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# ON A STRONGLY STARLIKENESS CRITERIA

BY

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**Abstract**. H. Silverman [Internat. J. Math. Math. Sci. 22(1999), 75-79] investigated and obtained some results for the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. In this paper, we obtain a sufficient condidtion of functions for strongly starlikeness of order $\beta$.

**1. Introduction.** Let $\mathcal{S}$ denote the class of functions $f(z)$ normalized by $f(0) = f'(0) - 1 = 0$ that are analytic and univalent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{S}$ is said to be starlike of order $\alpha$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$$

for some $\alpha(0 \leq \alpha < 1)$, and for all $z \in \mathbb{U}$. The class of starlike functions of order $\alpha$ is denoted by $\mathcal{S}^*(\alpha)$. Further, a function $f(z) \in \mathcal{S}$ is said to be convex of order $\alpha$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$$

for some $\alpha(0 \leq \alpha < 1)$, and for all $z \in \mathbb{U}$. Also we denote by $\mathcal{C}(\alpha)$ the

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subclass of $\mathcal{S}$ consisting of all convex functions of order $\alpha$ in $\mathbb{U}$.

On the other hand, a function $f(z)$ in $\mathcal{S}$ is said to be strongly starlike of order $\beta$ if it satisfies

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\beta$$

for some $\beta(0 < \beta \leq 1)$, and for all $z \in \mathbb{U}$. We say that $f(z) \in \mathcal{SS}^*(\beta)$ if $f(z)$ is strongly starlike of order $\beta$ in $\mathbb{U}$

Silverman [2] investigated the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. Let $\mathcal{G}_b$ be the subclass of $\mathcal{S}$ consisting of functions $f(z) \in \mathcal{S}$ which satisfy

$$\left|\left\{\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right\}-1\right| < b \quad (z \in \mathbb{U})$$

for some real $b$.

For this class $\mathcal{G}_b$, Silverman obtained the following rsult.

**Theorem A**([2]). If $0 < b \leq 1$, *then*

$$\mathcal{G}_b \subset \mathcal{S}^*(\frac{2}{1+\sqrt{1+8b}}).$$

*The result is sharp for all* $b$.

In this paper, we consider the strongly starlikeness for functions $f(z)$ belonging to $\mathcal{G}_b$.

**2. Strongly Starlikeness.** To discuss the strongly starlikeness of functions $f(z)$ in $\mathcal{G}_b$, we have to recall here the following result by Nunokawa [1].

**Lemma.** *Let* $p(z)$ *be analytic in* $\mathbb{U}$ *with* $p(0) = 1$ *and* $p(z) \neq 0 (z \in \mathbb{U})$.

*Suppose that there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg(p(z))| < \frac{\pi}{2}\beta \quad (|z| < |z_0|)$$

*and*

$$|\arg(p(z_0))| = \frac{\pi}{2}\beta,$$

*where* $\beta > 0$. *Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = iK\beta,$$

*where*

$$K \geq \frac{1}{2}\Big(a + \frac{1}{a}\Big) \quad \textit{when} \quad \arg(p(z_0)) = \frac{\pi}{2}\beta$$

*and*

$$K \leq -\frac{1}{2}\Big(a + \frac{1}{a}\Big) \quad \textit{when} \quad \arg(p(z_0)) = -\frac{\pi}{2}\beta,$$

*where* $p(z_0)^{\frac{1}{\beta}} = \pm ia$ *and* $a > 0$.

Our main result is contained in

**Theorem 1.** *If* $f(z)$ *belongs to the class* $\mathcal{G}_{b(\beta)}$ *with*

$$b(\beta) = \frac{\beta}{\sqrt{(1-\beta)^{1-\beta}(1+\beta)^{1+\beta}}} \quad (0 < \beta \leq 1),$$

*then* $f(z) \in \mathcal{SS}^*(\beta)$.

*Proof.* Let us define the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)}.$$

Then it follows that

$$\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}-1=\frac{zp'(z)}{p(z)^2}.$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\beta \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\beta,$$

then, applying Lemma, we have that

$$\begin{aligned}\left|\frac{z_0p'(z_0)}{p(z_0)^2}\right| &= \left|iK\beta\frac{1}{(\pm ia)^{\beta}}\right| = \beta|K|a^{-\beta}\\ &\geq \frac{\beta}{2}\left(a^{1-\beta}+\frac{1}{a^{1+\beta}}\right).\end{aligned}$$

Define the function $g(a)$ by

$$g(a) = a^{1-\beta}+\frac{1}{a^{1+\beta}} \quad (a > 0;\ 0 < \beta \leq 1).$$

Since

$$g'(a) = \frac{1}{a^{2+\beta}}((1-\beta)a^2-(1+\beta)),$$

$g(a)$ takes its minimum value at $a=\sqrt{\frac{1+\beta}{1-\beta}}$. This implies that

$$\begin{aligned}&\left|\frac{z_0p'(z_0)}{p(z_0)^2}\right| \geq \frac{\beta}{2}g\left(\sqrt{\frac{1+\beta}{1-\beta}}\right)\\ &= \frac{\beta}{2}\left\{\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1-\beta}{2}}+\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}}\right\}\\ &= \frac{\beta}{\sqrt{(1-\beta)^{1-\beta}(1+\beta)^{1+\beta}}},\end{aligned}$$

which contradicts our condition $f(z) \in \mathcal{G}_{b(\beta)}$ of the theorem. Thus we complete the proof of the theorem.

Considering the case of $\beta = 1$ in the proof of Theorem 1, we have

**Corollary 1.** *If* $f(z) \in \mathcal{G}_b$ *with* $b = \frac{1}{2}$*, then* $f(z) \in \mathcal{SS}^*\ (1)$*, or* $f(z)$ *is strongly starlike in* $\mathbb{U}$.

*Taking* $\beta = \frac{1}{2}$ *in Theorem* 1*, we have*

**Corollary 2.** *If* $f(z) \in \mathcal{G}_b$ *with* $b = \frac{1}{\sqrt{3\sqrt{3}}} = 0.438691\ldots$*, then* $f(z) \in \mathcal{SS}^*(\frac{1}{2})$.

## References

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