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CRITERIA OF CONSTRUCTIBILITY FOR REAL NUMBERS

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The purpose of this paper is to prove two theorems and a conjecture (Conjecture II) announced in section 15 an earlier paper of the author's (cited as "*CT*"),[1] and to compare them briefly with related results of Specker.[2] Familiarity with both papers is assumed; the terminology of the former is used throughout. On two points however clarification of the usage of *CT* is in order, and to this chore we must first proceed.

A *half-section* is the lower half of a Dedekind cut; if the cut is rational, the half section is to include the rational corresponding to the real defined by the cut. A *whole-section* is the relation which holds between any member of the lower and any member of the upper half of some Dedekind cut. If the cut is rational the corresponding rational is to be a member of both halves.

A real number α is said to be approximable in K to any required number of decimal places if it is possible to define the predicates '$x < \alpha$', '$x \leqq \alpha$', '$x > \alpha$', and '$x \geqq \alpha$' (x rational) in K. In view of section 7 of *CT* this will mean that every true inequation between α and a terminating decimal will be provable in *K*.

We need certain preliminary augmentations of the expressive powers of *K*; specifically, we need to show that bound class-variables can be introduced into *K* by definition, in positions both of abstraction and of existential quantification. Henceforth we shall use letters $x\, y\, z$ for sequences, $\alpha\, \beta\, \gamma$ for classes of sequences, and A B C for classes of classes of sequences, both in the object-language and the metalanguage, even where this conflicts with the usage of *CT*. The definition of abstraction is obvious. Where α is a class-name, we write

*D*18. $\alpha \in \hat{\beta}(\ldots\beta---)$ *for* $\ldots\alpha---$

(or a relettered version thereof so chosen as to avoid collision of variables).

We capitalize on the results of section 11 of *CT* to quantify over classes; thus we write

*D*19. $(\mathrm{E}\alpha)p$ *for* $(\mathrm{E}x)(\mathrm{E}y_1') \ldots (\mathrm{E}y_n')(\mathrm{E}z_1) \ldots (\mathrm{E}z_{n+1})(Class\ x\,.\,Nom\ (y_1', y_1)\,.$ $\ldots\,.\,Nom\ (y_n', y_n)\,.\,\mathrm{subst}\ (z_1, y_1', b_1, a)\,.\,\mathrm{subst}\ (z_2, y_2', b_2, z_1)\,.\,\ldots\,.$ $\mathrm{subst}\ (z_n, y_n', b_n, z_{n-1})\,.\,\mathrm{subst}\ (z_{n+1}, x, c, z_n)\,.\,Thm(K)z_{n+1}),$

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[1] *A complete theory of natural, rational and real numbers*, this JOURNAL, vol. 15 (1950), pp. 185–198. On page 196, line 19, after "definable" add "as a half-section"; line 21, after "definable" add "as a whole section". The author has been unable to prove Conjecture I, and it is probably false.

[2] ERNST SPECKER, *Nicht konstruktiv beweisbare Sätze der Analysis*, this JOURNAL, vol. 14 (1949), pp. 145–158.

where $y_1, \ldots, y_n$ exhaust the free variables of p, and $a, b_1, \ldots, b_n, c$ designate the Gödel numbers of $p, y_1, \ldots, y_n, \alpha$ respectively, and where '*Class x*' says 'x is the Gödel number of a class-name', '$Nom(x, y)$' says 'x is the Gödel number of the term designating y', and 'subst(w, x, y, z)' says 'w is the Gödel number of the result of writing the expression whose Gödel number is x for all free occurrences of the variable whose Gödel number is y in the expression whose Gödel number is z'. $(\mathsf{E}\alpha)p$ says there is a *definable* class α such that p.

We can now proceed to the proof of:

THEOREM I. *Every bounded class of half-sections definable in K has a least bound definable (as a half-section) in K.*[3]

Proof. If A designates the class, the least bound will be designated by

$$\hat{x}(\mathsf{E}\alpha)(x \in \alpha \,.\, \alpha \in \mathrm{A}).$$

THEOREM II. *Every whole-section definable in K can be approximated in K to any required number of decimal places.*

Proof. Let α be the whole-section in question. If α is rational, we let w be the corresponding sequence and define

$$x \begin{Bmatrix} \leq \\ < \\ > \\ \geq \end{Bmatrix} \alpha \text{ for } x \begin{Bmatrix} \leq \\ < \\ > \\ \geq \end{Bmatrix} w.$$

Otherwise we set

$$x \begin{Bmatrix} < \\ \leq \end{Bmatrix} \alpha \text{ for } (\mathsf{E}y)\alpha(x, y),$$

$$x \begin{Bmatrix} > \\ \geq \end{Bmatrix} \alpha \text{ for } (\mathsf{E}y)\alpha(y, x).$$

THEOREM III. (Conjecture II of *CT*.) *Not every half-section definable in K can be approximated in K to any required number of decimal places.*

Proof. Consider the class α of all rationals of the form $1/10^n$, where $Thm(K)n$. Form the class β of all sums of a finite number of distinct members of α. It is not hard to show that β and the half-section

$$\gamma = \hat{x}(\mathsf{E}y)(y \in \beta \,.\, x < y)$$

are definable in K.[4] This last is the real number

$$\sum_n h(n)/10^n,$$

[3] This theorem risks triviality because it is doubtful whether there exists such a bounded class.

[4] The sign '$<$' refers of course to the relation between rationals. We have not officially introduced this sign yet, but it is a routine chore to do so.

where $h(n) = 1$ if $Thm(K)n$, 0 otherwise. Now suppose we could define '$x > y$' in K, as say '$\phi(x)$'. We would have

$$x \tilde{\epsilon}\, Thm(K) \equiv (\mathsf{E}y)(y \epsilon \gamma \,.\, \phi(y + 1/10^x)),$$

and we would be landed in the paradox of Section 12 of *CT*.

We have thus two classes of reals, each of which might for all we know at this stage be called with propriety the class of constructible real numbers:

(i) The class K$-$ of half-sections definable in K.

(ii) The class K$\pm$ of domains of whole sections definable in K.

We shall find it convenient to consider in what follows:

(iii) The class K$+$ of real numbers whose *complements* (with respect to the class of rationals) are definable in K.

THEOREM IV. *Let* '$<(x, y, z, \alpha)$' *mean*

$$x = 0 \,.\, y/z < \alpha \,:\, \vee \,:\, x = 1 \,.\, -y/z < \alpha$$

where α *is a real number. Then* $\alpha \epsilon$ K$-$ *if and only if* $\hat{x}\hat{y}\hat{z}<(x, y, z, \alpha)$ *is of the form* $\hat{x}\hat{y}\hat{z}(\mathsf{E}w)R(x, y, z, w)$ *where* R *is recursive.*

Proof. I. If $\alpha \epsilon$ K$-$, α is definable in K and $<(x, y, z, \alpha) \equiv (\mathsf{E}w)(w$ is the Gödel number of a K-proof whose last line is $(x \,;\, y \,;\, z) \epsilon \alpha)$

II. If $<(x, y, z, \alpha) \equiv (\mathsf{E}w)R(x, y, z, w)$, α is defined in K by

$$\hat{x}\hat{y}\hat{z}(\mathsf{E}w)((x \,;\, y \,;\, z \,;\, w) \epsilon \varrho)$$

where R is defined by ϱ.

The following theorems are proved analogously.

THEOREM V. *Let* '$>(x, y, z, \alpha)$' *mean*

$$x = 0 \,.\, y/z > \alpha \,:\, \vee \,:\, x = 1 \,.\, -y/z > \alpha$$

where α *is a real number. Then* $\alpha \epsilon$ K$+$ *if and only if* $\hat{x}\hat{y}\hat{z}>(x, y, z, \alpha)$ *is of the form* $(\mathsf{E}w)R(x, y, z, w)$.

THEOREM VI. *If* α *is a real number,* $\alpha \epsilon$ K$\pm$ *if and only if both* $\hat{x}\hat{y}\hat{z}<(x, y, z, \alpha)$ *and* $\hat{x}\hat{y}\hat{z}>(x, y, z, \alpha)$ *are of the form* $(\mathsf{E}w)R(x, y, z, w)$; *hence if and only if* $\left\{\begin{array}{l}\textit{both are}\\ \textit{either is}\end{array}\right\}$ *of the form* $R'(x, y, z)$ (R' *recursive*); *also if and only if* $\alpha \epsilon$ K$+$ *and* $\alpha \epsilon$ K$-$.

The following is an immediate corollary of Theorems IV and V.

THEOREM VII. *If* α *is a real number,* α *is in* K$+$ *if and only if* $-\alpha$ *is in* K$-$.

The following is an immediate corollary of Theorems II, III and VI.

THEOREM VIII. *The class* K$\pm$ *is properly contained in each of the classes* K$+$ *and* K$-$; *the classes* K$+$ *and* K$-$ *overlap but neither is contained in the other.*

The relation between Specker's criteria and ours is made clear in the following theorem.[5]

THEOREM IX. *If in the definitions of Specker's classes* $\mathfrak{R}_1$, $\mathfrak{R}_2$, *and* $\mathfrak{R}_3$ *we supplant all references to "Rekursivität" by references to "Berechenbarkeit", we obtain three equivalent definitions of our class* K$\pm$.

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[5] (*Added August* 4, 1952.) Since this paper was submitted, the author has seen an outline proof of what is essentially our Theorem IX in R. M. Robinson's review of Rósza Péter's ***Rekursive Funktionen,*** this JOURNAL, vol. 16 (1951), pp. 280–282.