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On Riesz transforms for Laguerre expansions

Adam Nowak[1]

Instytut Matematyki, Politechnika Wrocławska, Wyb. Wyspiańskiego 27, Wrocław 50–370, Poland

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Abstract

We prove that Riesz transforms and conjugate Poisson integrals associated with the multi-dimensional Laguerre semigroup are bounded in L^p, $1<p<\infty$. Our main tools are appropriately defined square functions and the Littlewood–Paley–Stein theory.

MSC: primary 42C10; secondary 42B25

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1. Introduction

The aim of this paper is to study the Riesz transform $\mathscr{R}^\alpha = (R_1^\alpha, \dots, R_d^\alpha)$ naturally associated with multi-dimensional Laguerre polynomial expansions of type α. Our main result is contained in Theorem 13; we prove that if $\alpha \in [-\frac{1}{2}, \infty)^d$ then R_j^α, $j = 1, \dots, d$, are bounded operators in L^p with appropriate measure for $1<p<\infty$. Moreover, the corresponding L^p constants are independent of the dimension d and the type multi-index α. As a consequence we obtain boundedness and convergence results for the corresponding conjugate Poisson integrals, see Corollary 15 below.

Our methods are analytic and based on the Littlewood–Paley–Stein theory contained in monograph [12]. We construct appropriate square functions that relate a function and its Riesz transform, and then prove that these square functions are

E-mail address: anowak@im.pwr.wroc.pl.
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bounded in L^p, $1<p<\infty$. Noteworthy, the same scheme was exploited by Gutiérrez [2], who considered Riesz transforms associated with the multi-dimensional Hermite semigroup. Nevertheless, the case of Laguerre semigroup seems to be more involved.

Riesz transforms and conjugate Poisson integrals for the Laguerre semigroup were first studied by Muckenhoupt [9]. However, he worked in the one-dimensional setting and methods he used seem to be inapplicable in higher dimensions. Recently, a g-function and Riesz transforms associated with the multi-dimensional Laguerre semigroup were studied by Gutiérrez et al. [3]. The technique of transference exploited there allowed to obtain L^p, $1<p<\infty$, boundedness results only for a discrete set of half-integer multi-indices α. In the present paper, we remove this restriction and consider all intermediate multi-indices α. The case when $\alpha_i < -\frac{1}{2}$ for some $i = 1, \dots, d$ seems to require more subtle analysis and the corresponding Riesz transforms are not considered.

The paper is organized as follows. Section 2 contains basic facts and notation needed in the sequel. In Section 3, we define suitable square functions and prove necessary L^p inequalities. Main results of this section are contained in Theorems 6 and 7. Finally, in Section 4, we conclude the results concerning Riesz transforms and conjugate Poisson integrals.

2. Preliminaries

Given $\alpha > -1$, the one-dimensional *Laguerre polynomials* of type α are

$$L_k^\alpha(x) = \frac{1}{k!} e^x x^{-\alpha} \frac{d^k}{dx^k}(e^{-x} x^{k+\alpha}), \quad k \in \mathbb{N}, \ x > 0.$$

Note, that each L_k^α is a polynomial of degree k. Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha \in (-1, \infty)^d$, the d-dimensional Laguerre polynomials of type α are tensor products of the one-dimensional Laguerre polynomials, i.e.

$$L_k^\alpha(x) = \prod_{i=1}^{d} L_{k_i}^{\alpha_i}(x_i), \quad k \in \mathbb{N}^d. \ \in \mathbb{R}_+^d.$$

Laguerre polynomials have many interesting properties, see for example [6]. In particular, cf. [6, (4.18.6)],

$$\partial_{x_i} L_k^\alpha(x) = -L_{k-e_i}^{\alpha+e_i}(x), \quad i = 1, \dots, d, \tag{1}$$

e_i denoting the ith coordinate versor in $\mathbb{R}^d$. Here and later on we use the convention that $L_{k-e_i}^{\alpha+e_i} = 0$ if $k_i - 1 < 0$.

Consider the measure μ_α in $\mathbb{R}^d_+ = (0,\infty)^d$ given by $d\mu_\alpha(x) = \prod_{i=1}^d x_i^{\alpha_i} e^{-x_i}\,dx$. The *Laguerre differential operator*

$$\mathscr{L}^\alpha = -\sum_{i=1}^d \left[x_i \partial_{x_i}^2 + (\alpha_i + 1 - x_i)\partial_{x_i}\right]$$

is positive and symmetric in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$. Moreover, $\mathscr{L}^\alpha$ has a closure which is self-adjoint in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$ and which also will be denoted by $\mathscr{L}^\alpha$. Each Laguerre polynomial L_k^α is an eigenfunction of $\mathscr{L}^\alpha$ with the corresponding eigenvalue $|k|$. Furthermore, the system $\{L_k^\alpha : k \in \mathbb{N}^d\}$ constitutes an orthogonal basis in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$. Thus we have an orthogonal decomposition

$$L^2(\mathbb{R}^d_+, d\mu_\alpha) = \bigoplus_{n=0}^\infty \mathscr{H}_n,$$

where $\mathscr{H}_n = \operatorname{lin}\{L_k^\alpha : |k| = n\}$.

The semigroup generated by $\mathscr{L}^\alpha$ is called *Laguerre semigroup* and will be denoted by T_t^α. Given $f \in L^2(\mathbb{R}^d_+, d\mu_\alpha)$, it has the expansion

$$f = \sum_{k \in \mathbb{N}^d} a_k(f) L_k^\alpha,$$

where, with the notation $\langle f, L_k^\alpha \rangle_{d\mu_\alpha} = \int_{\mathbb{R}^d_+} f(y) L_k^\alpha(y)\,d\mu_\alpha(y)$,

$$a_k(f) = \langle f, L_k^\alpha \rangle_{d\mu_\alpha} / \|L_k^\alpha\|^2_{L^2(d\mu_\alpha)} = \langle f, L_k^\alpha \rangle_{d\mu_\alpha} \prod_{i=1}^d \frac{\Gamma(k_i + 1)}{\Gamma(k_i + \alpha_i + 1)},$$

hence

$$T_t^\alpha f = \sum_{k \in \mathbb{N}^d} a_k(f) e^{-t|k|} L_k^\alpha,$$

both series being convergent in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$. However, the definition of $T_t^\alpha f$ for $f \in L^p(\mathbb{R}^d_+, d\mu_\alpha)$, $1 \leqslant p \leqslant \infty$, by means of its Fourier–Laguerre expansion would be unsatisfactory, since the corresponding series may diverge for $p < 2$, see [8]. Therefore it is appropriate to use an integral definition, which turns out to be usable for all $p \in [1, \infty]$. We define

$$T_t^\alpha f(x) = \int_{\mathbb{R}^d_+} G_t^\alpha(x,y) f(y)\,d\mu_\alpha(y), \quad f \in L^p(\mathbb{R}^d_+, d\mu_\alpha), \tag{2}$$

where

$$G_t^\alpha(x,y) = \sum_{k \in \mathbb{N}^d} e^{-t|k|} L_k^\alpha(x) L_k^\alpha(y) / \|L_k^\alpha\|^2_{L^2(d\mu_\alpha)}.$$

The kernel $G_t^\alpha(x,y)$ may be computed explicitly by means of the Hille–Hardy formula [6, (4.17.6)]. The result is

$$G_t^\alpha(x,y) = \prod_{j=1}^{d} (1-u)^{-1} \exp\Big(-\frac{u}{1-u}(x_j+y_j)\Big)\sqrt{ux_jy_j}^{-\alpha_j} I_{\alpha_j}\bigg(\frac{2\sqrt{ux_jy_j}}{1-u}\bigg),$$

where $u = \mathrm{e}^{-t}$ and I_v denotes the modified Bessel function of the first kind and order v, cf. [6]. Noteworthy, $G_t^\alpha(x,y)$ is smooth and strictly positive for $(t,x,y) \in \mathbb{R}_+^{2d+1}$. By using standard estimates for I_v (see [8]) it is easily seen that the integral in (2) is absolutely convergent. It is well known that $\{T_t^\alpha\}$ is a *symmetric diffusion semigroup* (in fact $\{T_t^\alpha\}$ is a transition semigroup for the Laguerre diffusion process, which already received an attention due to some applications in financial mathematics). In particular, $T_t^\alpha \mathbf{1} = \mathbf{1}$ and

$$\|T_t^\alpha f\|_{L^p(d\mu_\alpha)} \leqslant \|f\|_{L^p(d\mu_\alpha)}, \quad 1 \leqslant p \leqslant \infty.$$

The corresponding Poisson semigroup $\{P_t^\alpha\}$ is defined by means of the subordination principle as the weighted average of T_t^α:

$$P_t^\alpha f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\mathrm{e}^{-u}}{\sqrt{u}} T_{t^2/(4u)}^\alpha f(x)\, du, \quad f \in L^p(\mathbb{R}_+^d, d\mu_\alpha).$$

Note, that for $f \in L^2(\mathbb{R}_+^d, d\mu_\alpha)$ we have

$$P_t^\alpha f = \sum_{k \in \mathbb{N}^d} a_k(f) \mathrm{e}^{-t|k|^{1/2}} L_k^\alpha.$$

By a general theory (see [12, p. 73]) it follows that the maximal operator $(P^\alpha)^* f(x) = \sup_{t>0} |P_t^\alpha f(x)|$ satisfies

$$\|(P^\alpha)^* f\|_{L^p(d\mu_\alpha)} \leqslant C_p \|f\|_{L^p(d\mu_\alpha)}, \quad f \in L^p(\mathbb{R}_+^d, d\mu_\alpha). \tag{3}$$

Let us emphasize that the constant C_p depends neither on the dimension d nor on the type multi-index α. The important consequence of (3) is

$$\lim_{t\to 0^+} P_t^\alpha f(x) = f(x) \quad \text{a.e.} \quad f \in L^p(\mathbb{R}_+^d, d\mu_\alpha), \ 1 < p < \infty,$$

the fact which will be used later without further mention.

We define the ith partial derivative associated with $\mathscr{L}^\alpha$ by

$$\delta_i = \sqrt{x_i}\,\partial_{x_i},$$

see [3]. The formal adjoint of δ_i in $L^2(\mathbb{R}_+^d, d\mu_\alpha)$ is given by

$$\delta_i^* = -\sqrt{x_i}\bigg(\partial_{x_i} + \frac{\alpha_i + \frac{1}{2} - x_i}{x_i}\bigg)$$

and we have

$$\mathscr{L}^\alpha = \sum_{i=1}^{d} \delta_i^* \delta_i.$$

The last equality may be written in a compact form

$$\mathscr{L}^\alpha = \operatorname{div}_\alpha \operatorname{grad}_\alpha,$$

where $\operatorname{grad}_\alpha = (\delta_1, \dots, \delta_d)$ and $\operatorname{div}_\alpha F = \sum_{i=1}^d \delta_i^* f_i$ for a vector-valued function $F(y) = (f_1(y), \dots, f_d(y))$.

The Riesz–Laguerre transform $\mathscr{R}^\alpha = (R_1^\alpha, \dots, R_d^\alpha)$ is then formally defined by (cf. [3])

$$\mathscr{R}^\alpha = \operatorname{grad}_\alpha (\mathscr{L}^\alpha)^{-1/2} \Pi_0, \tag{4}$$

where Π_0 denotes the orthogonal projection onto the orthogonal complement $\mathscr{H}_0^\perp$ of the eigenspace corresponding to the Laguerre eigenvalue 0. Note, that (4) makes sense for Laguerre polynomials (hence for all polynomials) and by (1) we have

$$R_i^\alpha L_k^\alpha = -|k|^{-1/2} \sqrt{x_i} L_{k-e_i}^{\alpha+e_i}, \quad |k|>0$$

and $R_i^\alpha L_k^\alpha = 0$ if $|k| = 0$. A crucial observation which should be made here is that $R_i^\alpha L_k^\alpha$ is not a Laguerre polynomial of the same type α (to make it worse, it is not a polynomial at all), which is a consequence of "bad" action of the Laguerre derivatives δ_i on L_k^α. This effect is absent in the Hermite setting and makes the present analysis more complex, including working with d auxiliary orthogonal systems and considering supplementary semigroups, which are not simply "translations" of P_t^α (as it was in the Hermite case, see [3]).

Now, we introduce additional semigroups $\{\tilde{P}_t^{\alpha,i}\}$, $i = 1, \dots, d$, which are generated by "slight" modifications of the operator $(\mathscr{L}^\alpha)^{1/2}$. As we shall see, they play an essential role in the study of Riesz transforms and conjugacy for Laguerre expansions. To proceed, we first define the operators

$$M_i^\alpha = \delta_i \delta_i^* + \sum_{j \neq i} \delta_j^* \delta_j = \mathscr{L}^\alpha + \frac{\alpha_i + \frac{1}{2} + x_i}{2x_i}, \quad i = 1, \dots, d.$$

Observe that each M_i^α is symmetric, and positive since for sufficiently regular functions f

$$\langle M_i^\alpha f, f \rangle_{d\mu_\alpha} = \int \left((\delta_i^* f)^2 + \sum_{j \neq i} (\delta_j f)^2 \right) d\mu_\alpha.$$

We set (formally)

$$\tilde{P}_t^{\alpha,i} = \mathrm{e}^{-t(M_i^\alpha)^{1/2}}, \quad i = 1, \dots, d.$$

To make this definition more explicit we will need the following.

Lemma 1. *Given* $i=1,\dots,d$ the functions $\sqrt{x_i}L_{k-e_i}^{\alpha+e_i}(x)$ *are eigenfunctions of* M_i^α, *with the corresponding eigenvalues* $|k|$. *Moreover, the system* $\{\sqrt{x_i}L_k^{\alpha+e_i}(x): k\in\mathbb{N}^d\}$ *forms an orthogonal basis in* $L^2(\mathbb{R}_+^d, d\mu_\alpha)$.

Proof. The first part follows by a direct computation and using the identity

$$\mathscr{L}^{\alpha+e_i}L_{k-e_i}^{\alpha+e_i} = (|k|-1)L_{k-e_i}^{\alpha+e_i}.$$

The second part is a consequence of the fact that the system $\{L_k^{\alpha+e_i}: k\in\mathbb{N}^d\}$ is an orthogonal basis in $L^2(\mathbb{R}_+^d, d\mu_{\alpha+e_i})$. $\square$

By the above lemma, given $i=1,\dots,d$, any $f\in L^2(\mathbb{R}_+^d, d\mu_\alpha)$ has the expansion

$$f=\sum_{k\in\mathbb{N}^d} a_k^i(f)\sqrt{x_i}L_k^{\alpha+e_i},$$

hence

$$\tilde{P}_t^{\alpha,i}f = \sum_{k\in\mathbb{N}^d} a_k^i(f)\mathrm{e}^{-t(|k|+1)^{1/2}}\sqrt{x_i}L_k^{\alpha+e_i}. \tag{5}$$

Unfortunately, such definition is not appropriate for general $f\in L^p(\mathbb{R}_+^d, d\mu_\alpha)$, $1\leqslant p\leqslant\infty$, the reason for that being similar as in the case of T_t^α. Therefore, we will use an integral representation. Let $f\in L^p(\mathbb{R}_+^d, d\mu_\alpha)$ and define

$$\tilde{T}_t^{\alpha,i}f(x) = \int_{\mathbb{R}_+^d} \tilde{G}_t^{\alpha,i}(x,y)f(y)\,d\mu_\alpha(y), \quad i=1,\dots,d, \tag{6}$$

where

$$\begin{aligned}\tilde{G}_t^{\alpha,i}(x,y) &= \sum_{k\in\mathbb{N}^d} \mathrm{e}^{-t(|k|+1)}\sqrt{x_i}L_k^{\alpha+e_i}(x)\sqrt{y_i}L_k^{\alpha+e_i}(y)/\|\sqrt{y_i}L_k^{\alpha+e_i}\|_{L^2(d\mu_\alpha)}^2\\ &= \mathrm{e}^{-t}\sqrt{x_iy_i}\sum_{k\in\mathbb{N}^d}\mathrm{e}^{-t|k|}L_k^{\alpha+e_i}(x)L_k^{\alpha+e_i}(y)/\|L_k^{\alpha+e_i}\|_{L^2(d\mu_{\alpha+e_i})}^2\\ &= \mathrm{e}^{-t}\sqrt{x_iy_i}G_t^{\alpha+e_i}(x,y).\end{aligned}$$

Similarly as in (2), the integral in (6) is absolutely convergent and for $f\in L^2(\mathbb{R}_+^d, d\mu_\alpha)$ we have

$$\tilde{T}_t^{\alpha,i}f = \sum_{k\in\mathbb{N}^d} a_k^i(f)\mathrm{e}^{-t(|k|+1)}\sqrt{x_i}L_k^{\alpha+e_i}. \tag{7}$$

Now, using the subordination formula, we set

$$\tilde{P}_t^{\alpha,i}f(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\mathrm{e}^{-u}}{\sqrt{u}}\tilde{T}_{t^2/(4u)}^{\alpha,i}f(x)\,du, \quad i=1,\dots,d.$$

Note, that the above definition makes sense for $f \in L^p(\mathbb{R}^d_+, d\mu_\alpha)$, $1 \leqslant p \leqslant \infty$, and if $p = 2$ it coincides with (5). Observe also, that the operators $\tilde{P}^{\alpha,i}_t$ are positive and symmetric by the corresponding properties of $\tilde{G}^{\alpha,i}_t(x, y)$. To obtain the relevant L^p inequalities for $\tilde{P}^{\alpha,i}_t$ we prove the following.

Lemma 2. *Given $i = 1, \ldots, d$ the quantity*

$$\inf\{\lambda : \tilde{G}^{\alpha,i}_t(x, y) \leqslant \lambda e^{-t/2} G^\alpha_t(x, y),\ \ x, y \in \mathbb{R}^d_+,\ \ t > 0\}$$

is the one-dimensional, finite function of α_i, which is further denoted by Λ.

The function $\Lambda(v)$ is nonincreasing for $v \in (-1, \infty)$, $\Lambda(v) = 1$ for $v \geqslant -\frac{1}{2}$, $\Lambda(v) > 1$ for $v \in \left(-1, -\frac{1}{2}\right)$ and $\Lambda(v) \to \infty$ as $v \to -1^+$.

Proof. By the explicit formula for $G^\alpha_t(x, y)$ we get

$$\tilde{G}^{\alpha,i}_t(x, y) = \Theta_{\alpha_i}\left(\frac{2\sqrt{u x_j y_j}}{1-u}\right) e^{-t/2} G^\alpha_t(x, y),$$

where the function Θ_v is given by

$$\Theta_v(z) = \frac{I_{v+1}(z)}{I_v(z)}, \quad z > 0.$$

Quotients of this type are of independent interest and have been studied by many authors. In particular, it is known (see [4]) that $\Theta_v(z)$ is a decreasing function of $v > -1$ for every fixed $z > 0$. Hence

$$\Lambda(v) = \sup_{z>0}\ \Theta_v(z)$$

is nonincreasing in interval $(-1, \infty)$. Consequently, we have

$$\Lambda(v) \leqslant \Lambda\left(-\tfrac{1}{2}\right) = \sup_{z>0}\ \tanh(z) = 1, \quad v \geqslant -\tfrac{1}{2}.$$

Thus we obtain $\Lambda(v) = 1$ for $v \geqslant -\frac{1}{2}$, because (cf. [6, (5.11.10)])

$$I_v(z) = (2\pi z)^{-1/2} e^z [1 + O(1/z)], \quad z \to \infty.$$

To treat $\Lambda(v)$ in the range $-1 < v < -\frac{1}{2}$ we shall use the estimate

$$B_v(z) \leqslant \Theta_v(z), \quad z > 0,\ \ v > -1,$$

where

$$B_v(z) = \frac{z}{v + \frac{1}{2} + \sqrt{z^2 + (v + \frac{3}{2})^2}}.$$

For this and other bounds for Θ_v, see [10] and references therein. A simple calculus shows that

$$\sup_{z>0} B_v(z) = \frac{v+\frac{3}{2}}{\sqrt{2(v+1)}}, \quad -1<v<-\tfrac{1}{2},$$

hence $\Lambda(v)>1$ for $v\in\left(-1,-\frac{1}{2}\right)$, and $\Lambda(v)\to\infty$ as $v\to -1^+$.

Let us note, that more precise description of the behavior of Λ is possible by using another estimates for $\Theta_v(z)$, cf. [10]. For example one can show that

$$\Lambda(v) = O((v+1)^{-1/2}), \quad v\to -1^+. \qquad \square$$

Corollary 3. *Given $i=1,\dots,d$ and a reasonable nonnegative function f we have*

$$\tilde{P}_t^{\alpha,i} f(x) \leqslant \Lambda(\alpha_i) P_t^\alpha f(x), \quad x\in\mathbb{R}_+^d, \quad t>0.$$

Thus, $\{\tilde{P}_t^{\alpha,i}\}$ is a uniformly bounded semigroup of operators (contractions, if $\alpha\in[-\frac{1}{2},\infty)^d$) in $L^p(\mathbb{R}_+^d, d\mu_\alpha)$, $1\leqslant p\leqslant\infty$.

Proposition 4. *Let $f\in L^2(\mathbb{R}_+^d, d\mu_\alpha)$ and $i\in\{1,\dots,d\}$. Then $T_t^\alpha f(x)$, $P_t^\alpha f(x)$, $\tilde{T}_t^{\alpha,i} f(x)$, $\tilde{P}_t^{\alpha,i} f(x)$ are C^∞ functions on $\mathbb{R}_+\times\mathbb{R}_+^d$. Moreover,*

$$(\partial_t + \mathscr{L}^\alpha) T_t^\alpha f(x) = 0 = (\partial_t + M_i^\alpha)\tilde{T}_t^{\alpha,i} f(x), \quad x\in\mathbb{R}_+^d,$$

$$(\partial_t^2 - \mathscr{L}^\alpha) P_t^\alpha f(x) = 0 = (\partial_t^2 - M_i^\alpha)\tilde{P}_t^{\alpha,i} f(x), \quad x\in\mathbb{R}_+^d.$$

Proof. We consider only $\tilde{T}_t^{\alpha,i} f(x)$ since treatment of the remaining functions is analogous. We will show that the series in (7) may be differentiated term by term. Observe that by Schwarz' inequality

$$|a_k^i(f)| \leqslant \frac{||f||_{L^2(d\mu_\alpha)}}{||\sqrt{y_i} L_k^{\alpha+e_i}||_{L^2(d\mu_\alpha)}} = \frac{||f||_{L^2(d\mu_\alpha)}}{\sqrt{k_i+\alpha_i+1}} \prod_{j=1}^d \left(\frac{\Gamma(k_j+1)}{\Gamma(k_j+\alpha_j+1)}\right)^{1/2},$$

hence $|a_k^i(f)|$ grows at most polynomially in $|k|$. Furthermore, for a fixed compact set $K\subset\mathbb{R}_+^d$ the quantity $\sup_{x\in K}|L_k^{\alpha+e_i}(x)|$ also has sub-polynomial growth in $|k|$, see [9, p. 405]. Therefore, the series defining $\tilde{T}_t^{\alpha,i} f(x)$ may be differentiated in t term by term. The result is

$$\partial_t^m \tilde{T}_t^{\alpha,i} f(x) = \sum_{k\in\mathbb{N}^d} a_k^i(f)(-1)^m(|k|+1)^m \mathrm{e}^{-t(|k|+1)} \sqrt{x_i} L_k^{\alpha+e_i}(x), \tag{8}$$

the right-hand side being continuous since the series converges almost uniformly in (t,x). Using (1) we see that also $\sup_{x\in K}|\partial_{x_j}(\sqrt{x_i}L_k^{\alpha+e_i}(x))|$ grows in $|k|$ not faster than polynomially and hence we may differentiate in x_j term by term the series in (8), the result being a continuous function since the convergence is again almost uniform in (t,x). The same arguments apply to higher derivatives, so $\tilde{T}_t^{\alpha,i}f(x)$ is smooth on $\mathbb{R}_+\times\mathbb{R}_+^d$. The corresponding heat equation is easily verified by differentiating term by term the series of $\tilde{T}_t^{\alpha,i}f(x)$. $\square$

Remark 5. Noteworthy, Proposition 4 is true for $f\in L^p(\mathbb{R}_+^d,d\mu_\alpha)$, $1\leqslant p\leqslant\infty$, which is proved by an analysis of the integral representations of the semigroups.

In what follows, we use the convention that constants may change their value (but not the dependence) from one use to the next. The notation c_p means that the constant depends *only* on p (in particular, c_p is independent of the dimension d and the type multi-index α). Constants are always strictly positive and finite.

3. Square functions results

Denote $\nabla_\alpha=(\partial_t,\mathrm{grad}_\alpha)$. We consider the following Littlewood–Paley–Stein type square functions:

$$g(f)(x)=\left(\int_0^\infty t|\nabla_\alpha P_t^\alpha f(x)|^2\,dt\right)^{1/2}$$

and

$$\tilde{g}_i(f)(x)=\left(\int_0^\infty t|\partial_t\tilde{P}_t^{\alpha,i}f(x)|^2\,dt\right)^{1/2},\quad i=1,\dots,d.$$

For a vector-valued function $F(x)=(f_1(x),\dots,f_d(x))$ we define

$$\tilde{g}(F)(x)=(\tilde{g}_1(f_1)(x),\dots,\tilde{g}_d(f_d)(x)).$$

The main results of this section read as follows.

Theorem 6. *Let $1<p<\infty$, $\alpha\in(-1,\infty)^d$ and Λ be the function from Lemma 2. There exists a constant c_p such that*

$$||g(f)||_{L^p(d\mu_\alpha)}\leqslant c_p\left[\Lambda\left(\min_i\alpha_i\right)\right]^2||f||_{L^p(d\mu_\alpha)}\tag{9}$$

for all $f\in L^p(\mathbb{R}_+^d,d\mu_\alpha)$.

Theorem 7. *Let* $1<p<\infty$ *and* $\alpha\in[-\frac{1}{2},\infty)^d$.

(a) *Given* $1\leqslant i\leqslant d$ *we have*

$$c_p^{-1}||f||_{L^p(d\mu_\alpha)}\leqslant||\tilde{g}_i(f)||_{L^p(d\mu_\alpha)}\leqslant c_p||f||_{L^p(d\mu_\alpha)}$$

for all $f\in L^p(\mathbb{R}^d_+,d\mu_\alpha)$.

(b) *Let* $\mathscr{A}=\#\{\alpha_i : i=1,\dots,d\}$ *be the number of distinct coordinates in the multi-index* α. *Then*

$$||\,|F|_{\ell^2}||_{L^p(d\mu_\alpha)}\leqslant c_p\mathscr{A}||\,|\tilde{g}(F)|_{\ell^2}||_{L^p(d\mu_\alpha)}$$

for all $F(x)=(f_1(x),\dots,f_d(x))$ *such that* $|F|_{\ell^2}\in L^p(\mathbb{R}^d_+,d\mu_\alpha)$.

3.1. Proof of Theorem 6

For a reasonable function $F=F(t,x)$ define

$$\mathbb{L}^\alpha F(t,x)=\partial_t^2F(t,x)-\mathscr{L}_x^\alpha F(t,x).$$

We will need several technical lemmas.

Lemma 8. *Let* $F=F(t,x)$ *be a* C^2 *function mapping* $\mathbb{R}_+\times\mathbb{R}^d_+$ *into* $(0,\infty)$ *such that* $\mathbb{L}^\alpha F=0$. *Then for any* $p\geqslant1$ *we have*

$$\mathbb{L}^\alpha(F^p)=p(p-1)F^{p-2}|\nabla_\alpha F|^2.$$

Proof. The result follows by a simple computation, see [3]. □

Lemma 9. *Let* $F:\mathbb{R}_+\times\mathbb{R}^d_+\to\mathbb{R}$ *be a* C^2 *function such that* $\mathbb{L}^\alpha F\geqslant0$ *or* $\int_0^\infty\int_{\mathbb{R}^d_+}t|\mathbb{L}^\alpha F(t,x)|\,d\mu_\alpha(x)\,dt<\infty$. *Assume that*

(a) $\sup\{|F(t,x)| : t>0,\ x\in\mathbb{R}^d_+\}<\infty$,
(b) $\sup\{|\nabla_x F(t,x)| : t>0,\ x\in\mathbb{R}^d_+\}<\infty$,
(c) $t|\partial_t F(t,x)|\leqslant c(1+|x|)\phi(t)$ *for all* $t>0$, *where the function* ϕ *is continuous, vanishes at* 0 *and* ∞, *and satisfies* $\int_0^\infty t^{-1}\phi(t)\,dt<\infty$.

Then

$$\int_0^\infty\int_{\mathbb{R}^d_+}t\mathbb{L}^\alpha F(t,x)\,d\mu_\alpha(x)\,dt=\int_{\mathbb{R}^d_+}F(0,x)\,d\mu_\alpha(x)-\int_{\mathbb{R}^d_+}F(\infty,x)\,d\mu_\alpha(x).$$

Proof. First observe that for each $x \in \mathbb{R}_+^d$ the limit

$$F(\infty, x) = \lim_{t \to \infty} F(t, x)$$

does exist. Indeed, by condition (c) we may write

$$\begin{aligned}|F(t+T,x) - F(t,x)| &\leqslant \sum_{j=0}^{N-1} \left| F\left(t + \frac{j+1}{N} T\right) - F\left(t + \frac{j}{N} T\right)\right| \\ &= \sum_{j=0}^{N-1} |\partial_t F(\theta_j, x)| \frac{T}{N} \leqslant C(1+|x|) \sum_{j=0}^{N-1} \frac{T}{N} \frac{\phi(\theta_j)}{\theta_j} \equiv C(1+|x|) S_N,\end{aligned}$$

with $\theta_j \in \left(t + \frac{jT}{N}, t + \frac{(j+1)T}{N}\right)$. Since $S_N \to \int_t^{t+T} r^{-1} \phi(r)\, dr$ as $N \to \infty$, the conclusion follows. Similar arguments show the existence of the limit

$$F(0, x) = \lim_{t \to 0^+} F(t, x).$$

Now, observe that

$$\mathbb{L}^\alpha F(t,x) = \partial_t^2 F(t,x) + \sum_{j=1}^d \mathrm{e}^{x_j} x_j^{-\alpha_j} \partial_{x_j} [\mathrm{e}^{-x_j} x_j^{\alpha_j+1} \partial_{x_j} F(t,x)].$$

Let $\mathscr{D}_T = (\mathrm{e}^{-T}, T) \times (\mathrm{e}^{-T}, T)^d$. Given $j \in \{1, \dots, d\}$ we have

$$\begin{aligned}&\int_{\mathrm{e}^{-T}}^T \mathrm{e}^{x_j} x_j^{-\alpha_j} \partial_{x_j} [\mathrm{e}^{-x_j} x_j^{\alpha_j+1} \partial_{x_j} F(t,x)]\, d\mu_{\alpha_j}(x_j) \\ &\quad = \mathrm{e}^{-T} T^{\alpha_j+1} \partial_{x_j} F(t,x)|_{x_j=T} - \exp(-\mathrm{e}^{-T}) \mathrm{e}^{-T(\alpha_j+1)} \partial_{x_j} F(t,x)|_{x_j=\mathrm{e}^{-T}},\end{aligned}$$

hence, by (b),

$$\left| \int_{\mathscr{D}_T} t \mathrm{e}^{x_j} x_j^{-\alpha_j} \partial_{x_j} [\mathrm{e}^{-x_j} x_j^{\alpha_j+1} \partial_{x_j} F(t,x)]\, d\mu_\alpha(x)\, dt \right| \leqslant c T^2 (\mathrm{e}^{-T} T^{\alpha_j+1} + \mathrm{e}^{-(\alpha_j+1)T}).$$

Therefore,

$$\int_{\mathscr{D}_T} t \sum_{j=1}^d \mathrm{e}^{x_j} x_j^{-\alpha_j} \partial_{x_j} [\mathrm{e}^{-x_j} x_j^{\alpha_j+1} \partial_{x_j} F(t,x)]\, d\mu_\alpha(x)\, dt \to 0, \quad T \to \infty.$$

This, together with the monotone (or the dominated) convergence theorem, implies

$$\int_0^\infty \int_{\mathbb{R}_+^d} t \mathbb{L}^\alpha F(t,x)\, d\mu_\alpha(x)\, dt = \lim_{T \to \infty} \int_{\mathscr{D}_T} t \partial_t^2 F(t,x)\, dt\, d\mu_\alpha(x).$$

On the other hand, integrating by parts we obtain

$$\int_{e^{-T}}^{T} t\partial_t^2 F(t,x)\,dt = F(e^{-T},x) - F(T,x) + t\partial_t F(t,x)\Big|_{t=e^{-T}}^{t=T}.$$

By (c) the absolute value of the last term is estimated by $c(1+|x|)[\phi(e^{-T})+\phi(T)]$. Since $\lim_{t\to 0^+}\phi(t) = \lim_{t\to\infty}\phi(t) = 0$ and $(1+|\cdot|)\in L^1(\mathbb{R}^d_+, d\mu_\alpha)$, the proof is finished with the aid of (a) and the dominated convergence theorem. □

Proposition 10. *Lemma* 9 *may be applied to the function*

$$F(t,x) = (P_t^\alpha f(x))^p,$$

where $p \geqslant 1$ *and* f *is an arbitrary nonnegative function from* $C_c^2(\mathbb{R}^d_+)$.

Proof. By the subordination principle and Proposition 4 we get

$$\begin{aligned}\partial_t P_t^\alpha f(x) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \partial_t [T^\alpha_{t^2/(4u)} f(x)]\,du \\ &= \frac{-1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} \mathscr{L}^\alpha [T^\alpha_{t^2/(4u)} f(x)]\,du.\end{aligned}$$

Interchanging the order of differentiation and integration above is justified by the dominated convergence theorem, using (see also the considerations below)

$$\partial_{x_j} T_t^\alpha f(x) = e^{-t} T_t^{\alpha+e_j}(\partial_{x_j} f)(x),$$

$$\partial^2_{x_j} T_t^\alpha f(x) = e^{-2t} T_t^{\alpha+2e_j}(\partial^2_{x_j} f)(x).$$

These identities are easily verified for Laguerre polynomials, and for functions from $C_c^2(\mathbb{R}^d_+) \subset L^2(\mathbb{R}^d_+, d\mu_\alpha)$ they are checked by term by term differentiation of the series defining $T_t^\alpha f$, see the proof of Proposition 4. Thus

$$\begin{aligned}\partial_t P_t^\alpha(x) &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} e^{-t^2/(2u)} \sum_{j=1}^d x_j T^{\alpha+2e_j}_{t^2/(4u)}(\partial^2_{x_j} f)(x)\,du \\ &\quad + \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} e^{-t^2/(4u)} \sum_{j=1}^d (\alpha_j+1) T^{\alpha+e_j}_{t^2/(4u)}(\partial_{x_j} f)(x)\,du \\ &\quad - \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} e^{-t^2/(4u)} \sum_{j=1}^d x_j T^{\alpha+e_j}_{t^2/(4u)}(\partial_{x_j} f)(x)\,du \\ &\equiv \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3.\end{aligned}$$

Since T_t^α are contractions on $L^\infty(\mathbb{R}_+^d)$ and f has bounded first- and second-order derivatives we have

$$|t\mathscr{I}_1| \leqslant ct \sum_{j=1}^d x_j \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} e^{-t^2/(2u)}\, du = cte^{-t\sqrt{2}} \sum_{j=1}^d x_j$$

and similarly

$$|t\mathscr{I}_2| + |t\mathscr{I}_3| \leqslant cte^{-t} \sum_{j=1}^d (\alpha_j + 1) + cte^{-t} \sum_{j=1}^d x_j.$$

Consequently,

$$|t\partial_t P_t^\alpha f(x)| \leqslant c(1+|x|)te^{-t}.$$

Since f is bounded and P_t^α are contractions on $L^\infty(\mathbb{R}_+^d)$ (which follows by the same property for T_t^α and the subordination principle), hypotheses (a) and (c) of Lemma 9 are satisfied (with $\phi(t) = te^{-t}$). Concerning (b), we have

$$\begin{aligned} |\partial_{x_j} P_t^\alpha f(x)| &= \left| \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-t^2/(4u)} T_{t^2/(4u)}^{\alpha+e_j} (\partial_{x_j} f)(x)\, du \right| \\ &\leqslant \frac{c}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-t^2/(4u)}\, du = ce^{-t}. \end{aligned}$$

The fact that $\mathbb{L}^\alpha[(P_t^\alpha f(x))^p] \geqslant 0$ is justified with the aid of Proposition 4 and Lemma 8 (an application of Lemma 8 is possible, because $P_t^\alpha f$ is strictly positive if $f(x_0) > 0$ for some $x_0 \in \mathbb{R}_+^d$, see the comment in the proof of Theorem 6 below). $\square$

Proposition 11. *Let $f \in C_c^2(\mathbb{R}_+^d)$ be nonnegative. There exists a constant c (depending on f) such that*

$$g(f)(x) \leqslant c(1+|x|), \quad x \in \mathbb{R}_+^d;$$

hence $g(f) \in L^p(\mathbb{R}_+^d, d\mu_\alpha)$ for all $1 \leqslant p < \infty$.

Proof. We apply the estimates on $|\partial_t P_t^\alpha f(x)|$ and $|\partial_{x_j} P_t^\alpha f(x)|$ obtained in the proof of Proposition 10. $\square$

Proposition 12. *Lemma 9 may be applied to the function*

$$F(t,x) = (P_t^\alpha f(x))^2 P_t^\alpha h(x),$$

with arbitrary nonnegative $f, h \in C_c^2(\mathbb{R}_+^d)$.

Proof. Items (a)–(c) are verified similarly as in the proof of Proposition 10. It remains to show the integrability condition.

Given reasonable functions $F, G : \mathbb{R}_+ \times \mathbb{R}^d_+ \to \mathbb{R}$, one has

$$\mathbb{L}^\alpha(FG) = (\mathbb{L}^\alpha F)G + (\mathbb{L}^\alpha G)F + 2\langle \nabla_\alpha F, \nabla_\alpha G\rangle. \tag{10}$$

Therefore,

$$|\mathbb{L}^\alpha[(P^\alpha_t f(x))^2 P^\alpha_t h(x)]| \leqslant \mathbb{L}^\alpha(P^\alpha_t f(x))^2 P^\alpha_t h(x) + 2|\nabla_\alpha(P^\alpha_t f(x))^2||\nabla_\alpha P^\alpha_t h(x)|,$$

since $\mathbb{L}^\alpha(P^\alpha_t h(x)) = 0$ by Proposition 4. Now, observe that

$$\int_0^\infty \int_{\mathbb{R}^d_+} t\mathbb{L}^\alpha(P^\alpha_t f(x))^2 P^\alpha_t h(x)\, d\mu_\alpha(x)\, dt \leqslant c \int_0^\infty \int_{\mathbb{R}^d_+} t\mathbb{L}^\alpha(P^\alpha_t f(x))^2\, d\mu_\alpha(x)\, dt$$

and the last term is finite by Proposition 10. Further, by Schwarz' inequality and the fact that $P^\alpha_t f(x)$ is bounded we obtain

$$\int_0^\infty \int_{\mathbb{R}^d_+} t|\nabla_\alpha(P^\alpha_t f(x))^2|\,\nabla_\alpha P^\alpha_t h(x)|\, d\mu_\alpha(x)\, dt \leqslant c \int_{\mathbb{R}^d_+} g(f)(x) g(h)(x)\, d\mu_\alpha(x),$$

the last integral being finite by Proposition 11. □

Proof of Theorem 6. *Case* $1 < p \leqslant 2$: Apart from minor changes, the proof relies on a classical reasoning, see [12]. Observe first, that it is sufficient to prove (9) for all nonnegative $f \in C^2_c(\mathbb{R}^d_+)$. Then the theorem is justified by standard arguments, that is decomposition into positive and negative parts, approximation of each part by an increasing sequence of nonnegative smooth compactly supported functions, and an application of Fatou's lemma.

Assume that $f \in C^2_c(\mathbb{R}^d_+)$, $f \geqslant 0$ and $f(x_0) \neq 0$ for some $x_0 \in \mathbb{R}^d_+$. Since the kernel $G^\alpha_t(x,y)$ is strictly positive we have $T^\alpha_t f(x) > 0$, $x \in \mathbb{R}^d_+$ and hence, by the subordination formula, $P^\alpha_t f(x) > 0$, $x \in \mathbb{R}^d_+$. An application of Lemma 8 gives

$$\begin{aligned}[g(f)(x)]^2 &= \int_0^\infty t|\nabla_\alpha(P^\alpha_t f(x))|^2\, dt \\ &= \frac{1}{p(p-1)} \int_0^\infty t(P^\alpha_t f(x))^{2-p}\mathbb{L}^\alpha(P^\alpha_t f(x))^p\, dt \\ &\leqslant \frac{1}{p(p-1)}[(P^\alpha)^* f(x)]^{2-p} \int_0^\infty t\mathbb{L}^\alpha(P^\alpha_t f(x))^p\, dt.\end{aligned}$$

Thus, by Hölder's inequality, (3) and Proposition 10 we obtain

$$||g(f)||^p_{L^p(d\mu_\alpha)} \leqslant c_p \int_{\mathbb{R}^d_+} [(P^\alpha)^* f(x)]^{p(1-p/2)} \left(\int_0^\infty t\mathbb{L}^\alpha (P^\alpha_t f(x))^p \, dt \right)^{p/2} d\mu_\alpha(x)$$

$$\leqslant c_p ||f||^{p(1-p/2)}_{L^p(d\mu_\alpha)} \left(\int_0^\infty \int_{\mathbb{R}^d_+} t\mathbb{L}^\alpha (P^\alpha_t f(x))^p \, d\mu_\alpha(x) \, dt \right)^{p/2}$$

$$= c_p ||f||^{p(1-p/2)}_{L^p(d\mu_\alpha)} \left[\int_{\mathbb{R}^d_+} f(x)^p \, d\mu_\alpha(x) - \int_{\mathbb{R}^d_+} (P^\alpha_\infty f(x))^p \, d\mu_\alpha(x) \right]^{p/2}$$

$$\leqslant c_p ||f||^p_{L^p(d\mu_\alpha)},$$

since $\mathbb{L}^\alpha (P^\alpha_t f(x))^p \geqslant 0$, see the proof of Proposition 10. The conclusion follows.

Case $p > 2$: We begin with some basic observations. Given a reasonable function f, by Schwarz' inequality we have

$$(P^\alpha_t f(x))^2 \leqslant P^\alpha_t (f^2)(x) P^\alpha_t \mathbf{1}(x) = P^\alpha_t (f^2)(x).$$

Similarly, by Corollary 3 we get

$$(\tilde{P}^{\alpha,i}_t f(x))^2 \leqslant \tilde{P}^{\alpha,i}_t (f^2)(x) \tilde{P}^{\alpha,i}_t \mathbf{1}(x) \leqslant [\Lambda(\alpha_i)]^2 P^\alpha_t (f^2)(x).$$

Assume that $f \in C^1_c(\mathbb{R}^d_+)$. Then

$$\partial_t P^\alpha_{t+s} f(x) = P^\alpha_s (\partial_t P^\alpha_t f)(x).$$

Moreover,

$$\delta_j (P^\alpha_t f)(x) = \tilde{P}^{\alpha,j}_t (\delta_j f)(x), \quad j = 1, \dots, d,$$

which is directly verified for Laguerre polynomials, and for $f \in C^1_c(\mathbb{R}^d_+)$ follows by differentiating term by term the series of $P^\alpha_t f$. Therefore

$$\delta_j (P^\alpha_{t+s} f)(x) = \tilde{P}^{\alpha,j}_t (\delta_j P^\alpha_s f)(x), \quad j = 1, \dots, d.$$

Let f, h be nonnegative functions from $C^2_c(\mathbb{R}^d_+)$ and denote $\mathscr{C}(\alpha) = [\Lambda(\min_i \alpha_i)]^2$. Using the above observations, symmetry of P^α_t and Lemma 8 with

$p = 2$ we write

$$\begin{aligned}\int_{\mathbb{R}_+^d} [g(f)(x)]^2 h(x)\, d\mu_\alpha(x) &= \int_{\mathbb{R}_+^d} \int_0^\infty t|\nabla_\alpha P_t^\alpha f(x)|^2 h(x)\, dt\, d\mu_\alpha(x) \\ &= \int_{\mathbb{R}_+^d} \int_0^\infty t\Bigg[(P_{t/2}^\alpha(\partial_t P_{t/2}^\alpha f)(x))^2 \\ &\quad + \sum_{j=1}^d (\tilde{P}_{t/2}^{\alpha,j}(\delta_j P_{t/2}^\alpha f)(x))^2 \Bigg] h(x)\, dt\, d\mu_\alpha(x) \\ &\leqslant \mathscr{C}(\alpha) \int_{\mathbb{R}_+^d} \int_0^\infty t\Bigg[P_{t/2}^\alpha([\partial_t P_{t/2}^\alpha f]^2)(x) \\ &\quad + \sum_{j=1}^d P_{t/2}^\alpha([\delta_j P_{t/2}^\alpha f]^2)(x) \Bigg] h(x)\, dt\, d\mu_\alpha(x) \\ &= \mathscr{C}(\alpha) \int_0^\infty t \int_{\mathbb{R}_+^d} |\nabla_\alpha P_{t/2}^\alpha f(x)|^2 P_{t/2}^\alpha h(x)\, d\mu_\alpha(x)\, dt \\ &= 2\mathscr{C}(\alpha) \int_0^\infty t \int_{\mathbb{R}_+^d} \mathbb{L}^\alpha [P_t^\alpha f(x)]^2 P_t^\alpha h(x)\, d\mu_\alpha(x)\, dt.\end{aligned}$$

Now, by (10) and the identity $\mathbb{L}^\alpha(P_t^\alpha h(x)) = 0$, the last double integral equals

$$\begin{aligned}&\int_0^\infty t \int_{\mathbb{R}_+^d} \mathbb{L}^\alpha[(P_t^\alpha f(x))^2 P_t^\alpha h(x)]\, d\mu_\alpha(x)\, dt \\ &\quad - 2\int_0^\infty t \int_{\mathbb{R}_+^d} \langle \nabla_\alpha P_t^\alpha h(x), \nabla_\alpha (P_t^\alpha f(x))^2 \rangle\, d\mu_\alpha(x)\, dt \equiv \mathscr{I}_1 - \mathscr{I}_2.\end{aligned}$$

In view of the estimates from the proof of Proposition 12 the integrals $\mathscr{I}_1$ and $\mathscr{I}_2$ are absolutely convergent. By Proposition 12

$$\mathscr{I}_1 \leqslant \int_{\mathbb{R}_+^d} f(x)^2 h(x)\, d\mu_\alpha(x)$$

and, by Schwarz' inequality,

$$|\mathscr{I}_2| \leqslant 4 \int_{\mathbb{R}_+^d} [(P^\alpha)^* f](x) g(f)(x) g(h)(x)\, d\mu_\alpha(x).$$

Thus,

$$\int_{\mathbb{R}^d_+} [g(f)(x)]^2 h(x)\, d\mu_\alpha(x) \leqslant 2\mathscr{C}(\alpha) \int_{\mathbb{R}^d_+} f^2(x) h(x)\, d\mu_\alpha(x) + 8\mathscr{C}(\alpha) \int_{\mathbb{R}^d_+} [(P^\alpha)^* f](x) g(f)(x) g(h)(x)\, d\mu_\alpha(x).$$

Assume that $p \geqslant 4$, $(2/p) + (1/q) = 1$ and $||h||_{L^q(d\mu_\alpha)} \leqslant 1$. By Hölder's inequality (for three functions), (3) and Theorem 6 for $q \leqslant 2$, we obtain

$$\langle g(f)^2, h \rangle_{d\mu_\alpha} \leqslant c_p \mathscr{C}(\alpha)(||f||^2_{L^p(d\mu_\alpha)} + ||g(f)||_{L^p(d\mu_\alpha)} ||f||_{L^p(d\mu_\alpha)}),$$

hence

$$||g(f)||^2_{L^p(d\mu_\alpha)} \leqslant c_p \mathscr{C}(\alpha)(||f||^2_{L^p(d\mu_\alpha)} + ||g(f)||_{L^p(d\mu_\alpha)} ||f||_{L^p(d\mu_\alpha)}).$$

This implies the desired estimate for $p \geqslant 4$. For $2 < p < 4$ the result follows by Marcinkiewicz' interpolation theorem. □

3.2. *Proof of Theorem 7*

To prove the upper bound in Theorem 7(a) one could try to adopt the reasoning used in the proof of Theorem 6, as was done by Gutiérrez [2] in the Hermite setting. Indeed, such an adaptation is (at least partially) possible, but the corresponding analysis is considerably more involved. Therefore, we prefer to invoke a general result, which immediately implies the desired inequalities.

Since for $\alpha \in [-\frac{1}{2}, \infty)^d$ the semigroups $\{\tilde{P}_t^{\alpha,i}\}$, $i = 1, \ldots, d$, form positive symmetric contraction semigroups (see Corollary 3), Theorem 7(a) is a consequence of the refinement of Stein's general Littlewood–Paley theory [12], which is due to Coifman et al. [1], see also [7, Theorem 2].

The case when $\alpha \notin [-\frac{1}{2}, \infty)^d$ seems to require more subtle treatment. Here are some details. Fix $i \in \{1, \ldots, d\}$. Notice that the function $x^{-1/2}$ belongs to $L^2(\mathbb{R}_+, d\mu_{\alpha_i+1})$ for all $\alpha_i \in (-1, \infty)$. Hence it has the Fourier–Laguerre expansion

$$x^{-1/2} = \sum_{k \in \mathbb{N}} a_k^i L_k^{\alpha_i+1}.$$

The coefficients a_k^i are computed to be (see [6, Section 4.24])

$$a_k^i = \frac{\Gamma(\alpha_i + \frac{3}{2})}{\Gamma(\alpha_i + 2)} \frac{\left(\frac{1}{2}\right)_k}{(\alpha_i + 2)_k},$$

where $(\cdot)_k$ is the Pochhammer symbol. Therefore **1** has the expansion

$$\mathbf{1} = \frac{\Gamma(\alpha_i + \frac{3}{2})}{\Gamma(\alpha_i + 2)} \sum_{k \in \mathbb{N}} \frac{(\frac{1}{2})_k}{(\alpha_i + 2)_k} \sqrt{x_i} L_k^{\alpha_i + 1},$$

the series being convergent in $L^2(\mathbb{R}_+, d\mu_{\alpha_i})$ since $(a_k^i) \in \ell^2$. It follows that

$$\tilde{T}_t^{\alpha,i}\mathbf{1} = \frac{\Gamma\left(\alpha_i + \frac{3}{2}\right)}{\Gamma(\alpha_i + 2)} \sum_{k \in \mathbb{N}} \frac{(\frac{1}{2})_k}{(\alpha_i + 2)_k} \mathrm{e}^{-t(k+1)} \sqrt{x_i} L_k^{\alpha_i + 1}.$$

Now, using suitable generating formula (see [11, Chapter 2, Section 2.5]) we obtain

$$\tilde{T}_t^{\alpha,i}\mathbf{1}(x) = \frac{\Gamma(\alpha_i + \frac{3}{2})}{\Gamma(\alpha_i + 2)} \mathrm{e}^{-t/2} \left(\frac{x_i \mathrm{e}^{-t}}{1 - \mathrm{e}^{-t}} \right)^{1/2} {}_1F_1\left(\frac{1}{2}; \alpha_i + 2; -\frac{x_i \mathrm{e}^{-t}}{1 - \mathrm{e}^{-t}} \right),$$

where ${}_1F_1$ denotes the confluent hypergeometric function. It can be shown that

$$\frac{\Gamma(\alpha_i + \frac{3}{2})}{\Gamma(\alpha_i + 2)} \sup_{z > 0} \sqrt{z}\, {}_1F_1\left(\tfrac{1}{2}; \alpha_i + 2; -z\right)$$

does not exceed 1 if $\alpha_i \geqslant -\frac{1}{2}$ and is greater than 1 if $\alpha_i < -\frac{1}{2}$. Thus, when $\alpha_i < -\frac{1}{2}$, $\tilde{T}_t^{\alpha,i}$ are not contractions on $L^\infty(\mathbb{R}_+^d)$ for t sufficiently small. Since $\|\cdot\|_{L^p(d\mu_\alpha)} \to \|\cdot\|_\infty$ as $p \to \infty$, we see that $\tilde{T}_t^{\alpha,i}$ are not contractions on $L^p(\mathbb{R}_+^d, d\mu_\alpha)$ for t sufficiently small and p sufficiently large, whenever $\alpha_i < -\frac{1}{2}$.

The above example shows, that $\alpha_i = -\frac{1}{2}$ is a "critical" point for the contraction properties of $\{\tilde{T}_t^{\alpha,i}\}$ (noteworthy, the same critical point appears in the logarithmic Sobolev inequality related to the Laguerre semigroup, see [5]). Nevertheless, it may happen that contraction properties of $\{\tilde{P}_t^{\alpha,i}\}$ are preserved for $\alpha_i < -\frac{1}{2}$. This, however, is by no means obvious and seems to require a distinct detailed analysis, which is beyond the scope of this paper. Let us also mention, that we were not able to treat the case $\alpha_i < -\frac{1}{2}$ by suitable adaptation of the proof of Theorem 6. The reason for that is, roughly speaking, that the difference

$$M_i^\alpha - \mathscr{L}^\alpha = \frac{\alpha_i + \frac{1}{2} + x_i}{2x_i}$$

is "negative" for small x_i if $\alpha_i < -\frac{1}{2}$.

The proof of Theorem 7(b) is a straightforward modification of the arguments given in [2]. We present it with details for the sake of completeness. To the end of this section we assume that $\alpha \in [-\frac{1}{2}, \infty)^d$.

Fix $i \in \{1, \dots, d\}$. If $f \in L^2(\mathbb{R}_+^d, d\mu_\alpha)$ has the expansion $f = \sum_k a_k^i(f) \sqrt{x_i} L_k^{\alpha + e_i}$, then

$$\tilde{P}_t^{\alpha,i} f = \sum_k a_k^i(f) \mathrm{e}^{-t(|k|+1)^{1/2}} \sqrt{x_i} L_k^{\alpha + e_i}$$

and therefore

$$\partial_t \tilde{P}_t^{\alpha,i} f = -\sum_k a_k^i(f)(|k|+1)^{1/2} \mathrm{e}^{-t(|k|+1)^{1/2}} \sqrt{x_i} L_k^{\alpha+e_i}.$$

Using Parseval's identity we get

$$\int_{\mathbb{R}_+^d} (\partial_t \tilde{P}_t^{\alpha,i} f(x))^2 \, d\mu_\alpha(x) = \sum_k a_k^i(f)^2 (|k|+1) \mathrm{e}^{-2t(|k|+1)^{1/2}} ||\sqrt{y_i} L_k^{\alpha+e_i}||^2_{L^2(d\mu_\alpha)}.$$

Hence, applying again Parseval's identity,

$$\begin{aligned} \int_{\mathbb{R}_+^d} [\tilde{g}_i(f)(x)]^2 \, d\mu_\alpha(x) &= \int_0^\infty t \int_{\mathbb{R}_+^d} [\partial_t \tilde{P}_t^{\alpha,i} f(x)]^2 \, d\mu_\alpha(x) \, dt \\ &= \sum_k a_k^i(f)^2 (|k|+1) ||\sqrt{y_i} L_k^{\alpha+e_i}||^2_{L^2(d\mu_\alpha)} \int_0^\infty t\mathrm{e}^{-2t(|k|+1)^{1/2}} \, dt \\ &= \frac{1}{4} \sum_k a_k^i(f)^2 ||\sqrt{y_i} L_k^{\alpha+e_i}||^2_{L^2(d\mu_\alpha)} = \frac{1}{4} ||f||^2_{L^2(d\mu_\alpha)}. \end{aligned}$$

This gives $2||\tilde{g}_i(f)||_{L^2(d\mu_\alpha)} = ||f||_{L^2(d\mu_\alpha)}$ and by polarization we obtain

$$4 \int_0^\infty \int_{\mathbb{R}_+^d} t[\partial_t \tilde{P}_t^{\alpha,i} f_1(x)][\partial_t \tilde{P}_t^{\alpha,i} f_2(x)] \, d\mu_\alpha(x) \, dt = \langle f_1, f_2 \rangle_{d\mu_\alpha} \tag{11}$$

for all real-valued $f_1, f_2 \in L^2(\mathbb{R}_+^d, d\mu_\alpha)$.

Let $F(x) = (f_1(x), \ldots, f_d(x))$ be a vector-valued function in $\mathbb{R}_+^d$. For $\xi \in \mathbb{R}^d$, $|\xi| = 1$, we define

$$F_\xi(x) = F(x) \circ \xi = \sum_{j=1}^d \xi_j f_j(x).$$

Assume that $1 < p < \infty$ and $1/p + 1/q = 1$. Let $h \in L^2(\mathbb{R}_+^d, d\mu_\alpha) \cap L^q(\mathbb{R}_+^d, d\mu_\alpha)$ be such that $||h||_{L^q(d\mu_\alpha)} \leqslant 1$. By (11) we get

$$\begin{aligned} \int_{\mathbb{R}_+^d} F_\xi(x) h(x) \, d\mu_\alpha(x) &= 4 \int_0^\infty \int_{\mathbb{R}_+^d} t \sum_{i=1}^d \xi_i [\partial_t \tilde{P}_t^{\alpha,i} f_i(x)][\partial_t \tilde{P}_t^{\alpha,i} h(x)] \, d\mu_\alpha(x) \, dt \\ &= 4 \int_0^\infty \int_{\mathbb{R}_+^d} t |V(t,x)|_{\ell^2} \cos(\xi, V(t,x)) \, d\mu_\alpha(x) \, dt, \end{aligned}$$

where

$$V(t,x) = [\partial_t \tilde{P}_t^{\alpha,i} f_i(x) \partial_t \tilde{P}_t^{\alpha,i} h(x)]_{i=1}^d.$$

Denote by σ the surface measure on the unit sphere $S^{d-1}=\{\xi\in\mathbb{R}^d : |\xi|=1\}$. Using the above and Minkowski's integral inequality we obtain

$$\begin{aligned}||\langle F_{\cdot},h\rangle_{d\mu_\alpha}||_{L^p(S^{d-1},d\sigma)} &\leqslant 4\int_{\mathbb{R}^d_+}\int_0^\infty t|V(t,x)|\ ||\cos(\cdot,V(t,x))||_{L^p(S^{d-1},d\sigma)}\,dt\,d\mu_\alpha(x)\\ &=4\mathscr{C}(d,p)\int_{\mathbb{R}^d_+}\int_0^\infty t|V(t,x)|\,dt\,d\mu_\alpha(x).\end{aligned}$$

Let $\alpha_{i_1},\dots,\alpha_{i_n}$ be all distinct values of coordinates in the multi-index α. Observe, that the last expression may be estimated from above by

$$4\mathscr{C}(d,p)\sum_{j=1}^n\int_{\mathbb{R}^d_+}\int_0^\infty t|\partial_t\tilde{P}_t^{\alpha,i_j}h(x)|\left(\sum_{i=1}^d[\partial_t\tilde{P}_t^{\alpha,i}f_i(x)]^2\right)^{1/2}dt\,d\mu_\alpha(x),$$

which, by Hölder's inequality and Theorem 7(a), is further bounded by

$$\begin{aligned}&4\mathscr{C}(d,p)\sum_{j=1}^n\int_{\mathbb{R}^d_+}\tilde{g}_{i_j}(h)(x)|\tilde{g}(F)(x)|_{\ell^2}\,d\mu_\alpha(x)\\ &\quad\leqslant 4\mathscr{C}(d,p)\sum_{j=1}^n||\tilde{g}_{i_j}(h)||_{L^q(d\mu_\alpha)}||\,|\tilde{g}(F)|_{\ell^2}||_{L^p(d\mu_\alpha)}\\ &\quad\leqslant 4c_p\mathscr{C}(d,p)n||\,|\tilde{g}(F)|_{\ell^2}||_{L^p(d\mu_\alpha)}.\end{aligned}$$

For each $\xi\in\mathbb{R}^d$, $|\xi|=1$, there exists a sequence $\{h_m^\xi\}\subset L^2(\mathbb{R}^d_+,d\mu_\alpha)\cap L^q(\mathbb{R}^d_+,d\mu_\alpha)$, $||h_m^\xi||_{L^q(d\mu_\alpha)}\leqslant 1$, such that

$$\lim_m|\langle F_\xi,h_m^\xi\rangle_{d\mu_\alpha}|=||F_\xi||_{L^p(d\mu_\alpha)}.$$

Thus, by Fatou's lemma, we get

$$\int_{|\xi|=1}||F_\xi||^p_{L^p(d\mu_\alpha)}\,d\sigma(\xi)\leqslant(4c_p\mathscr{C}(d,p)n||\,|\tilde{g}(F)|_{\ell^2}||_{L^p(d\mu_\alpha)})^p. \tag{12}$$

Now, observe that

$$|F_\xi(x)|^p=|F(x)|^p_{\ell^2}|\cos(\xi,V(t,x))|^p$$

and therefore

$$\int_{|\xi|=1}||F_\xi||^p_{L^p(d\mu_\alpha)}\,d\sigma(\xi)=\mathscr{C}(d,p)^p||\,|F|_{\ell^2}||^p_{L^p(d\mu_\alpha)}.$$

This, in view of (12), completes the proof. $\square$

4. Riesz transforms and conjugate Poisson integrals

Recall that the Riesz–Laguerre transform $\mathscr{R}^\alpha = (R_1^\alpha, \dots, R_d^\alpha)$ is formally given by

$$\mathscr{R}^\alpha = \text{grad}_\alpha (\mathscr{L}^\alpha)^{-1/2} \Pi_0,$$

which makes sense for a dense subset of $L^2(\mathbb{R}_+^d, d\mu_\alpha)$ of all polynomials, because for Laguerre polynomials we have

$$R_i^\alpha L_k^\alpha = -|k|^{-1/2} \sqrt{x_i} L_{k-e_i}^{\alpha+e_i}, \quad |k|>0,$$

and $R_i^\alpha L_k^\alpha = 0$ if $|k| = 0$. Given a suitable function f we define its *conjugate Poisson integrals* $U_t^{\alpha,i} f,\ i = 1, \dots, d,\ t>0$, by

$$U_t^{\alpha,i} f = \tilde{P}_t^{\alpha,i} R_i^\alpha f.$$

Such definition extends that given by Muckenhoupt [9] in the one dimensional setting and is well motivated by the following set of Cauchy–Riemann type equations:

$$\delta_j U_t^{\alpha,i} f = \delta_i U_t^{\alpha,j} f, \quad i,j = 1, \dots, d, \tag{13}$$

$$\delta_j P_t^\alpha f = -\partial_t U_t^{\alpha,j} f, \quad j = 1, \dots, d, \tag{14}$$

$$\sum_{j=1}^d \delta_j^* U_t^{\alpha,j} f = -\partial_t P_t^\alpha f. \tag{15}$$

Moreover,

$$\partial_t^2 U_t^{\alpha,j} f = M_j^\alpha U_t^{\alpha,j} f, \quad j = 1, \dots, d. \tag{16}$$

Some of the above equations, in one-dimensional version, may be found in [9]. Verification of (13)–(16) when f is a (Laguerre) polynomial is straightforward. Indeed, for $i = 1, \dots, d$ we have (see Section 2)

$$P_t^\alpha L_k^\alpha = e^{-t|k|^{1/2}} L_k^\alpha,$$

$$M_i^\alpha (\sqrt{x_i} L_{k-e_i}^{\alpha+e_i}) = |k| \sqrt{x_i} L_{k-e_i}^{\alpha+e_i},$$

$$\tilde{P}_t^{\alpha,i} (\sqrt{x_i} L_{k-e_i}^{\alpha+e_i}) = e^{-t|k|^{1/2}} \sqrt{x_i} L_{k-e_i}^{\alpha+e_i},$$

$$U_t^{\alpha,i} L_k^\alpha = -|k|^{-1/2} e^{-t|k|^{1/2}} \sqrt{x_i} L_{k-e_i}^{\alpha+e_i}, \quad |k|>0$$

and $U_t^{\alpha,i} L_k^\alpha = 0$ if $|k| = 0$. Now, identities (13) and (14) are easily verified with the aid of (1). To get (15) observe that $\delta_i^*(\sqrt{x_i} L_{k-e_i}^{\alpha+e_i}) = -k_i L_k^\alpha$, which follows by (1) and the fact that Laguerre polynomials L_k^α are eigenfunctions of $\mathscr{L}^\alpha$. Checking (16) makes no difficulties.

The main result of this paper is the following.

Theorem 13. *Assume that* $1 < p < \infty$ *and* $\alpha \in [-\frac{1}{2}, \infty)$. *Let* $\mathscr{A} = \#\{\alpha_i : i = 1, \dots, d\}$ *be the number of distinct coordinates in the multi-index* α. *There exists a constant* c_p (*depending neither on the dimension* d *nor on the type multi-index* α) *such that*

$$\| \, |\mathscr{R}^\alpha f|_{\ell^2} \|_{L^p(d\mu_\alpha)} \leqslant c_p \mathscr{A} \|f\|_{L^p(d\mu_\alpha)}$$

and

$$\|R_i^\alpha f\|_{L^p(d\mu_\alpha)} \leqslant c_p \|f\|_{L^p(d\mu_\alpha)}, \quad i = 1, \dots, d,$$

for all polynomials f *in* $\mathbb{R}_+^d$.

Proof. By (14) we have

$$[\partial_t \tilde{P}_t^{\alpha,1}(R_1^\alpha f), \dots, \partial_t \tilde{P}_t^{\alpha,d}(R_d^\alpha f)] = [-\delta_1 P_t^\alpha f, \dots, -\delta_d P_t^\alpha f],$$

hence

$$|\tilde{g}(\mathscr{R}^\alpha f)(x)|_{\ell^2} = |[\tilde{g}_1(R_1^\alpha f)(x), \dots, \tilde{g}_d(R_d^\alpha f)(x)]|_{\ell^2} \leqslant g(f)(x).$$

Ergo the conclusion follows by Theorems 6 and 7. □

Corollary 14. *Let* $\alpha \in [-\frac{1}{2}, \infty)$. *The operators* R_i^α *and* $U_t^{\alpha,i}$, $i = 1, \dots, d$, $t > 0$, *initially defined on a dense subset of* $L^2(\mathbb{R}_+^d, d\mu_\alpha)$, *extend uniquely to bounded linear operators in* $L^p(\mathbb{R}_+^d, d\mu_\alpha)$, $1 < p < \infty$.

Corollary 15. *Let* $1 < p < \infty$ *and* $\alpha \in [-\frac{1}{2}, \infty)$. *Then*

(a) *There exists a constant* c_p (*independent of* d *and* α) *such that*

$$\|U_t^{\alpha,i} f\|_{L^p(d\mu_\alpha)} \leqslant c_p \|f\|_{L^p(d\mu_\alpha)}, \quad t > 0, \quad i = 1, \dots, d$$

for all $f \in L^p(\mathbb{R}_+^d, d\mu_\alpha)$,

(b) *For all* $f \in L^p(\mathbb{R}_+^d, d\mu_\alpha)$ *and* $i \in \{1, \dots, d\}$

$$U_t^{\alpha,i} f \to R_i^\alpha f, \quad t \to 0^+,$$

the convergence being both in $L^p(\mathbb{R}_+^d, d\mu_\alpha)$ *and almost everywhere*,

(c) *For each* $i \in \{1, \dots, d\}$ *the family* $\{U_t^{\alpha,i}\}_{t>0}$ *is strongly continuous and uniformly bounded in* $L^p(\mathbb{R}_+^d, d\mu_\alpha)$.

Proof. Item (a) is a consequence of Theorem 13, (3) and the fact that

$$\sup_{t>0} |U_t^{\alpha,i} f(x)| \leqslant (P^\alpha)^*(R_i^\alpha f)(x), \quad x \in \mathbb{R}_+^d.$$

Statements (b) and (c) are justified by standard arguments with the aid of (a). □

Remark 16. When $p = 2$ we have

$$\| \, |\mathscr{R}^\alpha f|_{\ell^2} \|_{L^2(d\mu_\alpha)} = \|\Pi_0 f\|_{L^2(d\mu_\alpha)}, \quad f \in L^2(\mathbb{R}_+^d, d\mu_\alpha), \;\; \alpha \in (-1, \infty)^d.$$

Indeed, if $f \in L^2(\mathbb{R}_+^d, d\mu_\alpha)$ has the expansion $f = \sum_k a_k(f) L_k^\alpha$, then

$$R_i^\alpha f = -\sum_{|k|>0} a_k(f) |k|^{-1/2} \sqrt{x_i} L_{k-e_i}^{\alpha+e_i}$$

and therefore by Parseval's identity

$$\begin{aligned} \| \, |\mathscr{R}^\alpha f|_{\ell^2} \|_{L^2(d\mu_\alpha)}^2 &= \sum_{i=1}^d \sum_{|k|>0} a_k(f)^2 |k|^{-1} \| \sqrt{y_i} L_{k-e_i}^{\alpha+e_i} \|_{L^2(d\mu_\alpha)}^2 \\ &= \sum_{i=1}^d \sum_{|k|>0} a_k(f)^2 k_i |k|^{-1} \|L_k^\alpha\|_{L^2(d\mu_\alpha)}^2 \\ &= \sum_{|k|>0} a_k(f)^2 \|L_k^\alpha\|_{L^2(d\mu_\alpha)}^2 = \|\Pi_0 f\|_{L^2(d\mu_\alpha)}^2. \end{aligned}$$

Analogous computation shows that for each $i \in \{1, \ldots, d\}$ the operators $\{U_t^{\alpha,i}\}_{t>0}$ are contractions in $L^2(\mathbb{R}_+^d, d\mu_\alpha)$, $\alpha \in (-1, \infty)^d$.

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