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On Fully-Convex Harmonic Functions and their Extension

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abstract: Uniformly convex univalent functions that introduced by Goodman, maps every circular arc contained in the open unit disk with center in it into a convex curve. On the other hand, a fully-convex harmonic function, maps each subdisk $|z| = r < 1$ onto a convex curve. Here we synthesis these two ideas and introduce a family of univalent harmonic functions which are fully-convex and uniformly convex also. In the following we will mention some examples of this subclass and obtain a necessary and sufficient conditions and finally a coefficient condition is given as an aplication of some convolution results.

Key Words: Uniformly convex function, Fully-Convex function, Harmonic function, Convolution.

Contents

1. Introduction and Preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in complex plane. Let $\mathcal{A}$ be the familier class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disk $\mathbb{D}$. Let $\mathcal{S}$ denotes the family of all functions $f(z)$ of the form (1.1) that are univalent in $\mathbb{D}$ and normalized with $f(0) = 0$ and $f'(0) = 1$.

A conformal function $f(z)$ is said to be starlike if every point of its range can be connected to the origin by a radial line that lies entirely in that region. The class of all starlike functions in $\mathcal{S}$ is shown by $\mathcal{S}^*$ [9] and $f(z) \in \mathcal{S}^*$ if and only if $\mathbf{Re}\{z\dfrac{f'(z)}{f(z)}\} > 0$. Starlikeness is a hereditary property for conformal mappings, so if $f(z) \in \mathcal{S}$, and if f maps $\mathbb{D}$ onto a domain that is starlike with respect to the origin, then the image of every subdisk $|z| < r < 1$ is also starlike with respect to the origin.

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An analytic function $f(z)$ is said to be convex if its range $f(\mathbb{D})$ is a convex set. It has shown that every convex function f in $\mathcal{S}$ satisfy following analytic property

$$\mathbf{Re}\{1+z\frac{f''(z)}{f'(z)}\}>0$$

The class of all convex functions in $\mathcal{S}$ is denoted by $\mathcal{K}$ [9].

The subclass of uniformly starlike functions, $\mathcal{UST}$ introduced by Goodman [6] and studied in analytic and geometric view.

Definition 1.1. *[6] A function $f(z)\in\mathcal{S}^*$ is said to be uniformly starlike in $\mathbb{D}$ if it has the property that for every circular arc γ contained in $\mathbb{D}$, with center $\zeta\in\mathbb{D}$, the arc $f(\gamma)$ be starlike with respect to $f(\zeta)$. We denote the family of all uniformly starlike functions by $\mathcal{UST}$ and we have,*

$$\mathcal{UST}=\left\{f(z)\in\mathcal{S}:\mathbf{Re}\,\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}>0\ ,\ (z,\zeta)\in\mathbb{D}^2\right\} \tag{1.2}$$

It's clear that $\mathcal{UST}\subset\mathcal{S}^*$ and every function in $\mathcal{UST}$ maps each subdisk $\{|z-\zeta|<\rho\}\subset\mathbb{D}$ onto a domain starlike with respect to $f(\zeta)$. Goodman [5] also defined the subclass of convex functions with this property that map each disk $\{|z-\zeta|<\rho\}\subset\mathbb{D}$ onto a convex domain and called it uniformly convex function and denoted the set of all these functions by $\mathcal{UCV}$:

Definition 1.2. *[5] A function $f(z)\in\mathcal{K}$ is said to be uniformly convex in $\mathbb{D}$ if it has the property that for every circular arc γ contained in $\mathbb{D}$, with center $\zeta\in\mathbb{D}$, the arc $f(\gamma)$ be a convex arc. We have,*

$$\mathcal{UCV}=\left\{f(z)\in\mathcal{S}:\mathbf{Re}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right)\geq 0\ ,\ (z,\zeta)\in\mathbb{D}^2\right\} \tag{1.3}$$

A summary of early works on uniformly starlike and uniformly convex functions can be found in [10].

The complex-valued function $f(x,y)=u(x,y)+iv(x,y)$ is complex-valued harmonic function in $\mathbb{D}$ if f is continuous and u and v are real harmonic in $\mathbb{D}$. We denote H the family of continuous complex-valued functions which are harmonic in the open unit disk $\mathbb{D}$. In simply-connected domain $\mathbb{D}$, $f\in H$ has a canonical representation $f=h+\overline{g}$, where h and g are analytic in $\mathbb{D}$ [3,4]. Then, g and h have expansions in Taylor series as $h(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$, so we may represent f by a power series of the form

$$f(z)=h(z)+\overline{g(z)}=\sum_{n=0}^{\infty}a_nz^n+\overline{\sum_{n=0}^{\infty}b_nz^n} \tag{1.4}$$

The Jacobian of a function $f=u+iv$ is $J_f(z)=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}=|h'(z)|^2-|g'(z)|^2$, and $f(z)=h(z)+\overline{g(z)}$ is sense-preserving if $J_f(z)>0$. In 1984, Clunie and Sheil-Small

[3] investigated the class S_H, consisting of sense-preserving univalent harmonic functions $f(z) = h(z) + \overline{g(z)}$ in simply-connected domain $\mathbb{D}$ which normalized by $f(0) = 0$ and $f_z(0) = 1$ with the form,

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \tag{1.5}$$

The subclass $\mathcal{S}_H^0$ of $\mathcal{S}_H$ includes all functions $f \in \mathcal{S}_H$ with $f_{\overline{z}}(0) = 0$, so $\mathcal{S} \subset \mathcal{S}_H^0 \subset \mathcal{S}_H$. Clunie and Sheil-Small also considered convex functions in S_H, denoted by $\mathcal{K}_H$. The hereditary property of convexity for conformal maps does not generalize to univalent harmonic mappings. If f is a univalent harmonic map of $\mathbb{D}$ onto a convex domain, then the image of the disk $|z| < r$ is convex for each radius $r \leq \sqrt{2} - 1$, but not necessarily for any radius in the interval $\sqrt{2} - 1 < r < 1$. In fact, the function

$$\begin{aligned} f(z) &= \mathbf{Re}\,\frac{z}{1-z} + i\mathbf{Im}\frac{z}{(1-z)^2} \\ &= \frac{z - \frac{1}{2}z^2}{(1-z)^2} + \frac{-\frac{1}{2}\overline{z}^2}{(1-\overline{z})^2} \in \mathcal{K}_H \end{aligned} \tag{1.6}$$

is a harmonic mapping of the disk $\mathbb{D}$ onto the half-plane $\mathbf{Re}\, w > -\frac{1}{2}$, but the image of the disk $|z| \leq r$ fails to be convex for every r in the interval $\sqrt{2} - 1 < r < 1$ [4]. Thus we need a property to explain convexity of a map in a hereditary form in whole disk. We have following definition.

Definition 1.3. *[2] A harmonic mapping f with $f(0) = 0$ of the unit disk is said to be fully-convex if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a convex curve.*

For $f \in \mathcal{S}_H$, the family of fully-convex harmonic functions denotes by $\mathcal{FK}_H$. In 1980 Mocanu gave a relation between fully-starlikeness and a differential operator of a non-analytic function [7]. Let

$$Df = zf_z - \overline{z}f_{\overline{z}} \tag{1.7}$$

be the differential operator and

$$D^2 f = D(Df) = zzf_{zz} + \overline{zz}f_{\overline{zz}} + zf_z + \overline{z}f_{\overline{z}} \tag{1.8}$$

Lemma 1.4. *[7] Let $f \in C^1(\mathbb{D})$ is a complex-valued function such that $f(0) = 0$, $f(z) \neq 0$ for all $z \in \mathbb{D} - \{0\}$, and $J_f(z) > 0$ in $\mathbb{D}$ and* $\mathbf{Re}\,\dfrac{Df(z)}{f(z)} > 0$ *then f is univalent and fully-starlike in $\mathbb{D}$.*

Lemma 1.5. *Let $f \in C^2(\mathbb{D})$ is a complex-valued function such that $f(0) = 0$, $f(z) \neq 0$ for all $z \in \mathbb{D} - \{0\}$, and $J_f(z) > 0$ in $\mathbb{D}$ and* $\mathbf{Re}\,\dfrac{D^2 f(z)}{Df(z)} > 0$ *then f is univalent and fully-convex in $\mathbb{D}$.*

Since for a sense-preserving complex-valued function $f(z)$, $Df \neq 0$, If $f(z) \in \mathcal{S}_H$ and satisfies condition such as $\mathbf{Re}\,\dfrac{Df(z)}{f(z)} > 0$ or $\mathbf{Re}\,\dfrac{D^2f(z)}{Df(z)} > 0$ for all $z \in \mathbb{D} - \{0\}$, then f maps every circle $0 < |z| = r < 1$ onto a simple closed curve [7]. However, a fully-starlike mapping need not be univalent [2], we restrict our discussion to $\mathcal{S}_H$.

2. Definition and Examples

For a harmonic function $f(z) = h(z) + \overline{g(z)} \in \mathcal{S}_H$, and $\zeta \in \mathbb{D}$ we define the operator

$$\begin{aligned} \mathbf{D}f(z,\zeta) &= (z-\zeta)f_z(z) - \overline{(z-\zeta)}f_{\overline{z}}(z) \\ &= (z-\zeta)h'(z) - \overline{(z-\zeta)g'(z)} \end{aligned} \tag{2.1}$$

is harmonic also. For $\zeta = 0$ the operator $\mathbf{D}f(z,0) = zf_z - \overline{z}f_{\overline{z}} = zh' - \overline{zg'} = Df(z)$ is previous operator (1.7). Differentiating of the operator $\mathbf{D}f(z,\zeta)$ gives us

$$\begin{aligned} \mathbf{D}^2f(z,\zeta) &= \mathbf{D}(\mathbf{D}f(z,\zeta)) \\ &= \mathbf{D}((z-\zeta)h'(z) - \overline{(z-\zeta)g'(z)}) \\ &= (z-\zeta)^2h''(z) + \overline{(z-\zeta)^2g''(z)} + (z-\zeta)h'(z) \\ &\quad + \overline{(z-\zeta)g'(z)} \end{aligned} \tag{2.2}$$

For $\zeta = 0$ the operator $\mathbf{D}^2f(z,0) = z^2h''(z) + \overline{z^2g''(z)} + zh'(z) + \overline{zg'(z)} = D^2f(z)$ has described by Al-Amiri and Mocanu [1]. Similar to definition (1.1) we say that for an arbitrary function:

Definition 2.1. *A function $f \in \mathcal{S}_H$ is said to be uniformly fully-convex harmonic function in $\mathbb{D}$ if it has the property that for every circular arc γ contained in $\mathbb{D}$, with center $\zeta \in \mathbb{D}$, the arc $f(\gamma)$ is convex in $f(\mathbb{D})$.*

We denote the set of all uniformly fully-convex harmonic functions in $\mathbb{D}$ by $\mathcal{UFK}_H$. The following theorem gives analytic equivalency for above definition:

Theorem 2.2. *Let $f \in \mathcal{S}_H$. $f \in \mathcal{UFK}_H$ if and only if*

$$\mathbf{Re}\,\frac{\boldsymbol{D}^2f(z,\zeta)}{\boldsymbol{D}f(z,\zeta)} > 0\ ,\ \ (z,\zeta) \in \mathbb{D}^2 \tag{2.3}$$

Proof: Let $\gamma : \zeta + re^{i\theta}$ with $\theta_1 \leq \theta \leq \theta_2$ be a circular arc centered at ζ and contained in $\mathbb{D}$, then the image of γ under f is convex if the argument of the tangent to the image be a non-decreasing function of θ, that is,

$$\frac{\partial}{\partial\theta}\Big(\arg\frac{\partial}{\partial\theta}\{f(z) - f(\zeta)\}\Big) \geq 0$$

Hence

$$\mathbf{Im}\frac{\partial}{\partial\theta}\Big(\log\frac{\partial}{\partial\theta}\{f(z) - f(\zeta)\}\Big) \geq 0$$

But for a circular *arc* γ, set $z=\zeta+re^{i\theta}$, then $\frac{\partial}{\partial\theta}z=i(z-\zeta)$ and a brief computation will give us

$$\frac{\partial}{\partial\theta}\{f(z)-f(\zeta)\}=i\Big\{(z-\zeta)f_z(z)-\overline{(z-\zeta)}f_{\overline{z}}(z)\Big\}=i\,\mathbf{D}f(z,\zeta)$$

then

$$\begin{aligned}\frac{\partial}{\partial\theta}\log i\,\mathbf{D}f(z,\zeta) &= \frac{\partial}{\partial\theta}\log i\Big\{(z-\zeta)h'(z)-\overline{(z-\zeta)g'(z)}\Big\}\\ &= \frac{i\big[h'(z)+(z-\zeta)h''(z)\big]}{i\,\mathbf{D}f(z,\zeta)}i(z-\zeta)\\ &- \frac{i\overline{\big[g'(z)+(z-\zeta)g''(z)\big]}}{i\,\mathbf{D}f(z,\zeta)}\overline{i(z-\zeta)}\\ &= i\frac{\mathbf{D}^2f(z,\zeta)}{\mathbf{D}f(z,\zeta)}\end{aligned}$$

Therefore, we must have

$$\mathbf{Im}\frac{\partial}{\partial\theta}\log i\,\mathbf{D}f(z,\zeta) = \mathbf{Re}\,\frac{\mathbf{D}^2f(z,\zeta)}{\mathbf{D}f(z,\zeta)}\geq 0$$

as we want. □

It should be noted that $\frac{\mathbf{D}^2f(z,\zeta)}{\mathbf{D}f(z,\zeta)}(0,0)=1$, and

$$\mathcal{UFK}_H=\Big\{f(z)\in\mathcal{S}_H:\mathbf{Re}\,\frac{\mathbf{D}^2f(z,\zeta)}{\mathbf{D}f(z,\zeta)}>0\;,\;(z,\zeta)\in\mathbb{D}^2\Big\} \tag{2.4}$$

It's simple that one checks the rotations, $e^{-i\alpha}f(e^{i\alpha}z)$ for some real α, are preserve the class $\mathcal{UFK}_H$ and the transformation $\frac{1}{t}f(tz)$ preserves this class also, where $0<t\leq 1$. On the other hand, the class $\mathcal{UFK}_H$ includes all fully-convex functions and uniformly convex functions. With $g=0$ in (2.3), the analytic function $f(z)\in\mathcal{UFK}_H$ by (2.1) and (2.2) satisfies condition

$$\mathbf{Re}\,\frac{\mathbf{D}^2f(z,\zeta)}{\mathbf{D}f(z,\zeta)}=\mathbf{Re}\,\frac{(z-\zeta)^2h''(z)+(z-\zeta)h'(z)}{(z-\zeta)h'(z)}=\mathbf{Re}\Big(1+(z-\zeta)\frac{h''(z)}{h'(z)}\Big)\geq 0$$

where $(z,\zeta)\in\mathbb{D}^2$. Then

Corollary 2.3. *If $f\in\mathcal{UCV}$ be an analytic function, then $f\in\mathcal{UFK}_H$. So, $\mathcal{UCV}\subset\mathcal{UFK}_H\subset\mathcal{K}_H$. Goodman [5] shows the analytic function $f(z)=\frac{z}{1-Az}\in\mathcal{UCV}$ if and only if $|A|\leq\frac{1}{3}$, thus the convex function $f(z)=\frac{z}{1-z}\notin\mathcal{UFK}_H$.*

Example 2.1. For $|\beta| < 1$ the affine mappings $f(z) = z + \overline{\beta z} \in \mathcal{UFK}_H$, since

$$\mathbf{Re}\,\frac{(z-\zeta)+\overline{(z-\zeta)\beta}}{(z-\zeta)-\overline{(z-\zeta)\beta}} \geq 0$$

is equivalent to

$$\mathbf{Re}\Big((z-\zeta)+\overline{(z-\zeta)\beta}\Big)\Big(\overline{(z-\zeta)}-(z-\zeta)\beta\Big) \geq 0$$

that is $(1-|\beta|^2)|z-\zeta|^2 \geq 0$.

Corollary 2.4. *For $\zeta = 0$ in (2.3), the harmonic function $f \in \mathcal{UFK}_H$ will be univalent and fully-convex in $\mathbb{D}$ by Lemma 1.5. Thus it's clear any non fully-convex harmonic function is not in $\mathcal{UFK}_H$. The harmonic function $f(z) = \mathbf{Re}\,\frac{z}{1-z} + i\mathbf{Im}\frac{z}{(1-z)^2}$ isn't fully-convex ([4], p.46), then $f \notin \mathcal{UFK}_H$.*

In the following we will give a necessary and sufficient condition for that $f \in \mathcal{UFK}_H$. This condition is a generalization form of a theorem about fully-convex functions mentioned by Chuaqui et al. in [2], p139.

Theorem 2.5. *Let $f(z) \in \mathcal{S}_H$, $f \in \mathcal{UFK}_H$ if and only if*

$$\begin{aligned} &|(z-\zeta)h'(z)|^2\mathbf{Re}\,Q_h \geq \\ &|(z-\zeta)g'(z)|^2\mathbf{Re}\,Q_g + \mathbf{Re}\Big\{(z-\zeta)^3\big(h''(z)g'(z)-h'(z)g''(z)\big)\Big\} \end{aligned} \tag{2.5}$$

where $Q_h = 1+(z-\zeta)\dfrac{h''(z)}{h'(z)}$ and $Q_g = 1+(z-\zeta)\dfrac{g''(z)}{g'(z)}$ for (z,ζ) in polydisk $\mathbb{D}^2$.

Proof: According to the definition, $f \in \mathcal{UFK}_H$ if and only if $\mathbf{Re}\,\dfrac{\mathbf{D}^2 f(z,\zeta)}{\mathbf{D}f(z,\zeta)} > 0$ for $(z,\zeta) \in \mathbb{D}^2$, if and only if $\mathbf{Re}\Big\{\mathbf{D}^2 f(z,\zeta)\overline{\mathbf{D}f(z,\zeta)}\Big\} > 0$ for $(z,\zeta) \in \mathbb{D}^2$, then a simple calculation gives us (2.5). □

Lemma 2.6. *$f = h + \overline{\beta h} \in \mathcal{UFK}_H$ if and only if $h \in \mathcal{UCV}$, where $|\beta| < 1$.*

Proof: Let $f = h + \overline{g} \in \mathcal{S}_H$ and $g = \beta h$ with $|\beta| < 1$, then $f \in \mathcal{UFK}_H$ if and only if (2.5) holds. Since in this case, h and g satisfy equality $Q_h = Q_g$ so (2.5) holds if and only if $|(z-\zeta)h'(z)|^2\mathbf{Re}\,Q_h(1-|\beta|^2) \geq 0$, or $\mathbf{Re}\,Q_h \geq 0$ that shows $h \in \mathcal{UCV}$. □

Example 2.2. The analytic function $h = z + Az^2$ is in $\mathcal{UCV}$ if and only if $|A| \leq \frac{1}{6}$ [5]. By Lemma 2.6 we get $f(z) = z + Az^2 + \overline{\beta z + \beta A z^2} \in \mathcal{UFK}_H$ with $|\beta| < 1$ and $|A| \leq \frac{1}{6}$. For example, let $A = \frac{1}{6}$, $\beta = -\frac{i}{2}$ then $f = z + \frac{1}{6}z^2 - \overline{\frac{i}{2}z - \frac{i}{12}z^2} \in \mathcal{UFK}_H$. In Figure 1, the disk $|z-0.7| < 0.3$ is mapped under this uniformly fully-convex harmonic function to a convex elliptical shape with center $f(\zeta) = (0.78, 0.39)$.

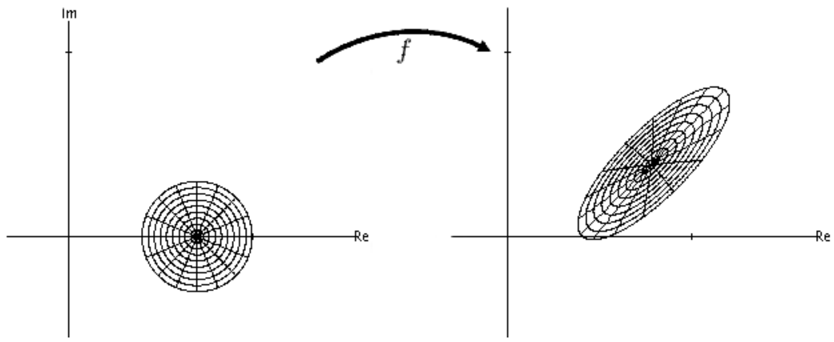

Figure 1: The image of $|z-0.7|<0.3$ under $f = z + \frac{1}{6}z^2 \overline{-\frac{i}{2}z - \frac{i}{12}z^2} \in \mathcal{UFK}_H$.

3. Convolution and a sufficient condition

The convolution or Hadamard product of two harmonic functions $f(z)$ and $F(z)$ with canonical representations

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n}\overline{z}^n \tag{3.1}$$

and

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n}\overline{z}^n \tag{3.2}$$

is defined as

$$(f * F)(z) = (h * H)(z) + \overline{g * G(z)} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n}\overline{z}^n \tag{3.3}$$

The right half-plane mapping $\ell(z) = \dfrac{z}{1-z}$ acts as the convolution identity and the Koebe map $k(z) = \dfrac{z}{(1-z)^2}$ acts as derivative operation over functions convolution. We have some properties for convolution over analytic functions f and g:

$$\begin{array}{lcl} f * g = g * f & , & \alpha(f * g) = \alpha f * g \\ f * \ell = f & , & z f'(z) = f * k(z) \end{array}$$

where $\alpha \in \mathbb{C}$. For a given subset $\mathcal{V} \subset \mathcal{A}$, its dual set $\mathcal{V}^*$ is defined by

$$\mathcal{V}^* = \left\{ g \in \mathcal{A} : \frac{f * g(z)}{z} \neq 0, \ \forall f \in \mathcal{V}, \ \forall z \in \mathbb{D} \right\} \tag{3.4}$$

Nezhmetdinov (1997) proved that class $\mathcal{UCV}$ is dual set for certain family of functions from $\mathcal{A}$. He proved ([8], Theorem 2, p.43) that the class $\mathcal{UCV}$ is the dual set of a subset of $\mathcal{A}$ consisting of functions $\varphi : \mathbb{D} \to \mathbb{C}$ given by

$$\varphi(z) = \frac{z}{(1-z)^3}\Big[1 - z - \frac{4z}{(\alpha+i)^2}\Big] \tag{3.5}$$

where $\alpha \in \mathbb{R}$. He determined the uniform estimate $|a_n(\varphi)| \leq n(2n-1)$ for the n-th Taylor coefficient of $\varphi(z)$:

Lemma 3.1. *[8] Let G is all function $\varphi \in \mathcal{A}$ of the form (3.5), then $\mathcal{UCV} = G^*$ and $|a_n(\varphi)| \leq n(2n-1)$ for all $n \geq 2$.*

For obtaining a sufficient condition in class $\mathcal{UFK}_H$, we define the dual set of a harmonic function. Let $\mathcal{A}_H$ be the class of complex-valued harmonic functions $f(z) = h(z) + \overline{g(z)}$ in simply connected domain $\mathbb{D}$ of the form (1.5) which are not necessarily sense-preserving univalent on $\mathbb{D}$. We define the dual set of a subset of $\mathcal{A}_H$:

Definition 3.2. *For a given subset $\mathcal{V}_H \subset \mathcal{A}_H$, the dual set $\mathcal{V}_H^*$ is*

$$\mathcal{V}_H^* = \Big\{F = H + \overline{G} \in \mathcal{A}_H : \frac{h*H}{z} + \frac{\overline{g*G}}{\overline{z}} \neq 0,\ \forall f = h + \overline{g} \in \mathcal{V}_H,\ \forall z \in \mathbb{D}\Big\} \tag{3.6}$$

Theorem 3.3. *Let $\alpha \in \mathbb{R}$, $|w| = 1$ and*

$$\begin{aligned} G_H &= \Big\{\varphi - \sigma\overline{\varphi} : \varphi(z) = \frac{z}{(1-z)^3}\Big(1 - \frac{w - i\alpha}{2 - w - i\alpha}z\Big), \\ &\qquad \sigma = \frac{\overline{(1-w)(2-w-i\alpha)}}{(1-w)(2-w-i\alpha)}, z \in \mathbb{D}\Big\} \end{aligned}$$

then $\mathcal{UFK}_H = G_H^*$. Furthermore If $\sum_{n=2}^{\infty} n(2n-1)|a_n| + n(2n-1)|b_n| < 1 - |b_1|$ then $f \in \mathcal{UFK}_H$.

It's clear that the analytic function φ is the same (3.5), but σ with $|\sigma| = 1$ isn't an arbitrary number and depend on both w and α in φ.

Proof: Let $f = h + \overline{g} \in \mathcal{UFK}_H$, that is

$$\mathbf{Re}\,\frac{(z-\zeta)^2h''(z) + \overline{(z-\zeta)^2g''(z)} + (z-\zeta)h'(z) + \overline{(z-\zeta)g'(z)}}{(z-\zeta)h'(z) - \overline{(z-\zeta)g'(z)}} > 0\ , \tag{3.7}$$

$(z,\zeta) \in \mathbb{D}^2$. For $\zeta = 0$ and then $z = 0$ we have $\dfrac{\mathbf{D}^2 f(z,\zeta)}{\mathbf{D}f(z,\zeta)} = 1$, hence the condition (3.7) may be write as

$$\begin{aligned} i\alpha\Big((z-\zeta)h'(z) - \overline{(z-\zeta)g'(z)}\Big) &\neq (z-\zeta)^2h''(z) + \overline{(z-\zeta)^2g''(z)} \\ &\quad + (z-\zeta)h'(z) + \overline{(z-\zeta)g'(z)} \end{aligned}$$

where $\alpha \in \mathbb{R}$. By the minimum principle for harmonic functions, it is sufficient to verify this condition for $|z| = |\zeta|$ and so, we may assume that $\zeta = wz$ with $|w| = 1$, then from the definition of the dual set for harmonic functions (3.6), with straightforward calculation we conclude that $\frac{h * \varphi}{z} + \sigma \frac{\overline{g * \varphi}}{\overline{z}} \neq 0$, so the first assertion follows.

For obtaining coefficients condition, let $f(z) = h(z) + \overline{g(z)}$ is of the form (3.1), and $\varphi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ be the series expansion of analytic function $\varphi(z)$, then $|\phi_n| \leq n(2n-1)$ for all $n \geq 2$, by Lemma 3.1. From previous part we see that

$$\begin{aligned} \left| \frac{h * \varphi}{z} + \sigma \frac{\overline{g * \varphi}}{\overline{z}} \right| &= \left| 1 + \sum_{n=2}^{\infty} a_n \phi_n z^{n-1} + \sigma \big(b_1 + \sum_{n=2}^{\infty} \overline{b_n \phi_n} \overline{z}^{n-1} \big) \right| \\ &\geq |1 + \sigma b_1| - \sum_{n=2}^{\infty} |a_n||\phi_n||z|^{n-1} - |\sigma| \sum_{n=2}^{\infty} |b_n||\phi_n||z|^{n-1} \\ &\geq |1 + \sigma b_1| - \sum_{n=2}^{\infty} n(2n-1)|a_n| - \sum_{n=2}^{\infty} n(2n-1)|b_n| \\ &> 0 \end{aligned}$$

when $\sum_{n=2}^{\infty} n(2n-1)|a_n| + n(2n-1)|b_n| < 1 - |b_1|$. □

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