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Classes of Uniformly Convex and Uniformly Starlike Functions as Dual Sets

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In this paper the classes of uniformly convex and uniformly starlike functions are presented as dual sets for certain function families (in the sense of convolution theory). The results are used to find some sharp sufficient conditions for functions, regular in the unit disk, to belong to the above classes.

1. INTRODUCTION

Denote by $\mathcal{A}$ the class of normalized functions $f(z) = z + \sum_{n=2}^{\infty} a_n(f)z^n$, regular in the unit disk $E = \{z : |z| < 1\}$. Consider also its subclasses S, ST, CV, consisting of the univalent, starlike, and convex functions, respectively. It is well known that for any $f \in ST$ not only $f(E)$ but the images of all circles centered at 0 and lying in E are starlike with respect to 0. B. Pinchuk posed a question whether this property is still valid for circles centered at other points of E. A. W. Goodman [1] gave a negative answer to this question and introduced the class UST of uniformly starlike functions $f \in ST$ such that for any circular arc γ lying in E and having the center at $\zeta \in E$ the image $f(\gamma)$ is starlike with respect to $f(\zeta)$. A necessary and sufficient condition for $f \in \mathcal{A}$ to be uniformly starlike takes the form

$$\operatorname{Re}\big[(z-\zeta)f'(z)/(f(z)-f(\zeta))\big] > 0, \qquad \forall z, \zeta \in E. \tag{1}$$

In a similar way, the class UCV of uniformly convex functions is defined to include the functions $f \in S$ such that any circular arc γ lying in E with the center $\zeta \in E$ is carried by f into a convex arc. A. W. Goodman [2]

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stated the criterion

$$\mathrm{Re}\left[1 + (z - \zeta) f''(z)/f'(z)\right] > 0, \qquad \forall z, \zeta \in E \Leftrightarrow f \in UCV. \quad (2)$$

Later F. Rønning (and independently W. Ma and D. Minda) [3] obtained a more suitable form of the criterion, namely

$$|zf''(z)/f'(z)| < \mathrm{Re}\left[1 + zf''(z)/f'(z)\right], \qquad \forall z \in E \Leftrightarrow f \in UCV. \quad (3)$$

In the survey [3] it is observed that the study of the properties of UST is impeded by the absence of a simpler counterpart of the condition (1). In the present paper we deduce criteria for f to lie in UST or UCV stated in terms of dual sets (see [4]).

For any $f, g \in \mathsf{A}$ define its convolution (or Hadamard product) as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n(f) a_n(g) z^n. \quad (4)$$

We follow [4] to introduce for any set $V \subset \mathsf{A}$ its dual set

$$V^* = \{g \in \mathsf{A} : (f * g)(z)/z \neq 0, \forall f \in V, \forall z \in E\}. \quad (5)$$

We show that both classes UST and UCV are dual sets for certain families of functions from A. In particular, these results can be used to obtain sharp forms of sufficient conditions for f to be uniformly convex or starlike.

2. UNIFORMLY CONVEX FUNCTIONS

THEOREM 1. *Define the function family* $\mathsf{G} = \{g \in \mathsf{A} : g(z) = z[1 - z - 4z/(\alpha + i)^2]/(1 - z)^3$, *where* $\alpha \in \mathbf{R}\}$. *Then* $UCV = \mathsf{G}^*$, *and in addition we have* $|a_n(g)| \leq n(2n - 1)$ *for all* $g \in \mathsf{G}$, $n \geq 2$.

Proof. The condition (3) is equivalent to the fact that the values of $w = u + iv = 1 + zf''(z)/f'(z), \forall z \in E$, lie in the domain bounded by the parabola $v^2 + 1 = 2u$. Clearly, it can be written as

$$1 + zf''(z)/f'(z) \neq (\alpha^2 + 1)/2 + \alpha i, \qquad \forall z \in E, \forall \alpha \in \mathbf{R}.$$

Hence, by the definition of the dual set, the first assertion of the theorem follows.

For any $g \in \mathsf{G}$ the coefficients can be written as $a_n(g) = n(\alpha^2 - 2n + 1 + 2i\alpha)/(\alpha + i)^2$. Consider the expression

$$|a_n(g)|^2 = n^2\left[(\alpha^2 - 2n + 1)^2 + 4\alpha^2\right]/(\alpha^2 + 1)^2 = B(t),$$

where $B(t) = t^{-2}n^2(t^2 - 4nt + 4n^2 + 4t - 4)$, $t = \alpha^2 + 1 \geq 1$. Then the function $B(t)$ decreases for $1 \leq t \leq 2n + 2$ and increases for $2n + 2 \leq t < \infty$, besides $B(1) = n^2(2n - 1)^2 > \lim_{t \to \infty} B(t) = n^2$. Thus we have the required estimate which is attained at $\alpha = 0$ for all $n \geq 2$ simultaneously.

COROLLARY 1. *Let $f(z) = z + cz^n$, $n \geq 2$. Then*

$$f \in UCV \Leftrightarrow |c| \leq 1/n(2n - 1).$$

Proof. If $f(z) = z + cz^n$, where $|c| \leq 1/n(2n - 1)$, then for any $g \in \mathsf{G}$ we get

$$|(f * g)(z)/z| = \left|1 + ca_n(g)z^{n-1}\right| \geq 1 - |ca_n(g)z| \geq 1 - |z| > 0,$$

and, by Theorem 1, $f \in UCV$. Conversely, suppose that $f(z) = z + cz^n \in UCV$. Consider $g_0(z) = \sum_{n=1}^{\infty} n(2n - 1)z^n \in \mathsf{G}$, so that $(f * g_0)(z)/z = 1 + cn(2n - 1)z^{n-1}$. Obviously, for $|c| > 1/n(2n - 1)$ there is a $\zeta \in E$ such that $(f * g_0)(\zeta)/\zeta = 0$, and, consequently, $f \notin UCV$.

Remark 1. In particular, when $n = 2$, this implies Lemma 2 from [3].

COROLLARY 2. *If $f \in \mathsf{A}$ satisfies the condition*

$$\sum_{n=2}^{\infty} n(2n - 1)|a_n(f)| \leq 1, \tag{6}$$

then $f \in UCV$. The constant 1 *on the right-hand side is the best possible.*

The condition (6) is verified directly by Theorem 1, whereas Corollary 1 implies the sharpness of the constant.

Remark 2. This result was earlier obtained in [5, Theorem 1], where the authors used quite an elementary estimate and verified the sharpness of the condition for the functions with negative coefficients.

We can also deduce a sufficient condition for $f \in \mathsf{A}$ to be in UCV of the form $\sum_{n=2}^{\infty} n^2|a_n(f)| \leq 1/2$, with the sharp constant $1/2$. It is of interest to compare this condition with a well-known convexity condition $\sum_{n=2}^{\infty} n^2|a_n(f)| \leq 1$ (see [6]).

3. UNIFORMLY STARLIKE FUNCTIONS

THEOREM 2. *Let* $H = \{h(z) \in A : h(z) = z[1 - (\omega + i\alpha)z/(1 + i\alpha)]/(1 - \omega z)(1 - z)^2$, *where* $\alpha \in \mathbf{R}$, $|\omega| = 1\}$. *Then* $UST = H^*$. *In addition, the uniform estimate* $C_n = \sup\{|a_n(h)| : h \in H\} \le dn$ *holds for all* $n \ge 2$ *with the sharp constant* $d = \sqrt{M} = 1.2557\ldots$, *where* $M = S(\theta_0) = 1.5770\ldots$ *is the maximal value of*

$$S(\theta) = (1/2)\left[1 + \theta^{-2}\sin^2\theta + \sqrt{(1 + \theta^{-2}\sin^2\theta)^2 - \theta^{-2}\sin^2 2\theta}\right]$$

on $0 \le \theta \le \pi$. *Here the extremal point* $\theta_0 = 0.9958\ldots$ *is the unique solution of the equation*

$$\theta^3(\cos\theta + \cos 3\theta) - \theta^2 \sin 3\theta + \sin^3\theta = 0,$$

on the segment $0.8 \le \theta \le 1.3$.

LEMMA 1. *Given any* a, $0 \le a \le 1$, *the function* $Q(x) = 1 + x + \sqrt{(1+x)^2 - 4ax}$ *increases on* $0 \le x \le 1$.

To prove the lemma, just observe that for any fixed $x \in [0, 1]$ the derivative $Q'(x) = 1 + (1 + x - 2a)/\sqrt{(1+x)^2 - 4ax}$ attains its minimum on $0 \le a \le 1$ at $a = 1$ and, therefore, is nonnegative.

LEMMA 2. *The function* $x^{-1}\sin x$ *decreases on the interval* $0 < x \le \pi$.

The proof follows by usual differentiation and in view of the inequality $0 < \cos x \le x^{-1}\sin x$, $0 < x \le \pi/2$.

Proof of Theorem 2. Let us write the criterion (1) of uniform starlikeness in the form

$$(z - \zeta)f'(z) \ne -i\alpha(f(z) - f(\zeta)), \qquad \forall\alpha \in \mathbf{R}, \forall z, \zeta \in E\ (z \ne \zeta).$$

As it is observed in [3], by the minimum principle it is sufficient to verify the condition for $|z| = |\zeta|$; therefore, we may assume that $\zeta = \omega z$, $|\omega| = 1$. Hence from the definition of the dual set the first assertion immediately follows.

Now, by setting $\omega = \exp(2i\tau)$, $-\pi/2 \le \tau \le \pi/2$, we write

$$\left|a_n(h)^2\right| = (A\alpha^2 + B\alpha + C)/(\alpha^2 + 1) = F(\alpha),$$

where $A = (\sin n\tau/\sin\tau)^2$, $B = -2n\sin n\tau\sin(n-1)\tau/\sin\tau$, $C = n^2$. Let us maximize $F(\alpha)$ for all $\alpha \in \mathbf{R}$. Note that $\lim_{\alpha\to\infty} F(\alpha) = A \le n^2$ for any τ. In the same way it is easy to see that for $B = 0$ we have

$F(\alpha) \le n^2$. Otherwise, if $B \neq 0$, then differentiation yields $F'(\alpha) = [-B\alpha^2 + 2(A - C)\alpha + B](\alpha^2 + 1)^{-2}$, whence the maximal value of $F(\alpha)$ on the whole line $-\infty < \alpha < \infty$ is attained for $\alpha = \alpha_0 = (A - C + \sqrt{D})/B$, where $D = (A - C)^2 + B^2$, and equals $F(\alpha_0) = (A + C + \sqrt{D})/2$.

Thus, $C_n/n \le \sqrt{\max_{0 \le \tau \le \pi/2} R_n(\tau)}$, where

$$R_n(\tau) = (1/2)\Big\{1 + (\sin n\tau/n \sin \tau)^2 + \sqrt{\left[1 + (\sin n\tau/n\sin\tau)^2\right]^2 - 4\left[\sin n\tau \cos(n-1)\tau/n \sin\tau\right]^2}\Big\}.$$

Note that $R_n(0) = 1$. Consider the case $0 < \tau \le \pi/(n-1)$. Then, setting $\theta = (n-1)\tau$, we get

$$\begin{aligned} 0 &< \sin n\tau/n \sin\tau = \sin(\theta + \tau)/n \sin\tau = (\cos\theta + \sin\theta \cot\tau)/n \\ &< (\cos\theta + \tau^{-1} \sin\theta)/n \le \theta^{-1}\sin\theta, \end{aligned} \tag{7}$$

because $\cot\tau < 1/\tau$, $0 < \tau \le \pi/2$, and $\cos\theta \le \theta^{-1}\sin\theta$, $0 < \theta \le \pi$. Now by virtue of Lemma 1 and (7) we obtain $R_n(\tau) \le S(\theta) \le M = \max_{0 \le \theta \le \pi} S(\theta)$.

It remains to consider the case when $\pi/(n-1) \le \tau \le \pi/2$. For $n = 2$ (or 3) it is void. If $n \ge 4$, then by Lemma 2 we have

$$\sin[\pi/(n-1)] \ge \pi(n-1)^{-1}(2/\pi)\sin(\pi/2) = 2(n-1)^{-1}.$$

Hence

$$|\sin n\tau/n\sin\tau| \le 1/n \sin[\pi/(n-1)] \le (n-1)/2n < 1/2,$$

and, obviously, from $R_n(\tau) \le 1 + (\sin n\tau/n\sin\tau)^2$ we infer that for $\pi/(n-1) \le \tau \le \pi/2$ the function $R_n(\tau) \le 1.25 < S(1) = 1.5769\ldots < M$, and the proof of the estimate $C_n \le \sqrt{M}n$ is complete.

Now we study $S(\theta)$ for $0 \le \theta \le \pi$. Since

$$\begin{aligned} S(\theta) &= (1/2)\Big\{1 + \theta^{-2}\sin^2\theta + \sqrt{(1 + \theta^{-2}\sin^2\theta)^2 - \theta^{-2}\sin^2 2\theta}\Big\} \\ &\le 1 + \theta^{-2}\sin^2\theta, \end{aligned}$$

then by Lemma 2 we have $S(\theta) \le 1 + 1.3^{-2}\sin^2 1.3 = 1.5493\ldots < M$ for $1.3 \le \theta \le \pi$. On the other hand, writing

$$S(\theta) = (1/2)\Big\{1 + \theta^{-2}\sin^2\theta + \sqrt{(1 - \theta^{-2}\sin^2\theta)^2 + 4\theta^{-2}\sin^4\theta}\Big\},$$

from the inequality $\sqrt{a^2+b^2} \le a+b$, where $a, b \ge 0$, we get $S(\theta) \le 1 + \theta^{-1}\sin^2\theta$. By a standard calculus routine we establish that $\theta^{-1}\sin^2\theta$ is an increasing function for $0 \le \theta \le \pi/4$, whence for $0 \le \theta \le 0.65$ it follows that $S(\theta) \le 1 + 0.65^{-1}\sin^2 0.65 = 1.5634\ldots < M$.

We need to make still narrower the interval containing the maximum point of $S(\theta)$. It can be easily seen that for $\alpha \le \theta \le \beta$ by Lemmas 1 and 2 we have

$$S(\theta) \le (1/2)\Big\{1 + \alpha^{-2}\sin^2\alpha + \sqrt{(1+\alpha^{-2}\sin^2\alpha)^2 - 4\alpha^{-2}\sin^2\alpha\cos^2\beta}\Big\}.$$

Now straightforward calculations show that

$$S(\theta) \le 1.5369\ldots < M, \qquad 0.65 \le \theta \le 0.7;$$

$$S(\theta) \le 1.5554\ldots < M, \qquad 0.7 \le \theta \le 0.75;$$

$$S(\theta) \le 1.5707\ldots < M, \qquad 0.75 \le \theta \le 0.8.$$

Thus the maximum of $S(\theta)$ is attained on the interval $0.8 \le \theta \le 1.3$. But from the equation $S'(\theta) = 0$ it follows that

$$T(\theta) \equiv \theta^3(\cos\theta + \cos 3\theta) - \theta^2\sin 3\theta + \sin^3\theta = 0.$$

Let us prove that the function $T(\theta)$ has the unique root on the interval $0.8 \le \theta \le 1.3$. We compute $T(0.8) = -0.08\ldots < 0$, $T'(0.8) = -0.07\ldots < 0$, and $T(1.3) = 1.04\ldots > 0$, $T'(1.3) = 6.30\ldots > 0$. We show that $T''(\theta) > 0$ on $[0.8; 1.3]$ which implies the uniqueness of the root of the equation alongside the convergence of the Newton method with the initial approximation $\theta = 1.3$. To this end, let us consider the following three cases. The second derivative equals

$$T''(\theta) = -\theta^3(\cos\theta + 9\cos 3\theta) - 3\theta^2(2\sin\theta + 3\sin 3\theta) + 6\theta(\cos\theta - \cos 3\theta) - \sin^3\theta.$$

(a) For $0.8 \le \theta \le 0.9$ the functions $\cos\theta$, $\cos 3\theta$, and $\sin 3\theta$ decrease, while $\sin\theta$ increases. In particular, we have

$$-(\cos\theta + 9\cos 3\theta) > -(\cos 0.8 + 9\cos 2.4) = 5.93\ldots > 0,$$

whence $-\theta^3(\cos\theta + 9\cos 3\theta) \ge -0.8^3(\cos 0.8 + 9\cos 2.4) = 3.04\ldots$. Similarly,

$$\theta^2(6\sin\theta + 9\sin 3\theta) \le 0.9^2(6\sin 0.9 + 9\sin 2.4) = 8.73\ldots;$$

$$6\theta(\cos\theta - \cos 3\theta) \ge 6\cdot 0.8(\cos 0.9 - \cos 2.4) = 6.52\ldots;$$

$$\sin^3\theta \le \sin^3 0.9 = 0.48\ldots.$$

Thus, $T''(\theta) \ge 0.35\ldots > 0$ for $0.8 \le \theta \le 0.9$.

(b) Let $0.9 \le \theta \le 1.1$. Reasoning as above, we obtain

$$T''(\theta) \ge -0.9^3(\cos 0.9 + 9\cos 2.7) - 1.1^2(6\sin 1.1 + 9\sin 2.7)$$
$$+6\cdot 0.9(\cos 1.1 - \cos 2.7) - \sin^3 1.1 = 0.97\ldots > 0.$$

(c) If $1.1 \le \theta \le 1.3$, then

$$T''(\theta) \ge -1.1^3(\cos 1.1 + 9\cos 3.9) - 1.3^2(6\sin 1.3 + 9\sin 3.3)$$
$$+6\cdot 1.1(\cos 1.3 - \cos 3.9) - \sin^3 1.3 = 6.38\ldots > 0.$$

This completes the proof of the theorem.

Now we apply this result to determine the greatest value of δ such that the condition

$$\sum_{n=2}^{\infty} n|a_n(f)| \le \delta \tag{8}$$

implies that $f \in UST$. A. W. Goodman showed in [1] that for $\delta = \sqrt{2}/2 = 0.7071\ldots$ this is valid; however, for the sufficiency of (8), δ must not exceed $\sqrt{3}/2 = 0.8660\ldots$. Finally, we have the following

COROLLARY 3. *The condition* (8) *implies the uniform starlikeness of f for $\delta_0 = 1/\sqrt{M} = 0.7963\ldots$, where M is determined in Theorem* 2. *If $\delta > \delta_0$, then there exists a function $f \notin UST$ satisfying* (8).

Proof. Set $\delta = \delta_0$ in (8). Then the estimate

$$|(f * h)(z)/z| > 1 - \sum_{n=2}^{\infty} |a_n(f)a_n(h)z^{n-1}| > 1 - |z|\sqrt{M}\sum_{n=2}^{\infty} n|a_n(f)|,$$

valid for any $h \in H$, and Theorem 2 imply that $f \in UST$.

Now, let $\delta > 1/\sqrt{M}$ with $M = S(\theta_0)$. Note that $R_n(\theta_0/n) \to S(\theta_0) > 1/\delta^2$ as $n \to \infty$. Therefore, a positive integer N can be chosen such that $R_N(\theta_0/N) > 1/\delta^2$. Put also $\alpha_0 = (A - C + \sqrt{D})/B$, where the expressions A, B, C, D are defined as in the proof of Theorem 2. Hence for a certain function $h_0 \in H$ the estimate $|a_N(h_0)| > \delta^{-1}N$ holds. Consequently, the function $f_0(z) = z + \delta N^{-1}z^N$ satisfies the condition (8); however, $(f_0 * h_0)(z)/z = 1 + \delta N^{-1}a_N(h_0)z^{N-1}$ vanishes at some point $\zeta \in E$, so $f \notin UST$.

Remark 3. Unlike its counterpart from Theorem 1, the estimate $C_n < dn$ is attained for no $n \ge 2$. We may consider specific sharp estimates $C_n = d_n n$, where $d_n = \sqrt{\max_{0\le\tau\le\pi/2} R_n(\tau)}$. We state the following

COROLLARY 4. *Let* $f(z) = z + cz^n$, $n \geq 2$. *Then* $f \in UST \Leftrightarrow |c| \leq 1/nd_n$, *with* d_n *defined as above.*

For $n = 2$ we have $d_2 = 2/\sqrt{3} = 1.155\ldots$ and the corollary gives a result from [1]. On the other hand, computations show that $d_3 = 1.1916\ldots$, $d_4 = 1.2087\ldots$, so d_n seems to be an increasing sequence; however, this is still to be proved.

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