

Optimal estimates for the Hardy averaging operator

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Dedicated to the memory of Erhard Schmidt

Let $Af(x) := \frac{1}{x} \int_0^x f(t) dt$ be the one-dimensional Hardy averaging operator. It is well-known that A is bounded on L^p whenever $1 < p \leq \infty$. We improve this result in the following sense: we introduce a pair of new function spaces, the ‘source’ space S_p , which is strictly larger than L^p , and the ‘target’ space T_p , which is strictly smaller than L^p , and prove that A is bounded from S_p into T_p . Moreover, we show that this result cannot be improved within the environment of solid Banach spaces. We present applications of this result to variable-exponent Lebesgue spaces $L^{p(x)}$.

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1 Introduction

Let $f \in L^1_{\text{loc}}[0, 1)$. We consider the *Hardy averaging operator* A , defined by

$$Af(x) = \frac{1}{x} \int_0^x f(t) dt, \quad x \in (0, 1),$$

and the *Hardy–Littlewood maximal operator* M , defined by

$$Mf(x) = \sup_{a < x < b} \frac{1}{b-a} \int_a^b |f(t)| dt, \quad x \in (0, 1).$$

We note that we can extend a function which is defined on $(0, 1)$ by zero outside $(0, 1)$, and obtain thereby Af and Mf defined on the entire \mathbb{R} .

It is well-known that both the operators M and A are bounded on L^p whenever $1 < p \leq \infty$. Of course, this result cannot be improved in an essential way within the Lebesgue spaces. However, it turns out that an improvement is possible when we consider other, more general function spaces and classes.

In this paper we focus on the question of how far we can improve this result within the framework of solid Banach spaces. The environment of solid Banach spaces is fairly general as it, for example, covers all the Banach function spaces. Our main aim can be described as follows: given $p \in (1, \infty)$, we construct a pair of new function spaces, namely the ‘source’ space S_p and the ‘target’ space T_p such that

(i) the Hardy averaging operator A satisfies

$$A : S_p \longrightarrow T_p;$$

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(ii) this result improves the classical estimate

$$A : L^p \longrightarrow L^p$$

in the sense that

$$T_p \hookrightarrow L^p \hookrightarrow S_p$$

and both the inclusions are strict;

(iii) this result cannot be improved any further, at least not within the environment of solid Banach spaces, in the sense that whenever Y is a solid Banach space strictly larger than S_p , then

$$A : Y \not\rightarrow T_p$$

and, likewise, when Z is a solid Banach space strictly smaller than T_p , then

$$A : S_p \not\rightarrow Z.$$

It turns out that the space T_p is one of the function spaces considered in connection with different matters by K.-G. Grosse–Erdmann [3]. The space S_p is, as far as we know, new.

We finally consider a closely related question of the action of the averaging operator on the so-called variable-exponent Lebesgue spaces (or $L^{p(x)}$ spaces), which have recently attracted a lot of attention. Therefore, naturally, we treat analogous questions for the variable-exponent Lebesgue spaces and obtain thereby several results of independent interest. The key ingredient here is a certain logarithmic control of the variation of the generating function $p(x)$, a notion which we call a *weak-Lipschitz property* and which is sometimes in the recent literature called also a *log-Hölder continuity*.

The paper is structured as follows. In Section 2 we collect some background material. In Section 3 we present a key equivalence between two variable-exponent Lebesgue spaces whose generating functions are ‘close’ in a certain sense. In Section 4 we introduce the spaces S_p and T_p . In Section 5 we present the first main result of this paper, namely, we show that A is bounded from $L^{p(x)}$ into T_p . A comparison of the new spaces to the classical ones is carried out in Section 6. The results obtained so far are then used in Section 7 to obtain sharp estimates for the action of A , and their optimality is proved in Section 8.

2 Preliminaries

We denote by \mathcal{B} the set of all measurable functions $p(\cdot)$ defined on $(0, 1)$ such that $1 \leq \operatorname{ess\,inf} p(x) \leq \operatorname{ess\,sup} p(x) < \infty$ and by \mathcal{B}_+ the set of all functions $p(\cdot)$ defined on $(0, 1)$ such that $1 < \operatorname{ess\,inf} p(x) \leq \operatorname{ess\,sup} p(x) < \infty$. By $\mathcal{M}(0, 1)$ we define the set of all Lebesgue-measurable functions on $(0, 1)$.

Definition 2.1 Given a function $p(\cdot) \in \mathcal{B}$, we define the functional

$$m_{p(\cdot)}(f) = \int_0^1 |f(x)|^{p(x)} dx, \quad f \in \mathcal{M}(0, 1),$$

the corresponding *Luxemburg norm*

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0; m_{p(\cdot)} \left(\frac{f(x)}{\lambda} \right) \leq 1 \right\}, \quad f \in \mathcal{M}(0, 1),$$

and the corresponding function space

$$L^{p(x)} = \{f \in \mathcal{M}(0, 1); \|f\|_{L^{p(\cdot)}} < \infty\}.$$

Remark 2.2 Under our assumptions on $p(\cdot)$, $m_{p(\cdot)}$ is a *convex modular* (see for example [8] for details), and $L^{p(\cdot)}$ is a Banach space under the Luxemburg norm. We will make use of the following simple but useful property of modulars ([7, Lemma 1.2]): whenever h is a positive function on $[0, \infty)$ which is bounded in some right neighborhood of zero, T is a linear operator, m_1, m_2 are two convex modulars on $\mathcal{M}(0, 1)$ and $\|\cdot\|_1, \|\cdot\|_2$ are the corresponding Luxemburg norms, then the modular inequality

$$m_2(Tf) \leq h(m_1(f)), \quad f \in \mathcal{M}(0, 1),$$

implies

$$\|Tf\|_2 \leq C\|f\|_1, \quad f \in \mathcal{M}(0, 1).$$

We shall now introduce a key notion of the weak-Lipschitz property.

Definition 2.3 Let $p(\cdot) : [0, 1] \rightarrow \mathbb{R}$. We say that $p(\cdot)$ is *weak-Lipschitz* if there is a $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{\ln \frac{e^2}{|x-y|}} \quad \text{for all } x, y \in [0, 1], \quad 0 < |x - y| \leq 1. \quad (2.1)$$

Moreover, given a $p \in (1, \infty)$, we say that $p(\cdot)$ is *weak-Lipschitz at zero with respect to p* if there exists a $\delta \in (0, 1)$ such that

$$|p(x) - p| \leq \frac{C}{\ln \frac{e^2}{x}} \quad \text{for all } x \in (0, \delta). \quad (2.2)$$

We note that the constant e^2 is used just for convenience. In particular, the function $x \mapsto \ln \frac{e^2}{|x-y|}$ is concave on the entire $(0, 1)$, which will be used in the proof of Lemma 5.3 below.

Remark 2.4 We emphasize that, in the preceding definition, the function $p(\cdot)$ can attain negative values. However, if $p(\cdot)$ is weak-Lipschitz at zero with respect to some $p \in (1, \infty)$ and, at the same time, $p(\cdot) \in \mathcal{B}$, then we can take $\delta = 1$ in (2.2). Indeed, this is a simple consequence of the fact that p is bounded on $(0, 1)$ and that $\ln \frac{e^2}{x}$ is bounded away from zero on $(\delta, 1)$ for every $\delta > 0$.

It has been known that the condition (2.1) plays a crucial role for the action of integral operators on $L^{p(\cdot)}$. In particular, Diening ([1]) showed that $M : L^{p(x)} \rightarrow L^{p(x)}$ whenever $p(\cdot) \in \mathcal{B}_+$ is weak-Lipschitz.

3 A key lemma

We shall now observe that if functions $p(\cdot)$ and $q(\cdot)$ in \mathcal{B} are in some sense close to each other, then the action of the corresponding variable-exponent Lebesgue norms on a non-increasing function is equivalent.

Lemma 3.1 Let $p(\cdot), q(\cdot) \in \mathcal{B}$. Assume that there are $0 < \delta < 1$ and $C > 0$ such that

$$|p(x) - q(x)| \leq \frac{C}{\ln \frac{e^2}{x}} \quad \text{for all } x \in (0, \delta). \quad (3.1)$$

Let f be a nonnegative and nonincreasing function on $(0, 1)$. Then

$$\int_0^1 f(x)^{p(x)} dx < \infty \quad \text{if and only if} \quad \int_0^1 f(x)^{q(x)} dx < \infty.$$

Moreover, there is an $A > 1$ such that

$$A^{-1}\|f\|_{q(\cdot)} \leq \|f\|_{p(\cdot)} \leq A\|f\|_{q(\cdot)}.$$

Proof. By symmetry, it suffices to prove just the ‘only if’ part. To this end, suppose that $\int_0^1 f(x)^{p(x)} dx < \infty$. Since $p(x) \geq 1$ on $(0, 1)$, we have $L^{p(x)} \hookrightarrow L^1$ (see e.g. [6]). Thus, $\int_0^1 f(x) dx < \infty$. Since f is non-increasing, we have $xf(x) \leq \int_0^1 f(y) dy =: K$, which gives

$$f(x) \leq \frac{K}{x}, \quad x \in (0, 1).$$

Moreover, since both $p(\cdot)$ and $q(\cdot)$ are bounded on $(0, 1)$, (3.1) holds in fact for every $x \in (0, 1)$. We thus have, for $x \in (0, 1)$,

$$f(x)^{\frac{C}{\ln \frac{e^2}{x}}} \leq \left(\frac{K}{e^2}\right)^{\frac{C}{\ln \frac{e^2}{x}}} \left(\frac{e^2}{x}\right)^{\frac{C}{\ln \frac{e^2}{x}}} \leq \left(\max\left\{1, \frac{K}{e^2}\right\}\right)^{\frac{C}{2}} e^C =: L.$$

Then,

$$\begin{aligned} \int_0^1 f(x)^{q(x)} dx &= \int_{\{x; f(x) \leq 1\}} f(x)^{q(x)} dx + \int_{\{x; f(x) > 1\}} f(x)^{p(x)+q(x)-p(x)} dx \\ &\leq 1 + \int_{\{x; f(x) > 1\}} f(x)^{p(x)+|q(x)-p(x)|} dx \\ &= 1 + \int_{\{x; f(x) > 1\}} f(x)^{p(x)} f(x)^{\frac{C}{\ln \frac{e^2}{x}}} dx \\ &\leq 1 + L \int_0^1 f(x)^{p(x)} dx. \end{aligned}$$

This shows the modular inequality

$$m_{q(\cdot)}(f) \leq h(m_{p(\cdot)}), \quad \text{where } h(t) = Lt + 1.$$

By Remark 2.2, we get the desired norm inequality. □

4 Spaces S_p and T_p and their elementary properties

We will now introduce two new function spaces.

Definition 4.1 Let f be a measurable function on $(0, 1)$. We then define for $p > 1$ two functionals

$$\|f\|_{S_p} = \left(\int_0^1 \operatorname{ess\,sup}_{t \in (x,1)} \left(\frac{1}{t} \int_0^t |f(s)| ds \right)^p dx \right)^{\frac{1}{p}}$$

and

$$\|f\|_{T_p} = \left(\int_0^1 \operatorname{ess\,sup}_{t \in (x,1)} |f(t)|^p dx \right)^{\frac{1}{p}},$$

and the corresponding spaces

$$S_p = \{f; \|f\|_{S_p} < \infty\}$$

and

$$T_p = \{f; \|f\|_{T_p} < \infty\}.$$

It is a routine matter to verify that S_p and T_p are Banach spaces.

We will now find certain useful equivalent norms on T_p .

Definition 4.2 Let $p(\cdot) \in \mathcal{B}$ be given. For a function f , we define the functional

$$m_{T_{p(\cdot)}}(f) = \int_0^1 \left(\operatorname{ess\,sup}_{t \in (x,1)} |f(t)| \right)^{p(x)} dx,$$

the norm

$$\|f\|_{T_{p(\cdot)}} = \inf \left\{ \lambda > 0; m_{T_{p(\cdot)}} \left(\frac{f}{\lambda} \right) \leq 1 \right\},$$

and the corresponding space

$$T_{p(\cdot)} = \{f; \|f\|_{T_{p(\cdot)}} < \infty\}.$$

Then, again, $m_{T_{p(\cdot)}}$ is a convex modular and $T_{p(\cdot)}$ is a Banach space with respect to the norm $\|\cdot\|_{T_{p(\cdot)}}$.

Theorem 4.3 Let $p \in (1, \infty)$ and let $p(\cdot) \in \mathcal{B}_+$ be weak-Lipschitz at zero with respect to p . Then the norms in $T_{p(\cdot)}$ and T_p are equivalent.

Proof. This is an immediate consequence of Lemma 3.1. □

5 Boundedness of A from $L^{p(\cdot)}$ into T_p

Our aim in this section is to prove that the average operator A is bounded from $L^{p(\cdot)}$ to T_p whenever $p \in (1, \infty)$ and the function $p(\cdot) \in \mathcal{B}_+$ is weak-Lipschitz at zero with respect to p . Our proof of this rather deep result will use three auxiliary lemmas, which we shall formulate and prove first, and also the following well-known Riesz's rising sun lemma, whose proof can be found for instance in [9].

Lemma 5.1 Assume that $h(\cdot)$ is a continuous function on an interval (a, b) . Set

$$U = \{x \in (a, b); \text{there exists } \xi \in (x, b) \text{ such that } h(\xi) > h(x)\}.$$

Then there is finite or infinite sequence of open pairwise disjoint intervals (a_j, b_j) , $j = 1, 2, \dots$, such that

$$U = \bigcup_j (a_j, b_j) \quad \text{and} \quad h(a_j) \leq h(b_j) \quad \text{for each } j = 1, 2, \dots$$

Lemma 5.2 Let $p(\cdot) \in \mathcal{B}_+$ be weak-Lipschitz. Then $A : L^{p(\cdot)} \rightarrow T_{p(\cdot)}$.

Proof. Let $0 \leq f \in L^1_{loc}[0, 1)$. We consider f extended by zero outside $(0, 1)$. Fix $x \in (0, 1]$ and let $x \leq t \leq 2x$. Then

$$Af(t) = \frac{1}{t} \int_0^t f(s) ds \geq \frac{1}{2x} \int_0^x f(s) ds = \frac{1}{2} Af(x).$$

Thus, for $0 < y \leq x$,

$$M(Af)(y) \geq \frac{1}{2x-y} \int_y^{2x} Af(t) dt \geq \frac{1}{2x} \int_x^{2x} Af(t) dt \geq \frac{1}{4} Af(x).$$

So,

$$Af(x) \leq 4M(Af)(y) \quad \text{for } y \leq x. \tag{5.1}$$

Set $g(x) = \operatorname{ess\,sup}_{y \in (x,1)} Af(y)$. Let

$$U = \{t; g(t) > Af(t)\}, \quad t \in (0, 1).$$

The function Af is continuous. Therefore, by Lemma 5.1, there exists a finite or countable sequence of pairwise disjoint intervals of the form $(x_n, x_n + \varepsilon_n)$, where $x_n \in (0, 1)$ and $\varepsilon_n > 0$ such that

$$U = \bigcup_{n=1}^{\infty} (x_n - \varepsilon_n, x_n).$$

Moreover,

$$g_n(x_n - \varepsilon_n) = g(x_n) = Af(x_n - \varepsilon_n) = Af(x_n).$$

It thus follows from the monotonicity of g that $g(x) = Af(x_n)$ for every $x \in (x_n - \varepsilon_n, x_n)$. Hence, by (5.1), for such x , $g(x) \leq 4M(Af)(x)$. As $g(x) = Af(x)$ outside U , we get, altogether,

$$g(x) \leq 4M(Af)(x), \quad x \in [0, 1].$$

By the result of Diening [1], mentioned at the end of Section 2, M is bounded on $L^{p(\cdot)}$ and, since $Af(x) \leq Mf(x)$, so is A . Thus, finally,

$$\|Af\|_{T_{p(\cdot)}} = \|g\|_{p(\cdot)} \leq 4\|M(Af)\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}. \quad \square$$

Lemma 5.3 *Let $C > 0$ and $p \in (1, \infty)$. Then the function $q(x) := p - \frac{C}{\ln \frac{e^2}{x}}$ is weak Lipschitz.*

Proof. Note first that every increasing concave function g defined on $[0, 1]$ with $g(0) = 0$ satisfies

$$g(x) - g(y) \leq g(x - y)$$

for $0 \leq y \leq x \leq 1$. Now, choose $0 < y < x < 1$. Since the function $x \mapsto \frac{1}{\ln \frac{e^2}{x}}$ is an increasing concave function on $[0, 1]$ vanishing at zero, we obtain

$$|q(x) - q(y)| = \left| -\frac{C}{\ln \frac{e^2}{x}} + \frac{C}{\ln \frac{e^2}{y}} \right| \leq \frac{C}{\ln \frac{e^2}{x-y}},$$

as desired. □

Lemma 5.4 *Let $p(\cdot), q(\cdot)$ be weak-Lipschitz with constants C_1 and C_2 , respectively. Then the function $h(x) = \max\{p(x), q(x)\}$ is weak-Lipschitz with the constant $\max\{C_1, C_2\}$.*

Proof. It is easy to see that $h(\cdot) \in \mathcal{B}_+$. Next, for $x \in [0, 1], 0 < |x - y| \leq 1$, we have

$$|h(x) - h(y)| \leq \max\{|p(x) - p(y)|; |q(x) - q(y)|\} \leq \max\{C_1; C_2\} \frac{1}{\ln \frac{e^2}{x-y}}. \quad \square$$

Now we are in a position to state and prove the main result of this section.

Theorem 5.5 *Let $p \in (1, \infty)$ and let $p(\cdot) \in \mathcal{B}_+$ be weak-Lipschitz at zero with respect to p . Then*

$$A : L^{p(\cdot)} \longrightarrow T_p.$$

Proof. By Remark 2.4, we can assume that (2.2) holds with $\delta = 1$. Thus,

$$|p(x) - p| \leq \frac{C}{\ln \frac{e^2}{x}}, \quad x \in (0, 1).$$

We now set $d := \text{ess inf } p(x)$ and

$$q(x) := \max \left\{ d, p - \frac{C}{\ln \frac{e^2}{x}} \right\}. \tag{5.2}$$

Then $q(x) \leq p(x)$ for almost every $x \in (0, 1)$. Moreover, $q(\cdot) \in \mathcal{B}_+$, whence $L^{p(\cdot)} \hookrightarrow L^{q(\cdot)}$ (see e.g. [6]). Next, by Lemmas 5.3 and 5.4, q is weak-Lipschitz. Thus, by Lemma 5.2, $A : L^{q(\cdot)} \rightarrow T_{q(\cdot)}$. Finally, by Lemma 3.1, $T_{q(\cdot)} \hookrightarrow T_p$. Altogether,

$$\|Af\|_{T_p} \leq C\|Af\|_{T_{q(\cdot)}} \leq C\|f\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)}. \quad \square$$

6 Further functional properties of S_p and T_p

In this section we shall collect some basic functional properties of the function spaces S_p and T_p . We focus on embeddings between these spaces and other familiar function spaces.

We will use without further reference the fact that, given a measurable function f on $(0, 1)$ such that $0 \leq f(x) < \infty$ almost everywhere, then, for almost every $x \in (0, 1)$,

$$f(x) \leq \operatorname{ess\,sup}_{y \in (x,1)} f(y).$$

Theorem 6.1 *Let $p \in (1, \infty)$ and let $p(\cdot) \in \mathcal{B}_+$ be weak Lipschitz at zero with respect to p . Then*

$$T_p \hookrightarrow L^{p(\cdot)}.$$

Proof. Let $f \in T_p$. By Theorem 4.3, we have $f \in T_{p(\cdot)}$, i.e.

$$\int_0^1 \left(\operatorname{ess\,sup}_{y \in (x,1)} |f(y)| \right)^{p(x)} dx < \infty.$$

Then

$$\int_0^1 |f(x)|^{p(x)} dx \leq \int_0^1 \left(\operatorname{ess\,sup}_{y \in (x,1)} |f(y)| \right)^{p(x)} dx,$$

which finishes the proof. □

Theorem 6.2 *Let $p \in (1, \infty)$ and let $p(\cdot) \in \mathcal{B}_+$ be weak-Lipschitz at zero with respect to p . Then*

$$L^{p(\cdot)} \hookrightarrow S_p \hookrightarrow L^1.$$

Proof. Let us first show that $L^{p(\cdot)} \hookrightarrow S_p$. Let $0 \leq f \in L^{p(\cdot)}$. By Theorem 5.5, we have $Af \in T_p$, i.e.

$$\int_0^1 \left(\operatorname{ess\,sup}_{y \in (x,1)} \frac{1}{y} \int_0^y f(t) dt \right)^p dx < \infty,$$

whence $f \in S_p$.

Now, we will prove $S_p \hookrightarrow L^1$. We have

$$\begin{aligned} \|f\|_{S_p}^p &= \int_0^1 \operatorname{ess\,sup}_{t \in (x,1)} \left(\frac{1}{t} \int_0^t |f(s)| ds \right)^p dx \\ &\geq \int_0^1 \lim_{t \rightarrow 1^-} \left(\frac{1}{t} \int_0^t |f(s)| ds \right)^p dx \\ &= \int_0^1 \left(\int_0^1 |f(s)| ds \right)^p dx \\ &= \left(\int_0^1 |f(s)| ds \right)^p \\ &= \|f\|_1^p. \end{aligned}$$

□

Remark 6.3 (i) Let us note that both the embeddings $L^{p(\cdot)} \hookrightarrow S_p$ and $T_p \hookrightarrow L^{p(\cdot)}$ are strict. Indeed, given $p(\cdot)$, take $q(\cdot)$ from (5.2). Lemmas 5.4 and 5.3 imply that $q(\cdot)$ is weak-Lipschitz and, consequently, using that $\lim_{x \rightarrow 0^+} q(x) = p$, $q(\cdot)$ is also weak-Lipschitz at zero with respect to p . Set

$$r(x) = \max \left(q(x) - \frac{C}{\ln \frac{e^2}{x}}, \operatorname{ess\,inf}_{x \in (0,1)} q(x) \right).$$

Clearly, $r(x) \leq q(x)$ almost everywhere in $(0, 1)$. Moreover, $r(x) < q(x)$ on an interval $(0, \eta)$ for some $\eta > 0$. Hence,

$$L^{p(\cdot)} \hookrightarrow L^{q(\cdot)} \not\hookrightarrow L^{r(\cdot)} \hookrightarrow S_p.$$

The sharpness of the embedding $T_p \hookrightarrow L^{p(\cdot)}$ can be shown in an analogous manner.

(ii) For every $p \in (1, \infty)$, we have

$$T_p \hookrightarrow S_p.$$

Indeed, by Theorems 6.1 and 6.2 we have

$$T_p \hookrightarrow L^p \hookrightarrow S_p.$$

Clearly, both the embeddings are strict.

7 Boundedness of A

As mentioned above, the operator A is bounded on L^p as long as $p \in (1, \infty]$. We shall now improve this result, at least for $p \in (1, \infty)$ in the sense that we will find a better (larger) source space than L^p , and likewise a better (smaller) range space than L^p that will still render the boundedness of A true.

Theorem 7.1 *Let $p \in (1, \infty)$. Then*

$$A : S_p \longrightarrow T_p.$$

Proof. Assume $f \in S_p$. By the definition of the spaces T_p and S_p , we have

$$\|Af\|_{T_p}^p = \int_0^1 \operatorname{ess\,sup}_{t \in (x,1)} \left| \frac{1}{t} \int_0^t f(s) \, ds \right|^p dx \leq \int_0^1 \operatorname{ess\,sup}_{t \in (x,1)} \left(\frac{1}{t} \int_0^t |f(s)| \, ds \right)^p dx = \|f\|_{S_p}^p. \quad \square$$

It is worth to mention some consequences of Theorem 7.1. All the assertions of the following corollary follow immediately from Theorems 7.1, 6.1, 6.2 and 6.3.

Corollary 7.2 *Let $p \in (1, \infty)$. Then*

(i)

$$A : S_p \longrightarrow S_p$$

and

(ii)

$$A : T_p \longrightarrow T_p.$$

Let moreover $r(\cdot), s(\cdot) \in \mathcal{B}_+$ be weak-Lipschitz at zero with respect to p . Then

(iii)

$$A : L^{r(\cdot)} \longrightarrow L^{s(\cdot)}.$$

It should be mentioned that the statements of Corollary 7.2 generalize some of the earlier results. Namely, in [5] it was shown that A is bounded on $L^{p(\cdot)}$ when p is weak-Lipschitz at zero and $p(x) \geq 0$. In [4], the authors were able to drop the assumption $p(x) \geq 0$. Finally, in [2], the sufficient condition was reduced to the weak-Lipschitz property at zero.

8 Optimality of S_p and T_p

In this section we shall prove that the spaces S_p and T_p that appear in Corollary 7.2 (i) and (ii) are sharp in a fairly general sense. We first need the notion of a solid Banach space of integrable functions.

Definition 8.1 A linear subset X of all Lebesgue measurable functions on $(0, 1)$ equipped with the norm $\|\cdot\|_X$ is called a *solid Banach space of integrable functions* if the following three conditions hold:

1. $(X, \|\cdot\|_X)$ is a Banach space,
2. $X \hookrightarrow L^1$,
3. if $g \in X$ and $|f| \leq |g|$, then $f \in X$.

Theorem 8.2 Let $p \in (1, \infty)$. Let $Z \subsetneq T_p$ be a solid Banach space of integrable functions. Then

$$A : T_p \not\rightarrow Z.$$

Proof. Take $g \in (T_p \setminus Z)$ and set $h(x) = \text{ess sup}_{t \in (x,1)} |g(t)|$. Then h is non-increasing, $h \geq |g|$ and $h \in T_p$. Since Z is a solid Banach space of integrable functions, we have $h \notin Z$. So, $h \in (T_p \setminus Z)$. Since h is non-increasing we have $Ah \geq h$ and so, $Ah \notin Z$. Consequently, $A : T_p \not\rightarrow Z$. □

Now we turn our attention to the optimality of S_p in Corollary 7.2 (i).

Theorem 8.3 Let $p \in (1, \infty)$. Let Z be a solid Banach space of integrable functions such that $S_p \subsetneq Z$. Then

$$A : Z \not\rightarrow S_p.$$

Proof. Take $0 \leq f \in (Z \setminus S_p)$. Since $Z \hookrightarrow L_1$, one has

$$K := \text{ess sup}_{z \in (1/e,1)} \left(\frac{1}{z} \int_0^z f \right)^p \leq \left(e \int_0^1 f \right)^p < \infty.$$

We estimate

$$\begin{aligned} \|Af\|_{S_p}^p &= \int_0^1 \text{ess sup}_{y \in (x,1)} \left(\frac{1}{y} \int_0^y Af(t) dt \right)^p dx \\ &= \int_0^1 \text{ess sup}_{y \in (x,1)} \left(\frac{1}{y} \int_0^y \frac{1}{t} \int_0^t f(s) ds dt \right)^p dx \\ &= \int_0^1 \text{ess sup}_{y \in (x,1)} \left(\frac{1}{y} \int_0^y f(s) \int_s^y \frac{dt}{t} ds \right)^p dx \\ &= \int_0^1 \text{ess sup}_{y \in (x,1)} \left(\frac{1}{y} \int_0^y f(s) \ln \frac{y}{s} ds \right)^p dx \\ &\geq \int_0^1 \text{ess sup}_{y \in (x,1)} \left(\frac{1}{y} \int_0^{y/e} f(s) \ln \frac{y}{s} ds \right)^p dx \\ &\geq \int_0^1 \text{ess sup}_{y \in (x,1)} \left(\frac{1}{y} \int_0^{y/e} f(s) ds \right)^p dx \\ &= e^{-p} \int_0^1 \text{ess sup}_{y \in (x,1)} \left(\frac{1}{y/e} \int_0^{y/e} f(s) ds \right)^p dx \\ &= e^{-p} \int_0^1 \text{ess sup}_{z \in (x/e,1/e)} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p dx. \end{aligned}$$

Fix $x \in (0, 1)$. Denote $a := \operatorname{ess\,sup}_{z \in (x/e, 1/e)} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p$. Due to the trivial inequality $a \geq \max(a, K) - K$ we have

$$\begin{aligned} & \operatorname{ess\,sup}_{z \in (x/e, 1/e)} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p \\ & \geq \max \left(\operatorname{ess\,sup}_{z \in (x/e, 1/e)} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p, \operatorname{ess\,sup}_{z \in (1/e, 1)} \left(\frac{1}{z} \int_0^z f \right)^p \right) - K \\ & = \operatorname{ess\,sup}_{z \in (x/e, 1)} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p - K \\ & \geq \operatorname{ess\,sup}_{z \in (x, 1)} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p - K \end{aligned}$$

and so, due to the fact $f \notin S_p$

$$\|Af\|_{S_p}^p \geq e^{-p} \int_0^1 \operatorname{ess\,sup}_{z \in (x, 1)} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p dx - e^{-p} K = \infty.$$

This gives $Af \notin S_p$ which finishes the proof. \square

Remark 8.4 It follows from Theorems 8.2 and 8.3 that the action of the operator $A : S_p \rightarrow T_p$ is optimal in the sense that neither the source space nor the target one can be essentially improved.

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