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# A SHORT PROOF OF CAUCHY'S POLYGONAL NUMBER THEOREM

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ABSTRACT. This paper presents a simple proof that every nonnegative integer is the sum of $m + 2$ polygonal numbers of order $m + 2$.

Let $m \geqslant 1$. The polygonal numbers of order $m + 2$ are the integers

$$p_m(k) = \frac{m}{2}(k^2 - k) + k$$

for $k = 0, 1, 2, \ldots$. Fermat [3] asserted that every nonnegative integer is the sum of $m + 2$ polygonal numbers of order $m + 2$. For $m = 2$, Lagrange [5] proved that every nonnegative integer is the sum of four squares $p_2(k) = k^2$. For $m = 1$, Gauss [4] proved that every nonnegative integer is the sum of three triangular numbers $p_1(k) = (k^2 + k)/2$, or, equivalently, that every positive integer $n \equiv 3 \pmod 8$ is the sum of three odd squares. Cauchy [1] proved Fermat's statement for all $m \geqslant 3$, and Legendre [6] refined and extended this result. For $m \geqslant 3$ and $n \leqslant 120m$, Pepin [8] published tables of explicit representations of $n$ as a sum of $m + 2$ polygonal numbers of order $m + 2$, at most four of which are different from 0 or 1. Dickson [2] prepared similar tables. Pall [7] obtained important related results on sums of values of a quadratic polynomial.

Uspensky and Heaslet [9, p. 380] and Weil [10, p. 102] have written that there is no short and easy proof of Cauchy's polygonal number theorem. The object of this note is to present a short and easy proof.

Because of Pepin's and Dickson's tables, it suffices to consider only $n \geqslant 120m$. For completeness, I also include a proof of Cauchy's lemma.

CAUCHY'S LEMMA. *Let $a$ and $b$ be odd positive integers such that $b^2 < 4a$ and $3a < b^2 + 2b + 4$. Then there exist nonnegative integers $s, t, u, v$ such that*

(1) $$a = s^2 + t^2 + u^2 + v^2,$$

(2) $$b = s + t + u + v.$$

PROOF. Since $a$ and $b$ are odd, it follows that $4a - b^2 \equiv 3 \pmod 8$, and so, by Gauss's triangular number theorem, there exist odd integers $x \geqslant y \geqslant z > 0$ such that

(3) $$4a - b^2 = x^2 + y^2 + z^2.$$

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Choose the sign of $\pm z$ so that $b + x + y \pm z \equiv 0 \pmod 4$. Define integers $s, t, u, v$ by

$$s = \frac{b + x + y \pm z}{4}, \qquad t = \frac{b + x}{2} - s = \frac{b + x - y \mp z}{4},$$
$$u = \frac{b + y}{2} - s = \frac{b - x + y \mp z}{4}, \qquad v = \frac{b \pm z}{2} - s = \frac{b - x - y \pm z}{4}.$$

Then equations (1) and (2) are satisfied, and $s \geqslant t \geqslant u \geqslant v$. To show these integers are nonnegative, it suffices to prove that $v \geqslant 0$, or $v > -1$. This is true if $b - x - y - z > -4$, or, equivalently, if $x + y + z < b + 4$. The maximum value of $x + y + z$ subject to the constraint (3) is $\sqrt{12a - 3b^2}$, and the inequality $3a < b^2 + 2b + 4$ implies that $x + y + z \leqslant \sqrt{12a - 3b^2} < b + 4$. This proves the lemma.

THEOREM 1. *Let $m \geqslant 3$ and $n \geqslant 120m$. Then $n$ is the sum of $m + 1$ polygonal numbers of order $m + 2$, at most four of which are different from 0 or 1.*

PROOF. Let $b_1$ and $b_2$ be consecutive odd integers. The set of numbers of the form $b + r$, where $b \in \{b_1, b_2\}$ and $r \in \{0, 1, \ldots, m - 3\}$, contains a complete set of residue classes modulo $m$, and so $n \equiv b + r \pmod m$ for some $b \in \{b_1, b_2\}$ and $r \in \{0, 1, \ldots, m - 3\}$. Define

$$a = 2\left(\frac{n - b - r}{m}\right) + b = \left(1 - \frac{2}{m}\right)b + 2\left(\frac{n - r}{m}\right). \tag{4}$$

Then $a$ is an odd integer, and

$$n = \frac{m}{2}(a - b) + b + r. \tag{5}$$

If $0 < b < \frac{2}{3} + \sqrt{8(n/m) - 8}$, then the quadratic formula implies that

$$b^2 - 4a = b^2 - 4\left(1 - \frac{2}{m}\right)b - 8\left(\frac{n - r}{m}\right) < 0$$

and so $b^2 < 4a$. Similarly, if $b > \frac{1}{2} + \sqrt{6(n/m) - 3}$, then $3a < b^2 + 2b + 4$. Since the length of the interval

$$I = \left(\frac{1}{2} + \sqrt{6\left(\frac{n}{m}\right) - 3}, \frac{2}{3} + \sqrt{8\left(\frac{n}{m}\right) - 8}\right) \tag{6}$$

is greater than 4, it follows that $I$ contains two consecutive odd positive integers $b_1$ and $b_2$. Thus, there exist odd positive integers $a$ and $b$ that satisfy (5) and the inequalities $b^2 < 4a$ and $3a < b^2 + 2b + 4$. Cauchy's Lemma implies that there exist $s, t, u, v$ satisfying (1) and (2), and so

$$n = \frac{m}{2}(a - b) + b + r = \frac{m}{2}(s^2 - s) + s + \cdots + \frac{m}{2}(v^2 - v) + v + r$$
$$= p_m(s) + p_m(t) + p_m(u) + p_m(v) + r.$$

This completes the proof.

Note that this result is slightly stronger than Cauchy's theorem. Legendre **[6]** proved that every sufficiently large integer is the sum of five polygonal numbers of order $m + 2$, one of which is either 0 or 1. This can also be easily proved.

THEOREM 2. *Let $m \geqslant 3$. If $m$ is odd, then every sufficiently large integer is the sum of four polygonal numbers of order $m+2$. If $m$ is even, then every sufficiently large integer is the sum of five polygonal numbers of order $m+2$, one of which is either* 0 *or* 1.

PROOF. There is an absolute constant $c$ such that if $n > cm^3$, then the length of the interval $I$ defined in (6) is greater than $2m$, and so $I$ contains at least $m$ consecutive odd integers.

If $m$ is odd, these form a complete set of residues modulo $m$, and so $n \equiv b \pmod{m}$ for some odd number $b \in I$. Let $r = 0$. Define $a$ by formula (4).

If $m$ is even and $n > cm^3$, then $n \equiv b + r \pmod{m}$ for some odd integer $b \in I$ and $r \in \{0, 1\}$. Define $a$ by (4).

In both cases, the theorem follows immediately from Cauchy's Lemma.

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