HISTORIA MATHEMATICA 1 (1974), 333-350

REVIEWS

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ORIGINS OF MODERN ALGEBRA. By Lubos Novy. Prague (Academia), 1973. 260 p.

Reviewed by Garrett Birkhoff
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For over 40 years, the phrase "modern algebra" has kept the meaning given to it in van der Waerden's famous book. Unlike "classical algebra," modern algebra is *not* primarily concerned with the real or complex field, or even with "numbers"; it is much more general, treating sets, permutations, linear transformations (matrices), etc. Its form is abstract and axiomatic.

Professor Novy's monograph constitutes a pioneer effort to trace the historical origins of modern algebra from 1770, when Lagrange first considered groups of permutations, to the publication in 1871 of the second edition of Dirichlet-Dedekind's *Zahlentheorie*. During the intervening century, much of the *content* of "modern" algebra was discovered, and the interests of algebraists and number theorists changed greatly; Novy's concern is with this change. However, in appraising his book, we should remember that the axiomatic *form* of modern algebra came well after 1871 [1].

Professor Novy puts the situation somewhat differently: "...the years 1770-1870 represent the first stage of the development of modern algebra, when structural algebraic thinking was established" (p. 4). In writing this, like Bourbaki, he is presumably using the word "structure" to mean algebraic system.

He also seems to use "realm" as synonymous with *Bereich*, which this reviewer would translate as "domain." Since Novy uses the words "structure" and "realm" very often, these remarks are worth keeping in mind.

Professor Novy's excellent bibliography includes 411 references: research articles, textbooks, and surveys by other historians of mathematics. A serious omission is F. Cajori, 1928-1929, *A History of Mathematical Notations*, Chicago (Open Court), 2 vols. From his references, he tries to extract the most essential conclusions. I shall try here to review briefly a few of the most interesting and important.

First, in his Introduction, Novy singles out three main "milestones" in the evolution of algebra during the period 1770-1840: the work of Waring, Lagrange and Vandermonde around 1770; that of Ruffini and Gauss about 1800; and that of Abel and Galois near 1830. He then comments on historical analyses by E.T. Bell, M.J. Crowe, R. Graves, G.A. Miller, V.H. Molodshii, J Vuillemin, H. Weber, H. Wussing and others which treat algebraic developments since 1770. Most of these are restricted to special topics or individual mathematicians. (Somewhat inconsistently, his comments on Felix Klein's analysis of algebraic number theory appear in a footnote on p. 93.)

The next chapter deals with eighteenth century algebra; Novy's general conclusion is that "From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange, Vandermonde and Waring in the years 1770-71, the evolution of algebra was... hardly dramatic" (p. 23). He then goes on to qualify this sweeping statement, which must be interpreted in the limited sense of *conceptual* changes in the objectives of algebra, as indeed his own frequent references to contributions by Newton, Leibniz and Euler show. (See Novy's author index; his book unfortunately has no subject index.)

Chapter 3 on "Solvability of equations" takes up the influence of Lagrange, Waring and Vandermonde, whose ideas were widely disseminated in Lacroix's popular algebra texts, which went through many editions in the years 1799-1863. The most "modern" idea was probably Lagrange's correlation of the radicals used to solve cubic and quartic equations with subgroups of permutations of the roots, by means of Euler's "resolvents." This inspired Ruffini's investigations of 1799-1802, and was the starting point of the truly revolutionary "Galois theory" invented by Galois around 1830 in very obscure (if excited) language [2].

The best contemporary German texts were those of Martin Ohm, especially his *Versuch Eines Vollkommenen Consequenten Systems der Mathematik* (1828-29). It is hard to reconcile Novy's criticism of Ohm (p. 70-71) for "failing to understand the more modern trends in algebra" with his later statement (p. 83-92, esp. p. 87) that Ohm's pioneer use of axioms in algebra was "appreciated by Peacock" (p. 89, ftnt. 18). Ohm also influenced Bolzano's work on the foundations of algebra (p. 90-92), whose influence however seems to have been minimal. Finally, one can speculate that Ohm's emphasis on the distinction between *Zahlenlehre* and *Grössenlehre* may have influenced Helmholtz's emphasis on *Zahlen* and *Messen* as the two most basic mathematical skills from a psychological standpoint.

These discussions are near the beginning of a long chapter on "the structure of numerical realms." In the next section (Section 4-3), this takes the reader through the *Disquisitiones Arithmeticae* (1801) and *Residuorum Biquadraticorum* (1825, 1831) of Gauss, and the pioneer work of Kummer on ideals, all the way to Dedekind's *Supplement X* to the second edition of Dirichlet-Dedekind. In this section, perhaps most interesting is Novy's observation that "It is now hard to say who was the first to point out the non-uniqueness of the factorization into prime factors.... It is... certain that Jacobi publically pointed out that a rational prime p can be factored in different ways into a product of different primes in a suitably chosen [domain]... not later than the beginning of 1839" (p. 104). It was, of course, this which led to the development of ideal theory.

Section 4-5 traces the origins (to 1871) of the concept of a field (*Körper*) as it evolved from work of virtually the same German mathematicians, and from the duality between subgroups and subfields of Galois theory. Actually, the work first appeared in Dedekind's *Supplement X* to the second (1871) edition of Dirichlet-Dedekind, and even in the third (1881) edition [3], it is still defined to mean a *subfield* of the complex field. In 1882, Dronecker used *Rationalitätsbereich* in the same sense, and *Integritätsbereich* to mean subdomain. The modern meaning of field and integral domain did not come for at least another decade, until the axiomatic approach became popular.

Between Sections 4-3 and 4-5, Novy sandwiches a review of the origins of modern ideas about the complex field $\mathbb{C}$. He begins with the suggestive representation of $\mathbb{C}$ as a plane, discovered independently by Wessel (1797) and Argand (1806), and utilized with such power by Cauchy and Riemann. He then reviews other philosophical ideas, including Cauchy's use (1847) of residue classes to construct $\mathbb{C}$ as the quotient-ring $\mathbb{R}[x]/(x^2 + 1)$. Hamilton's abstract treatment (1833-35) of complex numbers as ordered pairs of real numbers is finally taken as a model, perhaps because it generalizes so directly to the notion of a hypercomplex number system (see below).

The next chapter deals with "structures" of "untraditional" realms, presumably meaning those *not* embedded in $\mathbb{C}$. It begins with an inconclusive review of early work on permutations and combinations, which can be regarded as antecedents to the modern concepts of group and set (or subset) respectively. It continues with a review of the use by Lagrange and Gauss of the notion of being *congruent* mod n. (This underlies Cauchy's isomorphism $\mathbb{C} \simeq \mathbb{R}[x]/(x^2 + 1)$, mentioned above.)

In the same section, he also discusses Gauss' composition of quadratic forms, and extensions by Dirichlet and Kummer of this complicated construction. He omits to note that Dirichlet-Dedekind actually give priority to Lagrange's earlier notion of the "equivalence" of quadratic forms. What is even more interesting, on p. 168 of Dirichlet-Dedekind it is pointed out for the first time that Kummer's ideals are just congruence modules for the ring of integers, thus binding together for all time the concepts of unique factorization and congruence.

Novy then turns to the origins of the related concepts of determinant, vector, matrix, and hypercomplex number systems. He reminds us that "Cramer's Rule" was discovered first by Maclaurin, and briefly reviews the work on determinants by Laplace, Vandermonde and Gauss. He notes that it was not until the middle 1850's that Cayley defined the algebra of matrices, more than a decade after Hamilton had introduced quaternions, shortly after which Graves and Cayley discovered "octaves" (now called Cayley numbers). De Morgan and Graves had joined Hamilton in the search for such a "hypercomplex" number system, but without success. (De Morgan's efforts are reviewed in detail by Novy in his short Chapter 6; see below.) Novy notes that the contemporary efforts of Grassman (1844) to establish what we today recognize as vector algebra were so obscure as to have little influence. Novy might have mentioned that Grassman's second (1862) edition was much more influential. In this reviewer's opinion, he should also have emphasized the fundamental 1870 paper of Benjamin Peirce on "linear associative algebras." Even the founding by Willard Gibbs of a "practical vector analysis" in 1881-84 was contemporary with Kronecker's definition of *Integritätsbereich*.

However, these minor criticisms should not distract attention from the great value of Professor Novy's scholarly book, considered as a whole. It contains by far the best available study of the development of "modern" ideas about algebra in the century 1770-1870. If the picture it gives is not too coherent, this is probably because the algebraists of that period did not realize their mission: to pave the way for the ideas of van der Waerden and Bourbaki. Had they realized this more clearly, they would doubtless have proceeded in a much more orderly manner, and this orderliness would have been reflected in Professor Novy's book!

NOTES

1. See G. Birkhoff, 1973, "Current trends in algebra," *Am Math Monthly* 80, 760-782, esp. ¶4. On Bourbaki, see *ibid*, ¶13.
2. See also G. Birkhoff, 1937, *Osiris* 3, 260-267, which also

escaped Novy's bibliography.

3. Here it became Supplement XI. See the *Geleitwort* by van der Waerden to Richard Dedekind *Über die Theorie der Ganzen Algebraische Zahlen*, Braunschweig (Vieweg), 1964.

MATHEMATICIANS FROM ANTIQUITY TO TODAY. A PRELIMINARY EDITION. VOL. I. By J. Fang, in collaboration with U. Dudley. Hauppauge, New York (Paideia Press), 1972. 341 p. $12.80.

Reviewed by Kurt-R. Biermann
Berlin, DDR

Dem Mathematik-Historiker, der sich über biographische Einzelheiten informieren will, haben sich in den letzten Jahren eine Reihe von Nachschlagemöglichkeiten eröffnet, die ihm das Auffinden der jeweils benötigten Details erleichtern; genannt seien hier nur das *Mathematiker-Lexikon* von Herbert Meschkowski (Mannheim und Zürich, Bibliographisches Institut, 1968), [1] das *Lexikon der Geschichte der Naturwissenschaften*, begründet von Josef Mayerhöfer (Wien, Hollinek, seit 1959, bisher: Aa... bis Dod...), das von Fritz Krafft und Adolf Meyer-Abich besorgte *Biographische Lexikon Grosser Naturwissenschaftler* (Frankfurt a.M., Fischer, 1970), der Supplement-Band zu Band VIIa von J.C. Poggendorffs *Biogr.-literar. Handwörterbuch der exakten Naturwissenschaften* (Berlin DDR, Akademie-Verlag, 1971) und das unter Leitung von Charles C. Gillispie herausgegebene *Dictionary of Scientific Biography (DSB)* (New York, Charles Scribner's Sons, seit 1970, bisher Bd. I bis VIII. Es stellt sich daher die Frage, ob ein Bedürfnis nach einem weiteren biographischen Auskunftsmittel, wie es das Werk von Fang darstellt, vorhanden ist. Die Antwort hierauf wird positiv ausfallen, wenn ein Vergleich mit dem inhaltsreichsten (unter Mitarbeit von Fachleuten aus vielen Ländern entstandenen) der genannten Werke, dem DSB, ergibt, dass Fang zusätzliche Informationen darbietet.

Eine Stichprobe zeigt folgendes: Von A.C. Clairaut bis K. Culmann gibt Fang 50 Biographien. Davon entfallen 19 Artikel auf z.Zt. der Redigierung lebende Personen, deren Berücksichtigung der Editionsplan des DSB von vornherein ausgeschlossen hat. Fang bringt weiter neun Biographien, die im DSB fehlen. Andererseits enthält DSB im betrachteten Abschnitt des Alphabets sieben Lebensläufe, die bei Fang fehlen. Ich glaube, es ist also ersichtlich, dass das Fangsche Nachschlagewerk eine willkommene Ergänzung darstellt, wenn auch seine Angaben an Umfang und Aussagekraft nicht mit dem DSB [2] zu vergleichen sind, handelt es sich doch bei ihm um ausgesprochene Kurzbiographien. [3] Der Autor hat seinem Buch umfangreiche Vorbemerkungen [4]