Indian J. pure appl. Math., 32(7) : 1027-1032, July 2001

ON MEROMORPHIC STARLIKE AND CONVEX FUNCTIONS

M. NUNOKAWA* AND O. P. AHUJA**

*Department of Mathematics, University of Gunma, Aramaki, Maebashi, 371 Japan
**Division of Mathematics, National Institute of Education, Nanyang Technological University, 469 Bukit Timah Road, Singapore 259 756

(*Received* 21 *February* 2000; *accepted* 11 *September* 2999)

In this paper, we show that the meromorphic convex univalent functions and meromorphic starlike univalent functions do not hold the same relationships as that of between the convex univalent functions and starlike univalent functions.

Key Words : Meromorphics; Starlike and Convex Functions; Conformal Mapping

1. INTRODUCTION

Let Σ denote the class of functions F of the form

$$F(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are analytic and univalent in the punctured disk $D = \{z : 0 < |z| < 1\}$. A function $F \in \Sigma$ is called meromorphic starlike of order $\alpha\,(\alpha < 1)$ if $F(z) \neq 0$ in D and

$$-Re\,\frac{zF'(z)}{F(z)} > \alpha,\ z \in E,$$

where $E = \{z : |z| < 1\}$. We denote by $MS^*(\alpha)$ the class of meromorphic starlike functions of order α. Similarly, a function $F \in \Sigma$ is called meromorphic convex of order $\alpha\,(\alpha < 1)$ if $F(z) \neq 0$ in D and

$$-\left(1 + Re\,\frac{zF''(z)}{F'(z)}\right) > \alpha,\ z \in E.$$

We denote by $MC(\alpha)$ the class of meromorphic convex functions of order α.

A function $F \in \Sigma$ is said to be γ-meromorphic convex of order β if $F(z) \neq 0$ in D

and $$-Re\left\{(1-\gamma)\frac{zF'(z)}{F(z)} + \gamma\left(1 + \frac{zF''(z)}{F'(z)}\right)\right\} > \beta,\ z \in E,$$

where $\gamma \geq 0$ and $\beta < 0$ are fixed arbitrary real numbers. Denote by $\sum_{\gamma}^{*}(\beta)$ the family of γ-meromorphic convex functions of order β. Note that $\gamma = 0$ gives precisely the meromorphic starlike functions of order β and $\gamma = 1$ yields the family of meromorphic convex functions of order β.

On the other hand, we let S be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in E. A function $f \in S$ is called starlike of order $\alpha\,(0 \leq \alpha < 1)$ if

$$Re\,\frac{zf'(z)}{f(z)} > \alpha,\ z \in E.$$

We denote by $S^*(\alpha)$ the class of starlike functions of order α. Furthermore, a function $f \in S$ is called convex of order $\alpha\,(0 \leq \alpha < 1)$ if

$$1 + Re\,\frac{zf''(z)}{f'(z)} > \alpha,\ z \in E.$$

We denote by $C(\alpha)$ the class of convex functions of order α.

It is well known that Marx[3] and Strochhäcker[6] showed that if $f \in C(0)$, then $f \in S^*\left(\frac{1}{2}\right)$; that is $C(0) \subset S^*\left(\frac{1}{2}\right)$. The function $f(z) = z / (1 - z)$ is convex function of order 0 and starlike function of order $\frac{1}{2}$. Therefore, Marx-Strochhäcker's result is sharp.

Jack[1] proved the following result :

Theorem A — *If $f \in C(\alpha)\ (0 \leq \alpha < 1)$, then $f \in S^*(\beta(\alpha))$ where*

$$\beta(\alpha) \geq \frac{2\alpha - 1 + \sqrt{9 - 4\alpha + 4\alpha^2}}{4}.$$

In fact, Jack[1] posed the more general problem: What is the largest real number $\beta = \beta(\alpha)$ so that $C(\alpha) \subset S^*(\beta(\alpha))$? Subsequently, MacGregor[2] and Wilken and Feng[7] proved the following.

Theorem B — *If $f \in C(\alpha)\ (0 \leq \alpha < 1)$, then $f \in S^*(\beta(\alpha))$ where*

$$\beta(\alpha) = \left\{ \begin{array}{ll} \dfrac{1 - 2\alpha}{2^{2-2\alpha}[1 - 2^{2\alpha - 1}]} & \text{if } \alpha \neq \dfrac{1}{2} \\ \dfrac{1}{2\log 2} & \text{if } \alpha = \dfrac{1}{2} \end{array} \right\}$$

and this result is sharp.

Analogus to the family $\sum_{\gamma}^{*}(\beta)$ is the well-known class, $M_\gamma(\beta)$, of the functions

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, *analytic in E which satisfy the conditions* $\left(\frac{f(z) f'(z)}{z}\right) \neq 0$ *and*

$$Re\left\{(1-\gamma)\frac{zf'(a)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta$$

for all $z \in E$. *It is well known that* $M_\gamma(0) \subset S^*(0)$ *for all real* γ *and* $M_\gamma(0) \subset C(0)$ *for* $\gamma \geq 1$ *In [5], Miller, Mocanu and Reade proved.*

Theorem C — $M_\gamma(0) \subset S^*(\delta(\gamma))$ *for* $\gamma \geq 1$ *where*

$$\delta(\gamma) = \frac{-\gamma + \sqrt{\gamma^2 + \delta\gamma}}{4}.$$

It is the purpose of the present paper to show that the meromorphic convex functions and meromorphic starlike functions do not hold the same relationships as the above between the convex functions and starlike functions.

2. Main Results

Lemma 1 — Let a function p be analytic in E, $p(0) = 1$ and suppose that

$$Re\left(p(z) - \frac{zp'(z)}{p(z)}\right) > \frac{\alpha(3-2\alpha)}{2(1-\alpha)}, z \in E$$

where $\alpha < 0$. Then we have $Re\ p(z) > \alpha$ in E.

PROOF : Let

$$p(z) = (1-\alpha)\frac{1 + w(z)}{1 - w(z)} + \alpha, \quad \text{... (2.1)}$$

where w is analytic in E and $w(0) = 0$. It suffices to show that $|w(z)| < 1$ for all $z \in E$.

If there exists a point $z_0 \in E$ such that $|w(z)| < 1$ for $|z| < |z_0|$ and $|w(z_0)| = 1$, then from Jack's Lemma [1, p. 470], we have $z_0 w'(z_0) = kw(z_0), k \geq 1$. Setting $w(z_0) = e^{i\theta}$, it follows from (2.1) that

$$\frac{z_0 p'}{p(z_0)} = \frac{\dfrac{2(1-\alpha) z_0 w'(z_0)}{(1 - w(z_0))^2}}{(1-\alpha)\dfrac{1 + w(z_0)}{1 - w(z_0)} + \alpha}$$

$$= -\frac{\dfrac{2(1-\alpha)k}{2(1-\cos\theta)}}{(1-\alpha)\dfrac{2i\sin\theta}{2(1-\cos\theta)} + \alpha}$$

$$= \frac{-(1-\alpha)k}{\alpha(1-\cos\theta) + i(1-\alpha)\sin\theta}$$

$$= \frac{-(1-\alpha)\,k\,(\alpha\,(1-\cos\theta) - i\,(1-\alpha)\sin\theta}{\alpha^2\,(1-\cos\theta)^2 + (1-\alpha)^2\sin^2\theta}.$$

Thus, we obtain

$$Re\,\frac{z_0\,p'(z_0)}{p(z_0)} = \frac{-\alpha\,(1-\alpha)\,(1-\cos\theta)\,k}{\alpha^2\,(1-\cos\theta)^2 + (1-\alpha)^2\sin^2\theta}. \qquad \ldots (2.2)$$

Letting $1 - \cos\,\theta = t, 0 \leq t \leq 2$ and writing

$g(t) = \dfrac{t}{\alpha^2 t^2 + (1-\alpha)^2\,(2t - t^2)}$, (2.2) may be written as

$$\text{Re}\,\frac{z_0\,p'(z_0)}{p(z_0)} = -\,\alpha\,(1-\alpha)\,k\,g(t).$$

But, a simple calculation shows that $g(t)$ takes its minimum value at $t = 0$, and

$$\lim_{t\to 0}\; g(t) = \frac{1}{2(1-\alpha)^2}.$$

Therefore, we have

$$\text{Re}\,\frac{z_0\,p'(z_0)}{p(z_0)} \geq \frac{-\,\alpha}{2(1-\alpha)}.$$

Hence

$$\text{Re}\left(p(z_0) - \frac{z_0\,p'(z_0)}{p(z_0)}\right) \leq \alpha + \frac{\alpha}{2(1-\alpha)} = \frac{\alpha\,(3-2\,\alpha)}{2\,(1-\alpha)}.$$

This contradicts the assumption and therefore we have $|\,w(z)\,| < 1$ in E. This shows that Re $p(z) > \alpha$ in E. ■

Applying Lemma 1, we have the following result :

***Theorem* 1** — *Let* $F \in MC\left(\dfrac{\alpha\,(3-2\,\alpha)}{2(1-\alpha)}\right)$, *then* $F \in MS^*\,(\alpha)$, *where* $\alpha < 0$.

PROOF : Let $p(z) = -\,zF'(z)/F(z), p(0) = 1$. Then we have

$$p(z) - \frac{zp'(z)}{p(z)} = -\left(1 + \frac{zF''(z)}{F'(z)}\right).$$

In view of Lemma 1, the result follows. ■

Putting $\beta = \alpha\,(3-2\,\alpha)/2(1-\alpha)$ in Theorem 1, we have

Corollary 1 — Let $F \in \Sigma$, $F(z) \neq 0$ in D, and suppose

$$-\left(1 + \operatorname{Re}\frac{zF''(z)}{F'(z)}\right) > \beta, z \in E$$

where $\beta < 0$. Then

$$-\operatorname{Re}\frac{zF'(z)}{F(z)} > \frac{1}{4}(2\beta + 3 - \sqrt{4\beta^2 - 4\beta + 9}), z \in E.$$

Letting $\beta \to 0$ in Corollary 1, we obtain

Corollary 2 — If $F \in MC(0)$ then $F \in MS^*(0)$.

Remark 1 : Setting $F(z) = (1-z)^2/z$, we find that $F(z) \neq 0$ in D,

$$-\frac{zF'(z)}{F(z)} = \frac{1+z}{1-z}$$

and

$$-\left(1 + \frac{zF''(z)}{F'(z)}\right) = \frac{1+z^2}{1-z^2}.$$

This shows that the result in Corollary 2 is sharp. Note that the external function is the reciprocal of the Koebe function

$$f(z) = \frac{2}{(1-z)^2}$$

Remark 2 : From Corollary 2, we can say that if F is a meromorphic convex function of order 0, then F is a meromorphic starlike function of order at least 0.

***Theorem* 2** — *Let* $F \in \sum_{\gamma}^{*}\left(\frac{2\beta - 2\beta^2 + \gamma\beta}{2(1-\beta)}\right)$ *then* $F \in MS^*(\beta)$ *where* $\beta < 0$ *and* $\gamma \geq 0$ *are fixed real numbers.*

PROOF : We define the function w in E by

$$-\frac{zF'(z)}{F(z)} = (1-\beta)\frac{1+w(z)}{1-w(z)} + \beta, w(z) \neq 1. \qquad \text{... (2.3)}$$

Since the value of $-zF'(z)/F(z)$ at $z = 0$ is 1, we notice that w is analytic in E and $w(0) = 0$. Taking the logarithmic differentiation of both sides of (2.3), we have

$$1 + \frac{zF''(z)}{F'(z)} = -(1-\beta)\frac{1+w(z)}{1-w(z)} - \beta + \frac{2(1-\beta)zw'(z)}{(1-w(z))^2\left[(1-\beta)\frac{1+w(z)}{1-w(z)} + \beta\right]},$$

Therefore, we have

$$(1-\gamma)\frac{zF'(z)}{F(z)} + \gamma\left(1 + \frac{zF''(z)}{F'(z)}\right) \qquad \text{... (2.4)}$$

$$= -(1-\beta)\frac{1+w(z)}{1-w(z)} - \beta + \frac{2\gamma(1-\beta)zw'(z)}{(1-w(z))^2\left[(1-\beta)\frac{1+w(z)}{1-w(z)} + \beta\right]}.$$

It suffices to show that $|w(z)|<1$ for all $z \in E$. Let, if possible, there exists a point $z_0 \in E$ such that $|w(z)|<1$ for $|z|<|z_0|$ and $|w(z_0)|=1$ $(w(z_0)=e^{i\theta})$. Then it follows from Jack's Lemma [1, p 470] that $z_0 w'(z_0)=kw(z_0), k \geq 1$. Therefore, (2.4) yields

$$(1-\gamma)\frac{z_0 F'(z_0)}{F(z_0)}+\gamma\left(1+\frac{z_0 F''(z_0)}{F'(z_0)}\right)$$

$$= -(1-\beta)\frac{i \sin\theta}{1-\cos\theta}-\beta-\frac{\gamma(1-\beta)k}{\beta(1-\cos\theta)+i(1-\beta)\sin\theta}.$$

Hence,

$$\text{'Re}\left[(1-\gamma)\frac{z_0 F'(z_0)}{F(z_0)}+\gamma\left(1+\frac{z_0 F''(z_0)}{F'(z_0)}\right)\right]$$

$$= -\beta-\frac{\beta\gamma(1-\beta)k(1-\cos\theta)}{\beta^2(1-\cos\theta)^2+(1-\beta)^2\sin^2\theta}.$$

Proceeding as in the proof of Lemam 1, we obtain

$$-\text{Re}\left[(1-\gamma)\frac{z_0 F'(z_0)}{F(z_0)}+\gamma\left(1+\frac{z_0 F''(z_0)}{F'(z_0)}\right)\right]$$

$$\leq \beta+\frac{\gamma\beta}{2(1-\beta)}=\frac{2\beta-2\beta^2+\gamma\beta}{2(1-\beta)}$$

which contradicts the hypothesis. Therefore, $|w(z)|<1$ for $z \in E$ and hence $F \in MS^*(\beta)$. ■

Putting $\beta \to 0$ in Theorem 2, we have

Corollary 3 — $\sum_{\gamma}^{*}(0) \subset MS^*(0)$.

REFERENCES

1. I. S. Jack, *J. London. Math. Soc.*, **3** (1971) 469-74.
2. T. H. MacGregor, *J. London Math. Soc.* **9** (1975) 530-36.
3. A. Marx, *Math. Anna.*, **107** (1932-33) 40-67.
4. S. S. Miller, P. T. Mocanu and M. O. Reade, *Rev. Roum. Math. pure appl.* **19** (1974) 213-24.
5. S. S. Miller, P. T. Mocanu and M. O. Reade, *Rev. Roum. Math. pure appl.*, **21** (2) (1976) 219-25.
6. E. Strohhacker, *Math. Zeit.*, **37** (1933) 356-80.
7. D. R. Wilken, J. Feng, *J. London Math. Soc.*, **21** (1980) 287-90.