ON CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS

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The meromorphic functions $f(z)$ to be considered in this paper are characterized by the fact that the points z_ν at which a function $f(z)$ takes the values 0, 1 or ∞ form a sub-set of a given closed set S of complex numbers. It is easy to see that, for $z \not\subset S$, these functions $f(z)$ form a normal family; indeed, if $z = p(x)$ denotes the analytic function, regular for $|x| < 1$, which maps $|x| < 1$ on the universal covering surface of the complement of S, then $g(x) = f[p(x)]$ differs from 0, 1 and ∞ for $|x| < 1$. By a classical result, the functions $g(x)$ form a normal family, and this is therefore also true of the functions $f(z)$ so long as z differs from the points of S (since, at these points, the representation $g(x) = f[p(x)]$, *i.e.* $f(z) = g[p^{-1}(z)]$, breaks down).

The family of functions $f(z)$ being normal in the complement of S, extremum problems with regard to continuous functionals $J = J\{f(z)\}$ are certain to possess a solution within the family $\{f(z)\}$. The exact determination of such solutions will generally depend on a detailed knowledge of the properties of $p(x)$. This is, of course, where the difficulty comes in, since $p(x)$ (apart from the case where S is simply-connected) is an automorphic function of x and, as such, not easily accessible to numerical treatment.

We shall deal here with two cases in which the function $p(x)$ can be expressed in terms of classical functions. These are: (a) the case in which the set S consists of all integers $(0, \pm 1, \pm 2, \ldots)$; ($b$) the case in which S consists of the numbers $m+in$, both m and n being integers.

It will be noted that, for the particular theorems to be proved below, not the automorphic function $p(x)$ but a slightly different function, connected with it by a simple transformation, is used. This is due to the fact that these results refer to the behaviour of the functions $f(z)$ in the points of S, and these are precisely the points not taken by $z = p(x)$ for $|x| < 1$. In order to study the properties of $f(z)$ at a point z_0 belonging to S, we shall therefore use a function $z = p^*(x)$ which takes the value z_0 for a particular $x = x_0$ and otherwise differs from all the points of S.

THEOREM I. *Let $w = f(z)$ be an analytic function, meromorphic in the entire finite z-plane, which takes the values $w = 0, 1, \infty$ only if z is an integer*

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($z = n$; $n = 0, \pm 1, \pm 2, \ldots$). *If* α_ν *denote the zeros of* $f(z)$, β_ν *the zeros of* $f(z)-1$ *and* γ_ν *the simple poles of* $f(z)$, *we have*

$$|f'(\alpha_\nu)| \leqslant 2\pi, \quad |f'(\beta_\nu)| \leqslant 2\pi, \quad \lim_{z\to\gamma_\nu} |(z-\gamma_\nu)f(z)| \geqslant (2\pi)^{-1}$$

In these inequalities the value 2π *is the best possible.*

In order to prove this proposition, we consider the function

$$p(x) = 2\pi^{-1} \arcsin \sqrt{\{I(x^2)\}}, \quad [|x| < 1],$$

where $I(x) = 16x + \ldots$ denotes the elliptic modular function defined by $I(x) = k^2(\tau)$, $x = e^{\pi i \tau}$, τ being the period ratio of the Jacobian elliptic functions and $k(\tau)$ their modulus. Since $\sqrt{\{I(x^2)\}}$, for $|x| < 1$, vanishes only at $x = 0$ and does not take the values ± 1 at all, it follows that $p(x)$ is regular for $|x| < 1$ and does not take there any value $0, \pm 1, \pm 2, \ldots$, except the value 0 for $x = 0$.

We now consider one of the points α_ν, say α, at which $f(z)$ vanishes. In the neighbourhood of this point, $f(z)$ may be expanded into a power series

$$f(z) = f'(\alpha)(z-\alpha) + \ldots.$$

Since α is an integer, we may consider instead the function $f_\alpha(z) = f(z+\alpha)$ which obviously is also one of the functions covered by our hypotheses. We have, therefore,

$$f_\alpha(z) = f'(\alpha)z + \ldots$$

in a certain neighbourhood of $z = 0$ (at least for $|z| < 1$).

After these preparations, we consider the function

$$g(x) = f_\alpha[p(x)] = f_\alpha[2\pi^{-1} \arcsin \sqrt{\{I(x^2)\}}] = 8\pi^{-1} f'(\alpha)x + \ldots. \tag{1}$$

As shown above, $p(x)$ does not take, for $|x| < 1$, any value $\pm n$ ($n = 0, 1, 2, \ldots$) except for its zero at $x = 0$; on the other hand $f(x)$, by hypothesis, does not take any of the values $0, 1, \infty$ as long as z differs from $0, \pm 1, \pm 2, \ldots$. We may therefore conclude that the function $g(x)$ is regular, and $g(x) \neq 1$, for $|x| < 1$; and also that $g(x) \neq 0$ for $0 < |x| < 1$. By a classical result, due to Hurwitz†, functions $g(x)$ possessing these

† See, for example, Littlewood, *Lectures on the Theory of Functions* (Oxford University Press, 1944), Theorem 229.

properties are subordinate to the elliptic modular function $I(x) = 16x+\ldots$, *i.e.* we have

(2) $$g(x) = I[\omega(x)], \quad |\omega(x)| \leqslant |x| \quad [|x| < 1].$$

In view of (1), this entails

(3) $$8\pi^{-1}|f'(a)| \leqslant 16, \quad |f'(a)| \leqslant 2\pi.$$

This proves the proposition for the zeros of $f(z)$. As for the zeros of $f(z)-1$ and the simple poles of $f(z)$, these cases can be disposed of by a slight modification of the same argument, the functions to be considered being $1-f(z)$ and $1/f(z)$ respectively.

In order to show that the value 2π is the best possible, we have to find a function $f_0(z)$ for which the sign of equality holds in (3). To this end we consider the case $\omega(x) \equiv x$ in (2). This yields

$$f_0\{2\pi^{-1} \operatorname{arc} \sin \surd[I(x^2)]\} = I(x).$$

We now substitute

$$2\pi^{-1} \operatorname{arc} \sin \surd\{I(x^2)\} = z,$$

whence we obtain

(4) $$\surd\{I(x^2)\} = \sin \tfrac{1}{2}\pi z, \quad f_0(z) = I(x).$$

But the elliptic modular function $I(x)$ satisfies the functional equation

(5) $$I(x) = \frac{4\surd\{I(x^2)\}}{[1+\surd\{I(x^2)\}]^2},$$

which, in view of (4), leads to

$$f_0(z) = \frac{4 \sin \frac{1}{2}\pi z}{(1+\sin \frac{1}{2}\pi z)^2} = 2\pi z+\ldots.$$

It now remains to be shown that $f_0(z)$ indeed satisfies our hypotheses; but it can easily be seen that $f_0(z)$ vanishes for $z = 2n$, $(n = 0, \pm 1, \ldots)$, and takes the values 1 and ∞ for $z = 4n+1$ and $z = 4n-1$ respectively. Apart from these points, $f_0(z)$ differs from 0, 1, ∞.

2. By writing $2\pi^{-1} \operatorname{arc} \sin \surd\{I(x^2)\} = z$, formulae (1) and (2) yield the parametric representation

$$f_a(z) = I[\omega(x)], \quad \sin \tfrac{1}{2}\pi z = \surd\{I(x^2)\},$$

which, in view of (5), may also be written

$$f_\alpha(z) = I[\omega(x)], \tag{6}$$

$$[|\omega(x)| \leqslant |x|, \ |x| < 1]$$

$$\frac{4 \sin \frac{1}{2}\pi z}{(1+\sin \frac{1}{2}\pi z)^2} = I(x). \tag{6a}$$

With the help of (6) and (6a), exact bounds for $|f(z)|$ may be obtained if z is real.

THEOREM II. *If $f(z)$, α_ν, β_ν and γ_ν have the same meaning as in Theorem I and t denotes a non-negative number smaller than* 1, *the following inequalities hold*:

$$(a) \qquad |f(\alpha_\nu \pm t)| \leqslant \frac{4 \sin \frac{1}{2}\pi t}{(1-\sin \frac{1}{2}\pi t)^2},$$

$$(b) \qquad \left(\frac{1-\sin \frac{1}{2}\pi t}{1+\sin \frac{1}{2}\pi t}\right)^2 \leqslant |f(\beta_\nu \pm t)| \leqslant \left(\frac{1+\sin \frac{1}{2}\pi t}{1-\sin \frac{1}{2}\pi t}\right)^2,$$

$$(c) \qquad \frac{(1-\sin \frac{1}{2}\pi t)^2}{4 \sin \frac{1}{2}\pi t} \leqslant |f(\gamma_\nu \pm t)|.$$

Proof. In (6), $z = 0$ entails $x = 0$. Now $I(x)$ is locally schlicht in the vicinity of the origin and $I(x)$ increases steadily from 0 to 1 if x increases from 0 to 1. Hence the values of x corresponding, by (6*a*), to values of z which are non-negative and smaller than 1, must necessarily satisfy $0 \leqslant x < 1$, *i.e.*,

$$I(x) = \frac{4 \sin \frac{1}{2}\pi t}{(1+\sin \frac{1}{2}\pi t)^2} \quad (0 \leqslant x < 1, \ 0 \leqslant t < 1). \tag{7}$$

On the other hand, $I(x)$ satisfies the functional equation

$$-I(-x) = \frac{I(x)}{1-I(x)}$$

and the inequality

$$|I(x)| \leqslant -I(-|x|).$$

Hence

$$|f_\alpha(t)| = |I[\omega(x)]| \leqslant -I(-|\omega(x)|) \leqslant -I(-x) = \frac{I(x)}{1-I(x)}.$$

By (7) we have, therefore,

$$|f_\alpha(t)| \leqslant \frac{I(x)}{1-I(x)} = \frac{4 \sin \frac{1}{2}\pi t}{(1-\sin \frac{1}{2}\pi t)^2},$$

which is inequality (a). (c) follows immediately by considering $1/f(\gamma_\nu+t)$. The right-hand side of (b) follows by observing that

$$|f(\beta_\nu+t)| \leqslant |1-f(\beta_\nu+t)|+1 \leqslant \frac{4\sin\frac{1}{2}\pi t}{(1-\sin\frac{1}{2}\pi t)^2}+1 = \left(\frac{1+\sin\frac{1}{2}\pi t}{1-\sin\frac{1}{2}\pi t}\right)^2,$$

and the left-hand side from the fact that $f^*(z) = 1/f(\beta_\nu+z)$ is also covered by the hypotheses of Theorem II and satisfies $f^*(0) = 1$.

3. An argument similar to that used in the proof of Theorem I leads to the following more general proposition.

THEOREM III. *Let $w = f(z)$ be an analytic function, meromorphic in the entire finite z-plane, which takes the values $w = 0, 1, \infty$ only at points of the set $z = m+in$ $(m, n = 0, \pm 1, \pm 2, \ldots)$. If α_ν denote the zeros of $f(z)$, β_ν the zeros of $f(z)-1$ and γ_ν the simple poles of $f(z)$, we have*

$$|f'(\alpha_\nu)| \leqslant (2/\pi)^{\frac{1}{2}}\Gamma^2(\tfrac{1}{4}); \quad |f'(\beta_\nu)| \leqslant (2/\pi)^{\frac{1}{2}}\Gamma^2(\tfrac{1}{4});$$

$$\lim_{z\to\gamma_\nu} |(z-\gamma_\nu)f(z)| \geqslant [(2/\pi)^{\frac{1}{2}}\Gamma^2(\tfrac{1}{4})]^{-1}.$$

The value $(2/\pi)^{\frac{1}{2}}\Gamma^2(\frac{1}{4})$ is the best possible.

The trigonometric function used in the proof of Theorem I is here replaced by the Jacobian elliptic function $\operatorname{sn} z = z+\ldots$ with the period ratio $(2K)/(4K') = \frac{1}{2}$. Since $K = K'$, all poles, zeros and k^{-1}-points (k being the modulus of the function) coincide with the corners of the quadratic lattice $z = K(m+in)$. Conversely, each such lattice-point coincides with either a zero, a pole, or a k^{-1}-point of $\operatorname{sn} z$. If instead of $\operatorname{sn} z$ we consider $\operatorname{sn} Kz$, the same is true for the lattice $z = m+in$. Since the expression of K' in terms of k' is the same as that of K in terms of k, we have (in view of $K = K'$ and $k^2+k'^2 = 1$) $k^2 = \frac{1}{2}$. The inverse function $z = K^{-1}\operatorname{sn}^{-1}(w)$ can be uniformly continued everywhere in the finite w-plane, apart from the points $w = \pm 1, \pm\sqrt{2}$, where it possesses an infinity of simple branch-points.

We next consider the function

$$w = h(x) = \sqrt{\left(\frac{2\sqrt{\{I(x^4)\}}}{1+I(x^4)}\right)} = 2\sqrt{2}\,x+\ldots,$$

where $I(x)$ denotes the elliptic modular function mentioned above. From the fact that $I(x)$ never takes the values $0, 1, \infty$ (apart from $I(0) = 0$), it follows that $w = h(x)$ never takes the values $w = 0, \pm 1, \pm\sqrt{2}, \infty$ ($h(0) = 0$ again excepted).

If a is a zero of $f(z)$, we write, as in the proof of Theorem 1,

$$f_a(z) = f(z+a) = f'(a)z + \ldots$$

and note that $f_a(z)$ is also one of the functions covered by our hypotheses. We then consider

$$g(x) = f_a\{K^{-1}\,\mathrm{sn}^{-1}[h(x)]\}.$$

Since $h(x)/x$ does not vanish and $w = h(x)$ never takes the values

$$w = \pm 1, \pm\sqrt{2}, \infty$$

which are exactly the critical points of $\mathrm{sn}^{-1}(w)$, it follows that the function $K^{-1}\,\mathrm{sn}^{-1}[h(x)]$ is uniform for $|x| < 1$ and does not take there any value $m+in$ $(m, n = 0, \pm 1, \pm 2, \ldots)$. Consequently, the function

$$g(x) = f_a\{K^{-1}\,\mathrm{sn}^{-1}[h(x)]\} \tag{8}$$

is regular for $|x| < 1$ and differs there from 0 and 1 (apart from $g(0) = 0$). From the latter property we conclude as before that $g(x)$ is subordinate to the elliptic modular function $I(x)$, *i.e.*,

$$g(x) = I[\omega(x)], \quad |\omega(x)| \leqslant |x| \quad [|x| < 1]. \tag{9}$$

The expansions of $g(x)$ and $I(x)$ being $g(x) = 2^{\frac{3}{2}}K^{-1}f'(a)x + \ldots$ and $I(x) = 16x + \ldots$ respectively, this leads to

$$2^{\frac{3}{2}}K^{-1}|f'(a)| \leqslant 16, \quad |f'(a)| \leqslant 4\sqrt{2}K.$$

In virtue of the relations

$$K = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}} = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1-\frac{1}{2}\sin^2\phi)}} = \tfrac{1}{4}\pi^{-\frac{1}{2}}\Gamma^2(\tfrac{1}{4}),$$

we obtain finally

$$|f'(a)| \leqslant (2/\pi)^{\frac{1}{2}}\Gamma^2(\tfrac{1}{4}). \tag{10}$$

The results for the points at which $f(z) = 1$ or $f(z) = \infty$ follow again by considering the functions $1-f(z)$ and $1/f(z)$ respectively.

We now proceed to find a function $f_0(z)$ for which the sign of equality holds in (10). To this end we write $\omega(x) \equiv x$ in (9) and obtain, in view of (8),

$$f_0\left\{K^{-1}\,\mathrm{sn}^{-1}\sqrt{\left(\frac{2\sqrt{\{I(x^4)\}}}{1+\sqrt{\{I(x^4)\}}}\right)}\right\}=I(x).$$

By substituting

$$z=K^{-1}\,\mathrm{sn}^{-1}\sqrt{\left(\frac{2\sqrt{\{I(x^4)\}}}{1+\sqrt{\{I(x^4)\}}}\right)}$$

we obtain

$$f_0(z)=I(x),\quad \sqrt{\{I(x^4)\}}=\frac{\mathrm{sn}^2\,Kz}{2-\mathrm{sn}^2\,Kz}.$$

In view of the functional equation of $I(x)$ used above, we also obtain

$$I(x^2)=\frac{4\sqrt{\{I(x^4)\}}}{[1+\sqrt{\{I(x^4)\}}]^2}=\mathrm{sn}^2\,Kz(2-\mathrm{sn}^2\,Kz),$$

and, by using this equation once more,

$$I(x)=\frac{4\sqrt{\{I(x^2)\}}}{[1+\sqrt{\{I(x^2)\}}]^2}=\frac{4\,\mathrm{sn}\,Kz\sqrt{(2-\mathrm{sn}^2\,Kz)}}{[1+\mathrm{sn}\,Kz\sqrt{(2-\mathrm{sn}^2\,Kz)}]^2}.$$

Remembering that $I(x)=f_0(z)$, we obtain finally

$$f_0(z)=\frac{4\,\mathrm{sn}\,Kz\sqrt{(2-\mathrm{sn}^2\,Kz)}}{[1+\mathrm{sn}\,Kz\sqrt{\{2-\mathrm{sn}^2\,Kz\}}]^2}.$$

By introducing the Jacobian elliptic function $\mathrm{dn}\,Kz$ $[k^2\,\mathrm{sn}^2\,Kz+\mathrm{dn}^2\,Kz=1]$, and by observing that in our case we have $k=2^{-\frac{1}{2}}$, this can also be written

$$f_0(z)=\frac{4\sqrt{2}\,\mathrm{sn}\,Kz\,\mathrm{dn}\,Kz}{[1+\sqrt{2}\,\mathrm{sn}\,Kz\,\mathrm{dn}\,Kz]^2}=4\sqrt{2}\,Kz+\ldots.$$

This function does, in fact, satisfy our hypotheses. Thus, as may easily be verified, $f_0(z)$ vanishes at the points $z=2m+in$ and

$$2m+1+i(2n+1),$$

it has poles at $4m-1+2in$, and it takes the value $f_0(z)=1$ at the points $4m+1+2in$ $(m,\,n=0,\,\pm1,\,\ldots)$. At points other than $m+in$, $f_0(z)$ does not take any of the values $0,\,1,\,\infty$.

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