\mathcal{E}$ is modelled by the endomorphism ring of a projective $R$-resolution of the $R$-module $k$, or equivalently the cobar resolution. Its homology is the Ext algebra

$$H_*(\mathcal{E}) = \mathrm{Ext}_R^*(k, k). \quad \square$$

**Example 3.2.** *(Cochains on a space)* For a commutative ring $k$, consider the cochain complex $R = C^*(X; k)$ for a space $X$. If $k$ is a field of characeristic zero we may use the Thom-Sullivan PL de Rham complex as a commutative model, but the existence of non-trivial Steenrod operations is an unavoidable obstruction to a commutative cochain model. Instead, we identify $k$ with the corresponding Eilenberg-MacLane spectrum, viewed as a strictly commutative ring spectrum in the category of Elmendorf-Kriz-Mandell-May [11]. The cochain complex is then is the function spectrum of maps from $X$ to the $E_\infty$-ring $k$, and as such it is itself an $E_\infty$-$k$-algebra.

If the Eilenberg-Moore spectral sequence converges then

$$\mathcal{E} = C_*(\Omega X; k).$$

For instance this holds if $X$ is simply connected, or if $p^n = 0$ on $k$ for some $n$ and $X$ is connected with fundamental group a finite $p$-group [5]. $\square$

To begin work we need a mild finiteness condition, with something of the flavour of being Noetherian.

**Definition 3.3.** We say that $R$ is *proxy-regular* if There is an object $K$ with the following properties

- $K$ is small and
- $K$ may be constructed from $k$ in *finitely many* steps using triangles and retracts.
- $k$ may be constructed from $K$ using coproducts, triangles and retracts.

The local algebra example is proxy-regular for any Noetherian local ring, as one can see by taking $K$ to be the classical Koszul complex associated to a generating sequence for the maximal ideal. The ring $R = C^*(X)$ is very often proxy-regular, for example if $X$ is finite or $X = BG$.

From now on we assume $R \longrightarrow k$ is proxy-regular, and study conditions on $R$. We begin with the best possible behaviour.

**Definition 3.4.** We say that $R$ is *h-regular* if $k$ is small as an $R$-module.

This h-regularity assumption is very strong: for conventional rings it coincides with the conventional notion of regularity by Serre's theorem, and for $R = C^*(X; \mathbb{F}_p)$ it coincides with requiring $X$ to be the classifying space of a $p$-compact group.

Under fairly mild completeness hypotheses, $R$ is h-regular if and only if $H_*(\mathcal{E})$ is finite dimensional over $k$.

At the weaker end of the spectrum, we may say that $R$ is *h-Gorenstein* if $\mathrm{Hom}_R(k, R) \simeq \Sigma^a k$ for some $a$. It is well known that if $R$ is a classical Gorenstein commutative ring, it is h-Gorenstein with $a$ the Krull dimension of $R$. It is shown in [7] that $R = C^*(X; k)$ is h-Gorenstein if $X$ is an orientable manifold or the classifying space $BG$ for a compact Lie group.

Furthermore, under mild hypotheses, $R$ is h-Gorenstein if and only if $\mathcal{E}$ is h-Gorenstein.

## 4. Complete intersections

The first project for the new proposal is to study the much more subtle question of analogues of complete intersections. The first thought would be to take Avramov's proof of Quillen's conjecture that complete intersections are detected by finiteness of André-Quillen cohomology. One might then hope homotopy complete intersections were detected by finiteness of topological André-Quillen cohomology, but work of Dwyer and Mandell shows, for that for any simply connected $p$-complete space $X$, the ring $C^*(X; \mathbb{F}_p)$ satisfies the condition, whereas it is usually not even Gorenstein. Instead we take a different approach.

4.A. **A definition.** We say that an object $M$ in $D(R)$ is *small* if it is finitely built from $R$. We say that $M$ is *virtually small* if the thick subcategory it generates contains a non-trivial small object. Thus if $R$ is a commutative local ring with residue field $k$, then $k$ is always virtually small, but it is only small if $R$ is regular.

**Definition 4.1.** We say that $R$ is *quasi-ci* if every object containing the maximal ideal in its support is proxy-small.

The reason for the name is that if $R$ is a complete intersection then it is quasi-ci [9]. Conversely, if $R$ is quasi-ci then it is Gorenstein; there are examples of Gorenstein rings which are not quasi-ci, and no known examples of quasi-ci rings which are not ci.

**Problem 4.2.** Show that a quasi-ci commutative local ring is ci (or provide a counterexample).

Irrespective of the outcome of this, we can consider quasi-ci ring spectra, and it is natural to look in the known classes of Gorenstein ring spectra, namely $C^*(X; k)$ with $X$ an orientable manifold or $X = BG$ for a compact Lie group $G$.

4.B. **Growth rates.** We now turn to the Morita counterpart of the ci condition. In the case of commutative algebra, Gulliksen's theorem states that $R$ is ci if and only if the Ext algebra $\mathrm{Ext}_R^*(k, k)$ has polynomial growth. We therefore say $R$ is *g-ci* if $\mathrm{Hom}_R(k, k)$ has polynomial growth, so that ci and g-ci coincide for complete local rings.

When $R = C^*(BG; \mathbb{F}_p)$, R. Levi [18] has studied the growth rate of $\mathrm{Hom}_R(k, k) = C_*(\Omega(BG_p^\wedge))$. If $G$ is a $p$-group, $BG_p^\wedge = BG$ so that $C_*(\Omega(BG_p^\wedge)) = kG$, but the homology of $\Omega(BG_p^\wedge)$ is typically infinite dimensional. Levi has shown there is a dichotomy with the growth rate either being 'polynomial' or 'exponential'. The inverted commas are there because Levi's result allows the smaller growth rate to be a little more than polynomial, and the larger growth rate to be a little less than exponential; we will say $R$ is *gl-ci* if $\mathrm{Hom}_R(k, k)$ has 'polynomial' growth.

**Problem 4.3.** Show (under suitable hypotheses) that $R$ is quasi-ci if and only if it is gl-ci. Investigate the relationship between g-ci and gl-ci.

Gulliksen's proof is very indirect and uses many methods of commutative algebra that do not adapt to the more general context, so that entirely new methods are needed.

We are particularly interested in the case $R = C^*(BG)$, and ask what are the implications for the group cohomology ring $H^*(BG)$ if $R$ is ci (in any one of the above senses).

## 5. Orientability and Pic.

There are a number of different notions of orientability for Gorenstein ring spectra. We focus on the simplest. If $R$ is Gorenstein $\mathrm{Hom}_R(k, R) \simeq \Sigma^a k$, and thus $k$ acquires a *right* $\mathcal{E}$-action (not to be confused with the left action that it has by definition, since $\mathcal{E}$ is the endomorphism ring of $k$). We write $k^\gamma$ for $k$ with this Gorenstein action. In many examples, there is also a preferred right $\mathcal{E}$-action which might be considered 'trivial'. We write $k^\tau$ for $k$ wit this action. We say that $R$ is *orientably Gorenstein* if $R$ is Gorenstein and $k^\gamma \cong k^\tau$.

For example if $R = C^*(X; k)$, for a compact manifold $X$ and $k = \mathbb{Z}/2^n$, one may show $R$ is Gorenstein, and the action of $\mathcal{E} = C_*(\Omega X)$ factors through the (classical) orientation action $\pi_0(\Omega X) = \pi_1(X) \longrightarrow k^\times$. If $n = 1$, $R$ is always orientable. If $n \geq 2$, $R$ is orientable if and only if the manifold $X$ is orientable.

**Problem 5.1.** If $R$ is regular, show that it is orientable as a Gorenstein ring (or give a counterexample).

The question of orientability raises the general problem of studying right $\mathcal{E}$-actions on $k$. In fact there are three ways to consider this that have been fruitful. First, we may use the Morita adjunction between right $\mathcal{E}$-modules and torsion $R$-modules. Second we may use the Morita adjunction between right $\mathcal{E}$-modules and complete $R$-modules [6]. The second is more useful because the symmetric monoidal product (completed tensor over $R$) on complete $R$-modules is easier to understand, and it is well related to that on $\mathcal{E}$-modules (tensor product over $k$). The right $\mathcal{E}$-module $k$ is invertible, so it is natural to study the Picard group $Pic(R)$ of invertible complete $R$-modules. There is precedent from chromatic homotopy theory starting with the work of Hopkins, Mahowald and Sadofsky [17]. Most recently there is further progress from Goerss, Henn and Rezk.

**Problem 5.2.** Formalize the relationship between $Pic(R)$ and $Pic(\mathcal{E})$. Give methods of calculation, and implement it in examples. Apply this to classify right $\mathcal{E}$-actions on $k$.

To start with, we restrict attention to $\mathcal{E}$-actions $\alpha$ arising from an $R$-module $M(\alpha)$ (which we may suppose is complete) this covers the motivating examples. In the proxy-regular case, the correspondence between the actions $\alpha$ and the $R$-modules $M(\alpha)$ is bijective. The correspondence is such that

$$\mathrm{Hom}_{\mathcal{E}}(k^\gamma, k^\alpha) \simeq M(\alpha),$$

so the homotopy of $M(\alpha)$ can be calculated by an Adams spectral sequence:

$$\mathrm{Ext}^{*,*}_{\pi_*(\mathcal{E})}(k^\gamma, k^\alpha) \Rightarrow \pi_*(M(\alpha)).$$

Experience in the topological cases shows that the homotopy of $M(\alpha)$ is a powerful invariant, leaving little indeterminacy. The $E_2$-term of the Adams spectral sequence only involves the action of the *homotopy* $\pi_*(\mathcal{E})$ on $k$, and connectivity usually shows this action is unique. This shows the $E_2$-term is independent of $\alpha$, so the difference in $\pi_*(M(\alpha))$ is a result of the differentials in the Adams spectral sequence. Topological classifications proceed by identifying possible differentials. The project aims to study these phenomena systematically in more algebraic cases.

## 6. Completions and cellularizations.

Many of the questions described above are asked under hypotheses such as proxy-regularity or completeness. It is natural to investigate what happens when these are relaxed.

6.A. **Cellular approximation.** The question arises very generally, but we restrict attention to the rather algebraic case when $R = C^*(BG; \mathbb{F}_p)$, so that we have tools of group theory and representation theory. The corresponding endomorphism ring $\mathcal{E} = C_*(\Omega(BG_p^\wedge)$ is rather complicated, although the Eilenberg-Moore spectral sequence can be used to calculate it. Of course if $G$ is a $p$-group the ring is simply $\mathbb{F}_pG$, and its module theory is simply representation theory. We propose to study the general case in this spirit. One available tool is its Morita counterpart $C^*(BG)$, but we may also hope to build purely algebraic models as in the work of Dwyer [4] and the thesis of Shamir [20].

6.B. **DC-completion.** One of the facts often used silently in the above is that under proxy-regularity assumptions we may pass from $R$ to $\mathcal{E}$ and back to $R$ with very little effect. In general, we may consider the dc-completion

$$R \longrightarrow \mathrm{Hom}_{\mathcal{E}}(k, k).$$

If $k$ is a field, or more generally if $R \longrightarrow k$ is proxy-regular, one may show that the dc-completion coincides with the conventional derived completion. However, even in commutative algebra, if $k = R/I$ and $I$ is not the maximal ideal, the dc-completion can be quite mysterious. There are various methods familiar from the study of convergence of Adams spectral sequences, and it would be interesting to investigate algebraic examples by adapting these methods.

## References


[1] D.J.Benson and J.P.C.Greenlees "Duality in topology and modular representation theory." Submitted for publication.

[2] R.R.Bruner and J.P.C.Greenlees "The connective K-theory of finite groups." Mem. American Math. Soc. Vol. 165 (2003) number 785. 127pp

[3] R.R.Bruner and J.P.C.Greenlees "The real connective K-theory of finite groups." (In preparation) 59pp

[4] W.G.Dwyer "Homological localization of $\pi$-modules." J. Pure Appl. Algebra 10 (1977/78), no. 2, 135–151.

[5] W.G.Dwyer "Strong convergence of the Eilenberg-Moore spectral sequence" Topology **13** (1974) 255-265

[6] W.G.Dwyer and J.P.C.Greenlees "The equivalence of categories of complete and torsion modules." American J. Math. **124** (2002) 199-220

[7] W.G.Dwyer, J.P.C.Greenlees and S.B.Iyengar "Duality in algebra and topology." Advances in Maths. **200** (2006) 357-402

[8] W.G.Dwyer, J.P.C.Greenlees and S.B.Iyengar "Gross-Hopkins duality and the Gorenstein condition." (In preparation) 21pp

[9] W.G.Dwyer, J.P.C.Greenlees and S.B.Iyengar "Finiteness in derived categories of local rings." Comm. Math. Helv **81** (2006) 383-432

[10] W.G.Dwyer and C.Wilkerson "Homotopy fixed-point methods for Lie groups and finite loop spaces" Ann. Math. **139** (1994) 395-442

[11] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, Modules and Algebras in Stable Homotopy Theory*, Volume 47 of *Amer. Math. Soc. Surveys and Monographs*. American Mathematical Society, 1996.

[12] J.P.C.Greenlees "K-homology of universal spaces and local cohomology of the representation ring" Topology **32** (1993) 295-308.

[13] J.P.C.Greenlees "Commutative algebra for group cohomology." JPAA **98** (1995) 151-162

[14] J.P.C.Greenlees "Equivariant connective K theory for compact Lie groups" JPAA **187** 2004 129-152

[15] J.P.C.Greenlees and G.Lyubeznik "Rings with a local cohomology theorem with applications to cohomology rings of groups." JPAA **149** (2000) 267-285

[16] J.P.C.Greenlees and J.P.May "Localization and completion theorems for $MU$-module spectra." Annals of Maths. **146** (1997) 509-544
[17] M.J.Hopkins, M.E. Mahowald and H. Sadofsky "Constructions of elements in Picard groups." Topology and representation theory (Evanston, IL, 1992), 89–126, Contemp. Math., 158, Amer. Math. Soc., Providence, RI, 1994.
[18] R. Levi "On homological rate of growth and the homotopy type of $\Omega(BG_p^\wedge)$." Math. Z. **226** (1997) 429-444
[19] J.Rognes "Galois extensions of structured ring spectra." Mem AMS (to appear)
[20] S. Shamir "Cellularization over group-rings with applications to the Eilenberg-Moore spectral sequence." Thesis, Hebrew University of Jerusalem, 2006.

## 7. Summary

The project is on the boundary of algebraic topology, commutative algebra and representation theory. The time is ripe, both in terms of the early indications of fruitfulness, and in terms of available technology. Furthermore the PI perfectly placed to begin this development, having been already involved at the centre of many of the key developments, and having existing collaborative contacts with many relevant experts.

## 8. Diagrammatic project plan.

The calculational and theoretical aspects of the project will be developed in parallel. Patterns and structural features noticed during calculation feed in to the theoretical development, and calculation cannot proceed without theoretical support. Naturally, the outline here will be adapted in the light of experience as the project proceeds. Writing and publication is considered as an integral part of the process of research.

During the first 6 months the RA will become familiar with basic classical material in commutative algebra and structured ring spectra, moving on to more recent work cited above; this can be tested on the more routine tasks connecting different approaches. The RA will then move on to the more adventurous parts of the project as described above.

For the PhD student it is appropriate to details at a more tactical level. The student will spend the first 6 months on generic mathematical training, especially algebraic topology, commutative algebra and homotopy theory, and the second 6 months reading literature specifically related to this project. This will permit the student to begin calculations with zero dimensional Gorenstein rings, to decide which of them are qci in the above sense (extending the calculations in [8]). It is hoped that those that are qci will also be ci, but in any case the student will seek a pattern. Those which are not qci should be investigated for exotic dc-completions (extending results of Dwyer's). The subsequent work will depend on the patterns discovered. Further concrete calculations with accessible examples will illustrate the theory of orientations: small DGAs display some of the phenomena first discovered in more complicated settings. For example, the fact that DGAs with exterior homology over $\mathbf{F}_p$ on one generator of degree $-1$ are classified by complete DVRs with residue field $\mathbf{F}_p$ can be cast in terms of invertible modules, and there are several more substantial variants that would be fruitful to investigate.

## 9. Dissemination and beneficiaries.

Standard: publication in journals and on webpages. Presentation at conferences.

This research is likely to interest mathematicians working in algebraic topology, or commutative algebra, with group cohomology rings, in algebra and algebraic geometry.

## 10. Management and resources

The University of Sheffield would provide normal office, library and computing facilites. In addition to the usual use of email and word processing, the project will use computer experiment (Macaulay2 and Maple). The proposal therefore asks for a PC for the PDRA, a notebook for the PI and a printer. The University of Sheffield already has appropriate sitelicenses for the software. There is a sum (5% of computer officer's salary) included for computer support.

We have avoided designing the project for a particular RA because the EPSRC processes make it unwise. Nonetheless, several suitably qualified US-trained mathematicians have expressed interest in a postdoctoral position with the investigators in the area of the proposal.

The figure for travel is principally for both the RA and the investigators to go to international conferences in and algebraic topology, and for the RA to visit the US to consult experts. The PI will be able to claim some expenses from other sources, and so the figure is correspondingly reduced. A figure is included for substantial visits by L.Avramov (Lincoln, NA), S.Iyengar (Lincoln, NA), and W.G.Dwyer (Notre Dame) to discuss aspects of the project. We have also included an explicit sum for travel between Sheffield and Aberdeen, because D.Benson and R.Levi are both based there.