On Starlike Harmonic Functions

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Abstract

Uniformly starlike univalent functions introduced by Goodman and we develop this idea over harmonic functions. We introduce a subclass of harmonic univalent functions which are fully starlike and uniformly starlike also. In the following we will mention some examples of this subclass and obtain two necessary and sufficient conditions, one with the inequality form and other with convolution.

Keywords: Uniformly starlike function, Fully-Starlike function, Harmonic function, Convolution.

Mathematics Subject Classification (2010): Primary 30C45; Secondary 31C05, 31A05.

1 Introduction and Preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is open unit disk in complex plane. Let $\mathcal{A}$ be the familier class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are univalent in the open unit disk $\mathbb{D}$. Let $\mathcal{S}$ denotes the family of functions $f(z)$ in form (1) that are univalent in $\mathbb{D}$. An analytic function $f(z)$ is said to be starlike if it's range is starlike with respect to the origin. In geometric view of the range of $f(z)$, this means that every point of the range can be connected to the origin by a radial line that lies entirely in the region. In the other words, $arg\{f(e^{i\theta})\}$ will be a nondecreasing function of θ, or that

$$\frac{\partial}{\partial\theta} arg\{f(e^{i\theta})\} \geq 0$$

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The class of all starlike functions in $\mathcal{S}$ is shown by $\mathcal{S}^*$ [8]. Furthermore $f(z) \in \mathcal{S}^*$ iff $\mathbf{Re}\{z\dfrac{f'(z)}{f(z)}\} > 0$. Starlikeness is a hereditary property for conformal mappings, so if $f(z) \in \mathcal{S}$, and if f maps $\mathbb{D}$ onto a domain that is starlike with respect to the origin, then the image of every subdisk $|z| < r < 1$ is also starlike with respect to the origin.

The following subclass of uniformly starlike functions, $\mathcal{UST}$ introduced by Goodman [5] and studied in analytic and geometric view.

Definition 1.1. *A function $f(z) \in \mathcal{S}^*$ is said to be uniformly starlike in $\mathbb{D}$ if it has the property that for every circular arc γ contained in $\mathbb{D}$, with center $\zeta \in \mathbb{D}$, the arc $f(\gamma)$ be starlike with respect to $f(\zeta)$. We denote the family of all uniformly starlike functions by $\mathcal{UST}$ and we have [5],*

$$\mathcal{UST} = \left\{ f(z) \in \mathcal{S} : \mathbf{Re}\,\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} > 0\ , \quad (z,\zeta) \in \mathbb{D}^2 \right\} \tag{2}$$

It's clear that $\mathcal{UST} \subset \mathcal{S}^*$. Properties of the class $\mathcal{UST}$ are difficult to establish. Some examples for class $\mathcal{UST}$ are $f(z) = \dfrac{z}{1-Az}$ with $|A| \leq \dfrac{\sqrt{2}}{2}$, and more examples are rare.

The complex-valued function $f(x,y) = u(x,y) + iv(x,y)$ is *complex-valued harmonic function* in $\mathbb{D}$ if f is continuous and u and v are real harmonic in $\mathbb{D}$. Let H denote the family of continuous complex-valued functions which are harmonic in the open unit disk $\mathbb{D}$. In simply-connected domain $\mathbb{D}$, $f \in H$ has a canonical representation $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ [3, 4]. Then, g and h can be expanded in Taylor series as $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, so we may represent f by a power series of the form

$$f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^n} \tag{3}$$

The Jacobian of a function $f = u + iv$ is $J_f(z) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = |h'(z)|^2 - |g'(z)|^2$, and $f(z) = h(z) + \overline{g(z)}$ is sense-preserving if $J_f(z) > 0$. In 1984, Clunie and Sheil-Small [3] investigated the class S_H, consisting of complex-valued harmonic sense-preserving univalent functions $f(z) = h(z) + \overline{g(z)}$ in simply connected domain $\mathbb{D}$ which normalized by $f(0) = 0$ and $f_z(0) = 1$ with the form,

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \tag{4}$$

The subclass $\mathcal{S}_H^0$ of $\mathcal{S}_H$ includes all functions $f \in \mathcal{S}_H$ with $f_{\overline{z}}(0) = 1$, so $\mathcal{S} \subset \mathcal{S}_H^0 \subset \mathcal{S}_H$. Clunie and Sheil-Small also considered starlike functions in S_H, denote by $\mathcal{S}_H^*$. The subclass of all starlike functions in $\mathcal{S}_H^0$ denote by $\mathcal{S}_H^{0*}$. Starlikeness isn't a hereditary property for harmonic mappings, so the image of every subdisk $|z| < r < 1$ is not necessarily starlike with respect to the origin [4, 1]. Thus we need a property to explain starlikeness of a map in a hereditary form. We have folowing definition.

Definition 1.2. *[4] A harmonic mapping f with $f(0) = 0$ is said to be fully-starlike if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve that bounds a domain starlike with respect to the origin.*

For $f \in \mathcal{S}_H$, the family of fully-starlike functions denotes by $\mathcal{FS}_H^*$. In 1980 Mocanu gave a relation between fully-starlikeness and a differential operator of a non-analytic function [6]. Let

$$Df = zf_z - \overline{z}f_{\overline{z}} \tag{5}$$

be the differential operator and

$$D^2f = D(Df) = zf_z + \overline{z}f_{\overline{z}} + zzf_{zz} + \overline{zz}f_{\overline{zz}} \tag{6}$$

Lemma 1.1. *[6] Let $f \in C^1(\mathbb{D})$ is a complex-valued function such that $f(0) = 0$, $f(z) \neq 0$ for all $z \in \mathbb{D} - \{0\}$, and $J_f(z) > 0$ in $\mathbb{D}$ and $\mathbf{Re}\,\dfrac{Df(z)}{f(z)} > 0$ then f is univalent and fully-starlike in $\mathbb{D}$.*

Since for a sense-preserving complex-valued function $f(z)$, $Df \neq 0$, If $f(z) \in \mathcal{S}_H$ and satisfies condition such as $\mathbf{Re}\,\dfrac{Df(z)}{f(z)} > 0$ or $\mathbf{Re}\,\dfrac{D^2f(z)}{Df(z)} > 0$ for all $z \in \mathbb{D} - \{0\}$, then f maps every circle $0 < |z| = r < 1$ onto a simple closed curve [6]. This raised by

$$\frac{\partial}{\partial\theta} arg\{f(e^{i\theta})\} = \mathbf{Re}\,\frac{Df(z)}{f(z)} = \mathbf{Re}\,\frac{zh'(z) - \overline{zg'(z)}}{f(z)}$$

However, a fully-starlike mapping need not be univalent [2], we restrict our discussion to $\mathcal{S}_H$ class.

2 Main Results

For every circular arc $\gamma = \{z = \zeta + re^{i\theta}\}$ contained in $\mathbb{D}$, with center $\zeta \in \mathbb{D}$, the arc $f(\gamma) = \{f(z) : z \in \gamma\}$ is starlike with respect to $f(\zeta)$ iff

$$\frac{\partial}{\partial\theta} \arg\{f(z) - f(\zeta)\} \geq 0$$

and this yield iff

$$\mathbf{Im}\frac{\frac{\partial}{\partial\theta}f(z)}{f(z)-f(\zeta)}\geq 0$$

for z on γ. Set $z=\zeta+re^{i\theta}$, then $\dfrac{\partial}{\partial\theta}z=i(z-\zeta)$ and a brief computation gives

$$\mathbf{Re}\,\frac{(z-\zeta)f_z(z)-\overline{(z-\zeta)}f_{\overline{z}}(z)}{f(z)-f(\zeta)}>0$$

For a harmonic funtion $f(z)=h(z)+\overline{g(z)}\in\mathcal{S}_H$, the operator

$$\begin{aligned}\mathbf{D}f(z,\zeta) &= (z-\zeta)f_z(z)-\overline{(z-\zeta)}f_{\overline{z}}(z)\\ &= (z-\zeta)h'(z)-\overline{(z-\zeta)g'(z)}\end{aligned} \tag{7}$$

is harmonic also. For $\zeta=0$ the operator $\mathbf{D}f(z,0)=zf_z-\overline{z}f_{\overline{z}}=zh'-\overline{zg'}=Df(z)$ is previous operator (5). Like the definition 1.1 for analytic functions, we say that for harmonic functions

Definition 2.1. *A function $f(z)\in\mathcal{S}_H$ is said to be uniformly fully-starlike in $\mathbb{D}$ if it has the property that for every circular arc γ contained in $\mathbb{D}$, with center $\zeta\in\mathbb{D}$, the arc $f(\gamma)$ is starlike with respect to $f(\zeta)$, that is*

$$\mathcal{UFS}_H^*=\left\{f(z)\in\mathcal{S}_H:\mathbf{Re}\,\frac{\boldsymbol{D}f(z,\zeta)}{f(z)-f(\zeta)}>0\ ,\quad (z,\zeta)\in\mathbb{D}^2\right\} \tag{8}$$

It's simple that one check the rotations, $e^{-i\alpha}f(e^{i\alpha}z)$ for some real α, are preserve the $\mathcal{UFS}_H^*$ class. Other property that preserves this class is transformation $\dfrac{1}{t}f(tz)$, $0<t\leq 1$. $\mathcal{UFS}_H^*\subset\mathcal{FS}_H^*\subset\mathcal{S}_H^*$. With simplifying (8) and some terms of limiting, we obtain significant results:

Corollary 2.1. *It's clear that if $f\in\mathcal{UST}$ be an analytic function, then $f\in\mathcal{UFS}_H^*$. So, $\mathcal{UST}\subset\mathcal{UFS}_H^*$ and if any function does not satisfy (2), then it isn't in $\mathcal{UFS}_H^*$. Goodman [5] shows the analytic function $f(z)=\dfrac{z}{1-Az}\in\mathcal{UST}$ iff $|A|\leq\dfrac{\sqrt{2}}{2}$, thus the convex function $f(z)=\dfrac{z}{1-z}\notin\mathcal{UFS}_H^*$.*

Corollary 2.2. *For $\zeta=0$ in (8), the harmonic function $f\in\mathcal{UFS}_H^*$ will be univalent and fully-starlike in $\mathbb{D}$ by Lemma 1.1. Thus it's clear any non fully-starlike harmonic function is not in $\mathcal{UFS}_H^*$. The harmonic function $f(z)=\mathbf{Re}\,\dfrac{z}{(1-z)^2}+i\mathbf{Im}\dfrac{z}{1-z}$ isn't fully-starlike [4], then $f\notin\mathcal{UFS}_H^*$.*

Example 2.1. *For $|\beta|<1$ the affine mappings $f(z)=z+\overline{\beta z}\in\mathcal{UFS}_H^*$, since*

$$\mathbf{Re}\,\frac{(z-\zeta)-\overline{(z-\zeta)\beta}}{(z-\zeta)+\overline{(z-\zeta)\beta}}\geq 0$$

is equivalent to

$$\mathbf{Re}\left((z-\zeta)-\overline{(z-\zeta)\beta}\right)\left(\overline{(z-\zeta)}+(z-\zeta)\beta\right)\geq 0$$

that is $(1-|\beta|^2)|z-\zeta|^2 \geq 0$.

In the following we will give a necessary and sufficient condition for that $f \in \mathcal{UFS}_H^*$. This condition is a generalization form of a theorem about fully-starlike functions mentioned by Chuaqui et al. in [2], p139.

Theorem 2.3. *Let* $f(z) \in \mathcal{S}_H$, $f \in \mathcal{UFS}_H^*$ *iff*

$$\begin{aligned}&|h(z)-h(\zeta)|^2\mathbf{Re}\, Q_h \geq \\ &|g(z)-g(\zeta)|^2\mathbf{Re}\, Q_g + \mathbf{Re}\left\{\left(h(z)-h(\zeta)\right)\left(g(z)-g(\zeta)\right)\left(Q_g-Q_h\right)\right\}\end{aligned} \tag{9}$$

where $Q_h = \dfrac{(z-\zeta)h'(z)}{h(z)-h(\zeta)}$ *and* $Q_g = \dfrac{(z-\zeta)g'(z)}{g(z)-g(\zeta)}$ *for* (z,ζ) *in polydisk* $\mathbb{D}^2$.

Proof. According to the definition, $f \in \mathcal{UFS}_H^*$ iff $\mathbf{Re}\,\dfrac{\mathbf{D}f(z,\zeta)}{f(z)-f(\zeta)} > 0$ for $(z,\zeta) \in \mathbb{D}^2$, iff $\mathbf{Re}\,(\mathbf{D}f(z,\zeta))(\overline{f(z)}-\overline{f(\zeta)}) > 0$ for $(z,\zeta) \in \mathbb{D}^2$, then the simple calculation gives us (9). □

Example 2.2. *For* $h \in \mathcal{UST}$ *and* $g = \beta h$ *with* $|\beta| < 1$, *then* $f = h + \overline{g} \in \mathcal{UFS}_H^*$ *because in this case,* h *and* g *satisfy in* $Q_h = Q_g$ *so* (9) *holds. By this we may say* $f(z) = z + Az^2 + \overline{\beta z + \beta A z^2} \in \mathcal{UFS}_H^*$ *with* $|\beta| < 1$ *and* $|A| < \dfrac{\sqrt{2}}{2}$.

In last example, let $A = \dfrac{1}{2}$, $\beta = -\dfrac{i}{2}$ then $f = z + \dfrac{1}{2}z^2 \overline{-\dfrac{i}{2}z - \dfrac{i}{4}z^2} \in \mathcal{UFS}_H^*$. In Figure 1, a circle $|z + \dfrac{1-i}{2}| < \dfrac{1}{5}$ is mapped under this uniformly fully-starlike function to a starlike elliptical shape with center $f(\zeta) = (-0.37, 0)$.

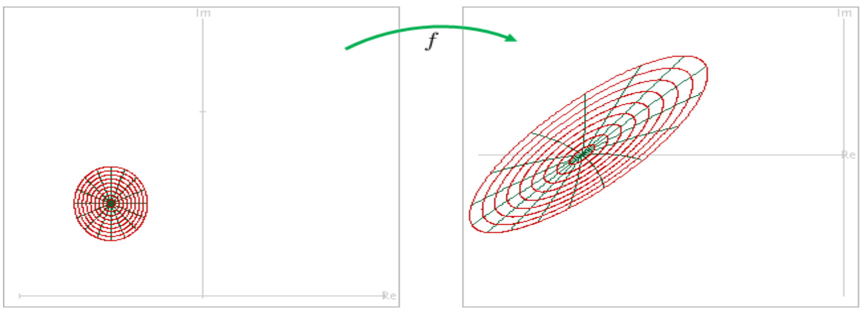

Figure 1: The image of $|z + \frac{1-i}{2}| < \frac{1}{5}$ under $f = z + \frac{1}{2}z^2 \overline{-\frac{i}{2}z - \frac{i}{4}z^2} \in \mathcal{UFS}_H^*$.

Corollary 2.4. *For fixed $\zeta \in \mathbb{D}$, set $z = 0$ in (8), then the harmonic function $f \in \mathcal{UFS}_H^*$ satisfies* $\textbf{Re}\, \dfrac{\zeta - \overline{\zeta} b_1}{f(\zeta)} > 0$. *Furthermore if $f \in \mathcal{UFS}_H^{*\,0} = \mathcal{S}_H^0 \cap \mathcal{UFS}_H^*$ then* $\textbf{Re}\, \dfrac{f(z)}{z} > 0$ *in $\mathbb{D}$.*

3 Convolution

The convolution or Hadamard product of two harmonic functions $f(z)$ and $F(z)$ with canonical representations

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} \overline{z}^n \tag{10}$$

and

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n} \overline{z}^n \tag{11}$$

is defined as

$$(f * F)(z) = (h * H)(z) + \overline{g * G(z)} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \overline{z}^n \tag{12}$$

The right half-plane mapping $\ell(z) = \dfrac{z}{1-z}$ acts as the convolution identity and the Koebe map $k(z) = \dfrac{z}{(1-z)^2}$ acts as derivative operation over functions convolution. We have some properties for convolution over analytic functions f and g:

$$\begin{aligned} &f * g = g * f \quad , \quad \alpha(f * g) = \alpha f * g \\ &f * \ell = f \quad\quad\;\; , \quad z f'(z) = f * k(z) \end{aligned}$$

where $\alpha \in \mathbb{C}$. For a given subset $\mathcal{V} \subset \mathcal{A}$, it's dual set $\mathcal{V}^*$ is defined by

$$\mathcal{V}^* = \left\{ g \in \mathcal{A} : \frac{f * g(z)}{z} \neq 0, \ \forall f \in \mathcal{V}, \ \forall z \in \mathbb{D} \right\} \tag{13}$$

Nezhmetdinov (1997) proved that class $\mathcal{UST}$ is dual set for certain family of analytic functions. He showed ([7], Theorem 2, p.43) that the class $\mathcal{UST}$ is the dual set of a subset of $\mathcal{A}$ consist of all functions $\varphi : \mathbb{D} \to \mathbb{C}$ given by

$$\varphi(z) = \frac{z}{(1-z)^2(1-wz)} \left(1 - \frac{w + i\alpha}{1 + i\alpha} z \right) \tag{14}$$

where $\alpha \in \mathbb{R}$, $w \in \mathbb{C}$ and $|w| = 1$. He determined the uniform estimate $|a_n(\varphi)| \leq dn$ for the n-th Taylor coefficient of $\varphi(z)$ with a sharp constant $d = \sqrt{M} \approx 1.2557$, where $M \approx 1.5770$ is the maximum value of a certain trigonometric expression. Using this, he showed that

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{1}{\sqrt{M}} \Longrightarrow f \in \mathcal{UST}$$

The bound $\dfrac{1}{\sqrt{M}}$ is sharp. Nezhmetdinov obtained a dual set for class $\mathcal{UST}$:

Lemma 3.1. *[7] Let G is all function φ of the form (14), then $\mathcal{UST} = G^*$ and $c_n = \sup_{\varphi \in G} |a_n| \leq dn$ for all $n \geq 2$, with the sharp constant $d = \sqrt{M} = 1.2557\ldots$ where $M = S(\theta_0) = 1.5770\ldots$ is the maxmimal value of*

$$S(\theta) = \frac{1}{2}\Big[1 + (\frac{\sin\theta}{\theta})^2 + \sqrt{\Big(1 + (\frac{\sin\theta}{\theta})^2\Big)^2 - (\frac{\sin 2\theta}{\theta})^2}\Big] \tag{15}$$

on $0 \leq \theta \leq \pi$. Here the extremal point $\theta_0 = 0.9958\ldots$ is the unique solution of the equation

$$\theta^3(\cos\theta + \cos 3\theta) - \theta^2 \sin 3\theta + \sin^3\theta = 0 \tag{16}$$

on the segment $0.8 \leq \theta \leq 1.3$.

To obtain a sufficient condition in class $\mathcal{UFS}_H^*$, we define the dual set of a family of harmonic functions. Let $\mathcal{A}_H$ be the class of complex-valued harmonic functions $f(z) = h(z) + \overline{g(z)}$ in simply connected domain $\mathbb{D}$ of the form (4) which not necessarily sense-preserving univalent on $\mathbb{D}$. We define the dual set of a subset of $\mathcal{A}_H$:

Definition 3.1. *For a given subset $\mathcal{V}_H \subset \mathcal{A}_H$, the dual set $\mathcal{V}_H^*$ is*

$$\mathcal{V}_H^* = \Big\{F = H + \overline{G} \in \mathcal{A}_H : \frac{h * H}{z} + \frac{\overline{g * G}}{\overline{z}} \neq 0,\ \forall f = h + \overline{g} \in \mathcal{V}_H,\ \forall z \in \mathbb{D}\Big\} \tag{17}$$

Theorem 3.1. *Let*

$$G_H = \{\varphi - \sigma\overline{\varphi} : \varphi(z) = \frac{z}{(1-z)^2(1-wz)}\Big(1 - \frac{w + i\alpha}{1 + i\alpha}z\Big), \sigma = \frac{\overline{(1-w)(1+i\alpha)}}{(1-w)(1+i\alpha)}, z \in \mathbb{D}\} \tag{18}$$

then $\mathcal{UFS}_H^* = G_H^*$. Futhermore If $\sum_{n=2}^{\infty} n|a_n| + n|b_n| < \dfrac{1-|b_1|}{\sqrt{M}}$ then $f \in \mathcal{UFS}_H^*$, where the constant $\sqrt{M} = 1.2557\ldots$ mentioned in lemma 3.1.

It's clear that the analytic function φ is the same (14), but σ with $|\sigma| = 1$ isn't an arbitrary number and depend on both w and α in φ.

Proof. Let $f = h + \overline{g} \in \mathcal{UFS}_H^*$, that is

$$\mathbf{Re}\,\frac{(z-\zeta)h'(z) - \overline{(z-\zeta)g'(z)}}{h(z) + \overline{g(z)} - h(\zeta) - \overline{g(\zeta)}} > 0\ , \quad (z,\zeta) \in \mathbb{D}^2 \tag{19}$$

For $\zeta = 0$ and then $z = 0$ we have $\dfrac{(z-\zeta)h'(z) - \overline{(z-\zeta)g'(z)}}{f(z) - f(\zeta)} = 1$, hence the condition (19) may be write

$$(z-\zeta)h'(z) - \overline{(z-\zeta)g'(z)} \neq i\alpha\Big(h(z) + \overline{g(z)} - h(\zeta) - \overline{g(\zeta)}\Big) \tag{20}$$

where $\alpha \in \mathbb{R}$. By the minimum principle for harmonic functions, it is sufficient to verify this condition for $|z| = |\zeta|$ and so, we may assume that $\zeta = wz$ with $|w| = 1$, then from the definition of the dual set for harmonic functions (17), with simple calculation we conclude $\dfrac{h * \varphi}{z} - \sigma\dfrac{\overline{g * \varphi}}{\overline{z}} \neq 0$, so the first assertion follows.

For the remaining coeficients condition, Let $f(z) = h(z) + \overline{g(z)}$ is of the form (10), and $\varphi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ be the series expansion of analytic function $\varphi(z)$, then $|\phi_n| \leq dn$ for all $n \geq 2$, with the sharp constant $d = \sqrt{M} = 1.2557\ldots$ mentioned in lemma 3.1. From previous part

$$\begin{aligned}
\left|\frac{h * \varphi}{z} - \sigma\frac{\overline{g * \varphi}}{\overline{z}}\right| &= \left|1 + \sum_{n=2}^{\infty} a_n\phi_n z^{n-1} - \sigma\big(b_1 + \sum_{n=2}^{\infty} \overline{b_n\phi_n}\,\overline{z}^{n-1}\big)\right| \\
&\geq |1 - \sigma b_1| - \sum_{n=2}^{\infty} |a_n||\phi_n||z|^{n-1} - |\sigma| \sum_{n=2}^{\infty} |b_n||\phi_n||z|^{n-1} \\
&\geq |1 - \sigma b_1| - \sum_{n=2}^{\infty} |a_n| dn - \sum_{n=2}^{\infty} |b_n| dn \\
&> 0
\end{aligned}$$

when $\sum_{n=2}^{\infty} n|a_n| + n|b_n| < \dfrac{1 - |b_1|}{\sqrt{M}}$. □

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