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Some new considerations on the Bernoulli numbers, the factorial function, and Riemann's zeta function

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Abstract

An outstanding problem connected with the Bernoulli numbers and their application is considered, and the context of its resolution is applied to a consequent reconsideration of the Riemann zeta function and the attendant problem of the nature of its zeros, together with some new findings on the gamma/factorial function. © 2000 Elsevier Science Inc. All rights reserved.

1. Introduction

The numbers named in the title were discovered by the talented Swiss mathematician Jacob Bernoulli (1654–1705) [1] in connection with expressing sums of real integer powers. Then the sums of reciprocal integer powers n^{-p} gave rise to the function cited in the title, which was first found by another great Swiss mathematician, Leonhard Euler (1707–1783). But the function came into its own as a profound mathematical reality only when the genius of Berhard Riemann investigated its full complex-variable form and investigated its properties in a ground-breaking paper of 1859 [2] and in his published notes [3]. It was not until 1932 [4], however, that Carl Siegel with unparalleled skill and insight resurrected them and found an otherwise lost, extraordinarily accurate asymptotic approximation to the numerical values of complex zeros of Riemann's recalcitrant function $\zeta(s) = \sigma + it,\ \sigma$ and t being

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real. In his single but momentous paper [2] on $\zeta(s)$, Riemann stated that it was very probable (sehr wahrscheinlich) that all its complex zeros ρ were of the form $Re(\rho) = 1/2$, where $\zeta(\rho) = 0$. And that is where, the matter has rested to this day.

A similar situation arose in connection with the zeta function and Bernoulli's numbers when Euler published [5] his formula for $\zeta(s)$ where $s = 2n$, n being a natural number – a formula justly celebrated in treatises and textbooks ever since. But Euler could not find a kindred expression for $s = 2n + 1$, a fact also noted in both text books and treatises. Picking one of each at random and in order, Louis Brand wrote in 1958 [6] that "when p is an odd integer, $\zeta(p)$ has not been simply expressed; to this day [and this held until now C.M.] no closed expression for $\zeta(3)$ is known". In a much later [1985] treatise on the zeta Function [7] Aleksandar Ivić voices the same sentiments: "No one has succeeded in obtaining a formula as simple as ... [Euler's] for $\zeta(2k + 1)$", significantly adding: "in fact almost nothing is known of the arithmetical nature of $\zeta(2k + 1)$ for $k \geqslant 1$". The following section will principally address the problem of expressing the zeta function for odd natural numbers $\geqslant 3$.

But with this function, mystery seems to be piled upon mystery, and its most outstanding problem devolves upon Riemann's now famous riddle that it appeared extremely likely that for all ρ, $\sigma = 1/2$ and $t = \pm\tau$, where τ is positive and real, also stating that ρ comprised an infinite set. If true, this theorem would possess ramified implications for and applications to number theory. But its validation is still awaited. Perhaps, like Rieman's magnificent asymptotic formula rescued by Carl Siegel from cryptic and barely legible jottings, the answer might lie in what Riemann himself thought and wrote.

2. Bernoulli's numbers (B_n)

The B_n are most directly defined as the coefficients in the polynomial expansion of $t/(\mathrm{e}^t - 1)$, which alternate in sign and are zero for all odd $n \geqslant 3$. Thus $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, etc. Using Riemann's viewpoint, we earlier noted that they actually can be considered as a sparse selection from a much larger set of values of an underlying complex function. Although before the present paper this had not been done, complex forms $B_n(z)$ for Bernoulli *polynomials* had been considered by Carlitz [8], Spira [9], and later by Dilcher [10] who even extended the prior findings to various generalisations of the $B_n(z)$. But the first generalisation of that sort, i.e. advancing the $B_n(z)$ to various orders, was discovered by the Danish mathematician Erik Niels Nørlund [11] shortly before 1920 (Dilcher evidently knew only of his lectures on differentiation that appeared four years later in Berlin). And then came an important contributor who, by the current erosion of cultural memory [12] [1] escaped both Dilcher's references and the editors of the Amer. Math. Soc. *Memoirs*, even though he was a mathematician of repute: Louis Melville Milne-Thomson, the expert in numerical approximation theory and application who taught at the Royal Naval Colege in Greenwich. He creatively re-presented Nørlund umbral notation and refined it elegantly. Indeed Milne-Thomson's work [13] [2] is an updated re-write of George Boole's classical and inspired treatise on the theory of finite differences and was consciously planned that way. Yet with all these investigators from Nørlund to Dilcher, the chief aim and interest lay in generalising the Bernoulli polynomials to higher orders and neglecting the real gems of the subjects – the Bernoulli *numbers* themselves. In many ways that choice was a not too seminal cul-de-sac.

A deeper path lies in allowing the polynomial order and variable to become, respectively, 1 and 0, whereupon the *B*-polynomials become the Bernoulli numbers of integer index n; and then generalising n, specifically by letting it to become an unrestrictedly real variable, and finally further allowing $n \to s$, where, as before, $s = \sigma + \mathrm{i}t$ with σ and t real. Then the Bernoulli numbers become an analytic *function* which we designate $\beta(s)$. (This truly Bernoulli function is not to be confused with Milne-Thomson's somewhat over-named

[1] A much more recent instance showing that the loss of cultural memory is alive and well, occurred with George Boole's creative treatise on finite differences, which was reprinted by Chelsea in 1970 and was then, after the decease of the owner of that ordinarily fine house, taken over by the American Mathematical Society with a notice to the effect that: "The present, fifth edition differs from the fourth edition (New York, 1958) primarily in certain improvements of notation and in the addition of a name and subject index. The third and earlier editions were published under the title: A Treatise on the Calculus of Finite Differences". However, the so-called improved 4th and 5th editions lacked the important errata-and-notes pages as well as the extremely useful 52 pages of hints on the solutions of the exercises by Edward B. Escott, which graced the 3rd edition (Stechert, New York, 1946). And if one were to believe that the errata and notes missing in the 4th and 5th 'improved' editions had been corrected and added it would be a sad mistake. Thus on page 297 of the 5th edition one sees $t_x = (x/\mu) + C$ instead of the correct $1/t_x =$ etc., given in the errata list in the 3rd edition; and the same goes for another 20 serious misprints by which readers of the 5th (and 4th) editions would be gravely confused and misled, not to mention much deprived by the absence of Escott's compendious hints on solving the exercises – so helpful to a teacher preparing classwork exams, not to mention readers wishing to follow in the footsteps of the self-taught Boole. Indeed all of us would gladly forego the index for the corrections, though it would have been nice to have had both! So much for the preserving of cultural memory on the part of its very custodians such as reprint publishers, not to mention the editors of scientific and cultural societies. Their only excuse – and one must sympathise – is a perceptual breakdown in the face of the memory-destroying flood of often mis-called "information" besetting all people of this and the last century. Later is not always, and by no means necessarily, better.

[2] Milne-Thomson published this just before his book [14] of the same year, which is a phenomenal and thorough updating of George Boole's classic.

"Bernoulli function" of order v, which is simply a mere ancillary periodic function of unit period that coincides with his vth order Bernoulli polynomial in x when $0 \leqslant x < 1$).

Although it may be evident to some already, we shall soon see that $\beta(s) = 0$ for all $s = \rho$, where ρ is a root of $\zeta(s)$. $\beta(s)$ has, of course, whenever $s = 2n + 1$, another infinite set of what might be called its trivial zeros, n being a natural number as throughout this paper. In analogous contrast, the so-called trivial zeros of $\zeta(s)$ occur whenever $s = -2n$, negative even values of s here countering its positive odd values in $\beta(s)$. They would not have been down-graded to triviality by any who knew their interesting arithmetic structure, which will later come up for discussion.

The Bernoulli numbers go mathematically very deep – more so than the Euler numbers that in ordinary texts are so often given equal billing as it were. Indeed this conclusion is implicit in the fact that in the profound and ubiquitously helpful Euler–Maclaurin summation it is the Bernoulli numbers that play a major rôle, expressing the difference between the operations of integration and summation, and so relating continuity and discontinuity. They are thus arguably, aside from the natural numbers themselves, one of the most if not the most important infinite set of rational numbers in all of mathematics. It is, then, not too surprising to find that the B_{2n} and the $\zeta(2n)$ are strongly linked, the latter being an infinite set of real transcendental numbers in π. Neither should it be to find that $\beta(s)$ and $\zeta(s)$ have close interconnections, as already adumbrated above.

Returning now to the interesting question of making the Bernoulli-number index continuous, we now proceed towards specifying the function $\beta(s)$ as a first step in resolving the old enigma of the odd-integer analogue of Euler's finding that

$$B_{2n} = (-1)^{n+1} 2(2n)!(2\pi)^{-2n}\,\zeta(2n). \tag{1}$$

The single exception to every term in Eq. (1) being a function of $2n$ is in the first factor on the right-hand side, $(-1)^{n+1}$. Transformation to a more precise trigonometric equivalent emerges from the fact that $(-1)^{n+1} = (-1)^n(-1)$, which can be re-written as $\cos(\pi/2)(2n)\cos\pi(1 - 2n)$. Substitution in (1) then yields

$$B_{2n} = 2(2n)!\,\zeta(2n)(2\pi)^{-2n}\cos\frac{\pi}{2}(2n)\cos\pi(1 - 2n) \tag{2}$$

which now contains nothing that depends on the fact that $2n$ is even. We can then transform further (note that "!" here means the factorial function of unrestricted argument), obtaining the function

$$\beta(s) = 2s!\,\zeta(s)(2\pi)^{-s}\cos\frac{\pi}{2}s\cos\pi(1 - s), \tag{3}$$

where s is the variable already defined. If $s = 2n$ we have Euler's formula, and if $s = 2n + 1$ we have its long-sought analogue

$$B_{2n+1} = 2(2n+1)!\,\zeta(2n+1)(2\pi)^{-(2n+1)}\cos\frac{\pi}{2}(2n+1) \tag{4}$$

showing at once that $B_{2n+1} = 0$ for all $n \geqslant 1$. But Eq. (4) also holds for $n = 0$, whereupon $B_1 = -1/2$. Thus, with $n = 0$ we have from (4): $B_1 = (\zeta(1)/\pi)\cos(\pi/2)$ at which point we need an important lemma derivable from the functional equation for $\zeta(s)$; namely, $\zeta(1)\cos(\pi/2) = -(\pi/2)$, whereupon Eq. (4) becomes $B_1 = -1/2$, which is otherwise and independently known. Since $\zeta(2n+1) > 0$ for all $n > 0$, we thus have

$$\zeta(2n+1) = \frac{B_{2n+1}(2\pi)^{2n+1}}{2(2n+1)!\cos(\pi/2)(2n+1)}. \tag{4a}$$

The 0/0 term, namely $B_{2n+1}/(\cos(\pi/2)(2n+1))$, is here determinated as a finite positive real magnitude, as Eq. (3), with $s = 2n + 1$ shows. It was the presence of this term in the analogue that prevented its being found by ordinary methods. Of this more presently. This will be discussed later. Similarly, for $s = 0$ Eq. (3) delivers: $-2\zeta(0)\cos 0$, which means that $B_0 = 1$, as is known.

We also see from Eq. (4) that although B_{2n+1} and, say, B_{2n+3} are both zero, their ratio is a real number of absolute value greater than zero. Specifically,

$$\frac{B_{2n+1}}{B_{2n+3}} = -\frac{2\pi^2\,\zeta(2n+1)}{(n+1)(2n+3)\,\zeta(2n+3)}. \tag{5}$$

Thus if $n = 1$, then

$$\frac{B_3}{B_5} = -\frac{\pi^2}{5} \tag{5a}$$

and also the interesting fact that

$$\lim_{n\to\infty}\frac{B_{2n+1}}{B_{2n+3}} = 0 \tag{5b}$$

– results next to impossible to find without knowing Eq. (3) and the function $\beta(s)$ which we will term the *Bernoulli Function*.

Now all this tells us something about zero, and specifically shows the error of the view that "there is only one zero in real analysis" (alas, stated *ex cathedra* by a professional who should have known better but suffered from invidious conditioning). In fact, "zero" is not a single number, but an infinite set of infinitesimal constants all of which are zero with respect to finite numbers or, more briefly put, $\circeq F$, where the modified equality sign denotes "equal with respect to" and F is the finite domain. Note that no object which is $\in 0$ is finite, even though the symbol 0, denoting an entire infinite class, also shows up as the difference of two identical numbers. Similarly, ∞ also denote an infinite class of objects, none of which shows up in F at all, unlike 0 which denotes the null point included in F. Thus if $x \in F$, and $z \in 0$, then $z/x \triangleq 0$ or $z/x \in 0$. By the same token, $x/z \in \infty$. The practical point here is that, when suitable specification is made, $0/0$ is no more "indeterminate" than f_1/f_2, where f_1, f_2 are unspecified members of F; and *mutatis mutandi* for members of ∞. Thus neither 0 nor its reciprocal ∞ is finite, and both these symbols denote infinite sets of non-finite values which are in principal specifiable. On the other hand, the result of an operation between two unspecified members of any set is indeterminate (what else *could* it be!). Conversely, when the members and the operation involving them are specified, the result will also be determinate. We now have a simple but far-reaching.

Theorem. *The indeterminacy of, say,* $\infty - \infty$ *or* $0/0$ *does not stem from the nature of zero or infinity, but from a lack of specification.*

Thus $F - F = F$, or $0 \pm 0 = 0$, $\infty + \infty = \infty$, or $0 \times \infty, \ldots$, are all shorthand expressions for operations between unspecified members of infinite sets. Thus, we have just seen that if $0/0 = B_3/B_5$ then $0/0 = -\pi^2/5$. With due specification, any paradoxes (which are simply misleadingly stated because of confusedly understood facts) disappear. Also then, fallacies such as the claim that a part is "as great as" the whole are seen for what they are: in this case, the fact that any two members of infinity are themselves infinite. That fact, however, by no means implies that either of them is equal to the entire class of infinite numbers – any more than the fact that $3 \in F$ and $5 \in F$ implies that either 3 and 5 are equal, or that either of them is equal to F, the entire set of finite numbers.

3. The function $\beta(s)$

On examining Eq. (3) further it becomes clear that (a) if $\sigma = (2n+1)/2$ and $t = 0$, where n may be a real integer or zero, then $\beta(s) = 0$; and (b) if σ, $t \neq 0$ with $s = \rho$ and $\beta(\rho) = 0$, then we have $\zeta(\rho) = 0$ and hence that if $\beta(s) = 0$ for complex s and the Riemann Vermutung holds, we have $\sigma = 1/2$. We shall consider these facts further in Section 5, as now are focusing on $\beta(s)$.

It is natural to ask, What is the Bernoulli function when the argument is a negative real integer? There are two cases, even and odd. Taking the first first, if $s = -2n$, then starting from Eq. (3) we have, after a straightforward but somewhat tedious simplification, paying careful attention to sign, and using some needed lemmas

$$B_{-2n} = 2n\,\zeta(2n+1). \tag{6}$$

Similarly, if $s = -(2n-1) = 1-2n$, we have

$$B_{1-2n} = (1-2n)\,\zeta(2n) \tag{7}$$

which, along with Eq. (6), is a new result. Two needed relations are the following.

Lemma.

1. *m and n being natural numbers,*

$$(-m)! = \frac{(-1)!}{(-1)^{m-1}(m-1)!}. \tag{8}$$

2.

$$(-1)!\cos\frac{\pi}{2}(2n \pm 1) = (-1)^{n+1}\frac{\pi}{2}. \tag{9}$$

They are easily proved and imply at once the useful relation

$$(-m)!\cos\frac{\pi}{2}(2n \pm 1) = \frac{(-1)^{m+n}}{(m-1)!}\frac{\pi}{2} \tag{10}$$

which actually is a specification of $\infty \times 0$. A kindred lemma sometimes needed is

$$\zeta(1)\cos\frac{\pi}{2} = -\frac{\pi}{2} \tag{10a}$$

which is really subsumed under Eq. (10) since it can be shown that

$$\zeta(1 \pm \alpha) = -(-1 \mp \alpha)! = -\Gamma(\mp\alpha), \tag{11}$$

where $\alpha \in 0$ and is specified by $\lim_{n\to\infty} n\alpha = 1$, where n as usual is a natural number. By the same token

$$\zeta(1+\alpha) + \zeta(1-\alpha) = 2\gamma \tag{12}$$

and

$$\zeta(1+\alpha) - \alpha^{-1} = \gamma \tag{12a}$$

as well as

$$\alpha^{-1} + \zeta(1-\alpha) = \gamma, \tag{12b}$$

where, as customary, γ denotes Euler's constant, itself specified by a subtraction between two numbers, both infinite with respect to ordinary numbers, yet one slightly larger than the other. Interestingly, in this connection, we can alpha-specify with: $\gamma = H_{\alpha^{-1}} - \log\alpha^{-1}$, where $H_n \equiv \sum_{k=1}^{n} k^{-1}$, with α defined in terms of n as before; and as is not so well-known and was noted above,

$\gamma = \zeta(1+\alpha) - \alpha^{-1}$. What makes the two γ-differences more interesting is that, though all their four terms are infinite numbers, yet the last two are infinitely greater than the first two and $\zeta(1+\alpha)/H_{\alpha^{-1}} \in \infty$, or

$$\frac{\sum_{n=1}^{\infty} n^{-1}}{\zeta(1)} = 0 \tag{13}$$

thus proving that, because it is a higher order infinity, $\zeta(s) \neq \sum_{n=1}^{\infty} n^{-s}$ for $s = 1$. Suffice it to say here, $\gamma(= 0.5772156649\ldots)$ is intimately connected both with the zeta and factorial functions, themselves two of the deepest in pure mathematics; and they in turn are both involved in $\beta(s)$ as we have already seen. Returning to the B-numbers, there is a more general theorem that delivers Eqs. (6) and (7) without labour; namely, where m is a natural number,

$$B_{-m} = (-1)^m m\, \zeta(m+1). \tag{14}$$

Similarly, the known fact that $\zeta(1-2n) = -B_{2n}/2n$ is a direct result of the more general theorem

$$\zeta(-m) = (-1)^m \frac{B_{m+1}}{m+1}. \tag{15}$$

Eq. (14) also at once yields the new result (holding even for $n = 0$):

$$\zeta(-2n) = \frac{B_{2n+1}}{2n+1} \in 0 \tag{16}$$

which shows the structure of the real zeros of $\zeta(s)$, and is much more explicit and hence more useful than the comparatively empty conventional statement that $\zeta(-2n) = 0$. Thus Eq. (16) at once furnishes the interesting fact that

$$\frac{\zeta(-2n)}{B_{2n+1}} = \frac{1}{2n+1} \tag{17}$$

and therefore, that $\zeta(-2n)$ is a zero of higher order than B_{2n+1}; indeed

$$\lim_{n\to\infty} \frac{\zeta(-2n)}{B_{2n+1}} = 0 \tag{18}$$

a result not otherwise easily accessible.

The function $\beta(s)$ for real s (i.e. $s = \sigma, t = 0$) is continuous with increasingly violent oscillations as s increases positively. As s extends negatively the function oscillates with increasing amplitude, but much more moderately; at $s = 0$, as is known, $\beta(s) = 1 = B_0$. But it has not been recognized that the 1 here is actually $\alpha\zeta(1+\alpha)$ since by Eq. (12a) $\zeta(1+\alpha) = 1 + \alpha\gamma \stackrel{\circ}{=} 1$, i.e. 1 with respect to the finite domain, $\alpha\gamma$ being infinitesimal. In this regard, Eq. (6) is valid even if $-2n \in 0$ and likewise Eq. (7) holds for $n = 0$ as then we have $B_{1-2n} = \beta(1-0) = (1-0)\zeta(0)$ or $\beta(1) = B_1 = -1/2$, an otherwise known

Table 1
B-numbers of negative integer index

n	B_{-2n}	To 7 places	B_{1-2n}
1	$2\zeta(3)$	2.4041138	$-\pi^2/6$
2	$4\zeta(5)$	4.1477110	$-\pi^4/30$
3	$6\zeta(7)$	6.0500956	$-\pi^6/189$
4	$8\zeta(9)$	8.0160671	$-\pi^8/1350$
n	$2n\zeta(2n+1)$	–	$(1-2n)\zeta(2n)$

value. Also note that if $s = 1 - 2n$, then even for fairly small n we have the asymptotic relation

$$|\zeta(1-2n)| \sim 2(2n-1)!(2\pi)^{-2n}.$$

Thus for $n = 12$, the above gives 3607.5103, and actually $\zeta(-23) = 3607.5105$ to four places. Since for real k, $\lim_{k\to\infty} \zeta(k) = 1$, we also now have the elegant asymptotic relations $B_{-2n} \sim 2n$ and $B_{1-2n} \sim 1 - 2n$.

Finally, aside from the zeros at all $s = 2n + 1,\ n > 0$, we note that $\beta(s) = 0$ for all half-integer values of s, and, as noted before, $\beta(s) = 0$ for all complex ρ such that $\zeta(\rho) = 0$. We end this section with some hitherto unknown numerical values in Table 1.

4. A re-examination of the factorial function

This function is deeply involved in the investigations of this paper. By $z!$ we mean Gauss' factorial function of unrestricted argument, and not simply for real integer z. For the reader's convenience, let us recall here Eq. (8), which springs directly from the basic nature of the function and which furnishes the specification for an infinite set of infinite numbers (m being a real integer)

$$(-m)! = \frac{(-1)!}{(-1)^{m-1}(m-1)!}.$$

Thus although $(-2)! = -(-1)!$ is correct as far as it goes, it is not explicit enough since, for example, $(-2+\alpha)!$ is infinitely removed from $(-2-\alpha)!$, α being the infinitesimal number already defined. So for complete accuracy we need the fully explicit relationship (where the $\pm$ signs are matched)

$$(-m \pm \alpha)! = \frac{(-1 \pm \alpha)!}{(-1)^{m-1}(m-1)!}. \tag{19}$$

We also now have a finite specification of ∞/∞ since

$$\frac{(-1 \pm \alpha)!}{(-m \pm \alpha)!} = (-1)^{m-1}(m-1)!. \tag{19a}$$

Note that even though Eq. (19a) is accurate as far as the finite quotient of these two infinite numbers is concerned, to ascertain their finite *difference* also is another matter as we shall presently see in Section 4. The factorial function is also closely connected with a deep trig function occurring in many applications. Their connection is given by

$$\frac{\sin\theta}{\theta} = \frac{1}{(-\theta/\pi)!(\theta/\pi)!},$$

valid whether θ be real or complex.

Continuing the main thread of the discussion, by a functional equation of the factorial function we have $(-m)!m! = \pi m/\sin\pi m$ and hence $(-m)! = \pi/((m-1)!\sin\pi m)$ which, when combined with Eq. (19) yields

$$\pm(-1)^{m-1}(-1\pm\alpha)!\sin\pi(m\pm\alpha) = (-1)^m\pi \tag{19b}$$

which is another finite specification of $\infty\times 0$. The $\pm$ signs are matched. For example, if $m-\alpha = 2-\alpha$, then $(-1-\alpha)!\sin\pi(2-\alpha)! = \pi$, which is easily machine-demonstrated by approximating α by, say, 0.00001 and then we have $3.14161 \simeq 3.14159$. The prominence of $(-1)!$ in these and related expressions stems from the fact that, in relation to all other negative real integers, -1 is like the integer-infinity, being the largest negative integer of the entire infinite set of them. Consequently, all $(-m)!$, where m is a natural number, are less in absolute value than $|(-1)!|$, even though all such absolute numbers $\in\infty$. We will see that the quotients and differences of these various infinite numbers are all precisely determinable.

It is interesting that the usual equation $0! = 1$ is not accurate infinitesimally since then $(-0)! = (+0)!$ is not true. Indeed that might have been deduced from the fact that often one must separate, say, $\int_{-t}^{-0} f(x)\,\mathrm{d}x$ and $\int_{+0}^{t} f(x)\,\mathrm{d}x$ or the fact that $\Gamma(-0)$ is infinitely far from $\Gamma(+0)$. Removing the ambiguity of the symbol "0" (which as we saw actually denotes an infinite set of infinitesimal numbers), and explicitizing by means of α, we have

$$\frac{(-\alpha)! - (\alpha)!}{\alpha} = 2\gamma$$

showing that $(-0)!$ is infinitesimally larger than $(+0)!$, their difference being $2\gamma\alpha \in 0$. In general, when the zero symbol in each instance has the same specification, then $((-0)! - (+0)!)/(+0) = 2\gamma$. Thus if $z \in Zero$ is any specified positive infinitesimal, we have

$$\frac{(-z)! - (z)!}{z} = 2\gamma \tag{20}$$

whether z be α, $\sin\pi(1-\alpha), -\cos(\pi/2)(1+\alpha), \ldots$

It is also to be observed that $(-1)!\sin\pi n = (-1)^n\pi$, which means $(-1\pm\alpha)!\sin\pi(n\pm\alpha) = (-1)^n\pi$, as well as $(-1)!\cos(\pi/2)(2n\pm 1) = (-1)^n(\pi/2)$, with a similar α-specification. Also, by the principal functional equation of the factorial function, namely $(z+1)z! = (z+1)!$, one has at once $0(-1)! = 0! = 1$, where 0 here is the integer zero, i.e. $1-1$. The α-specification of this equation is clearly $\alpha(-1+\alpha)! = 1$ and, indeed, $\pm\alpha(-1\pm\alpha)! = 1$. Thus $0! = 1$ is a theorem rather than merely a definition as is sometimes mis-stated. It is also seen that with uniform α-specification, $(1-1) < \cos(\pi/2)(2n\pm 1) > \sin\pi(n)$, despite the fact that all the terms of these inequalities $\in 0$. It is time the fact be recognized that the symbols $>$ and $<$ extend into the infinitesimal realm, and that the usual Weierstrassian "epsilontic" stipulation that $\in$ is "a small positive finite number", whereas it actually must be an infinitesimal to "pass to the limit", stands in need of revision for its inaccuracy and consequent evasion of the underlying mathematical reality.

Concluding this discussion of α-specification, note that its usefulness extends to higher orders of infinitestimal numbers. For example, $|\cot(\pi/2)(n+\alpha)| - |\cos(\pi/2)(n+\alpha)| = 2\alpha^3$. And also, contrary to conventional thinking, it is not true that $|\sin\pi(n+\alpha| = 2|\cos(\pi/2)(n+\alpha)|$, since the term on the right-hand side exceeds that on the left-hand side by an infinitesimal of higher order than α (approximately $3.7\alpha^3$) because $|\sin(\pi/2)(n+\alpha)| < 1$. These examples illustrate how unrealistic the notion of "absolute rigour" is in mathematics. In fact, it simply does not exist for the very good reason that the richness of mathematical reality cannot be arbitrarily circumscribed. Indeed, mathematicians in principle cannot sneer at engineers for accuracy within only a certain tolerance because they themselves are obliged to do the same, as our symbol $\mathring{=}$ meaning "equal with respect to finite numbers", directly implies. The fact that mathematical tolerances are ever so much smaller than engineering tolerances changes neither the principle nor the fact that tolerances must be deployed willy-nilly. The notion of "absolute rigour" is an illusion. It also has no place in sound pedagogy because it makes for inaccurate thinking and hence eventually mathematically erroneous statements.

4.1. And its poles

There is another set of numbers than those considered so far – those inherently involved in the factorial function. They are of particular interest to the present investigation – all the more so because they have not been as yet discussed in the literature, though first announced in a preliminary paper presented in Cambridge, England, at a joint meeting of the London and American Mathematical Societies in June 1993. Briefly put, they are the differences between poles that occur at $(-m\pm\alpha)!$, where m is a natural number and hence they may be called ∞-generated numbers. The fundamental such number, Euler's constant γ, which has already risen in the discussion, plays a fundamental role in their structure, as well as does the function most closely associated with it: $\Psi(z)$, the logarithmic derivative of the factorial function, whose translation one unit along the real axis is the familiar gamma function $\Gamma(z)$ so that, where ψ designates the logarithmic derivative of the gamma function, $\psi(z+1) = \Psi(z)$.

Already we have noted the jumping-off point to this enquiry; namely, an easy transformation of Eq. (12)

$$-(-1-\alpha)! - (-1+\alpha)! = 2\gamma \tag{21}$$

or equivalently, in terms of the gamma function $\Gamma(z)$, a unit translation of $z!$,

$$-\Gamma(-\alpha) - \Gamma(\alpha) = 2\gamma. \tag{21a}$$

Thus Eq. (21) answers our question when $m = 1$. For $m = 2$, we find that

$$(-2+\alpha)! - (-2-\alpha) = 2(1-\gamma) \tag{22}$$

and it is hence seen that it is not entirely accurate to say that $(-2!) = -(-1)!$. Rather, the equality sign be changed to $\doteq$ since an infinitesimal is involved. Actually $|(-2)!|/|(-1)!| = 1-\beta$, where β here is an infinitesimal given explicitly by $2\gamma - 1/|(-1)!|$. One can avoid the absolute values by making the appropriate α-specifications as the reader can easily ascertain.

We now go on to determine the general expression for $|(|(-m+\alpha)! | - |(-m-\alpha)!|)|$. It turns out to be remarkably simple. Calling this general difference Δ_m,

$$\frac{1}{2}\Delta_m = \frac{1}{(m-1)!}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m-1} - \gamma\right)$$

which can be more succinctly written as

$$\frac{1}{2}\Delta_m = \frac{\Psi(m-1)}{(m-1)!} = \frac{\psi(m)}{\Gamma(m)}, \tag{23}$$

where Ψ and ψ are as already defined. The value $(1/2)\Delta$ represents the range of magnitude on either side of what we shall term the alpha points (A_m) since

$$|(-m)!| = \frac{\alpha^{-1}}{m!} \pm \frac{1}{2}\Delta_m. \tag{24}$$

Thus each successive alpha point is an infinite number of lower order than the preceding one. Note that, as already seen, $\alpha^{-1} = \zeta(1+\alpha) - \gamma$. Table 2 exhibits the first few alpha points.

Table 2
The alpha points and their ranges. Each $|(-m \pm \alpha)!|$ is a pair of infinite numbers whose differences Δ_m are finite and can be calculated

m	A_m	$(1/2)\Delta_m$
1	$\alpha^{-1}/0!$	$\gamma = -\psi(0)$
2	$\alpha^{-1}/1!$	$1-\gamma = 1+\psi(0)$
3	$\alpha^{-1}/2!$	$\frac{1}{2}\left(\frac{3}{2}-\gamma\right) = \frac{3}{2}+\psi(0)$
4	$\alpha^{-1}/3!$	$\frac{1}{6}\left(\frac{11}{6}-\gamma\right) = \frac{11}{6}+\psi(0)$
5	$\alpha^{-1}/4!$	$\frac{1}{24}\left(\frac{25}{12}-\gamma\right) = \frac{25}{12}+\psi(0)$
6	$\alpha^{-1}/5!$	$\frac{1}{120}\left(\frac{137}{60}-\gamma\right) = \frac{1}{120}+\psi(0)$
m	$\alpha^{-1}/(m-1)!$	$\frac{\lvert\Psi(m-1)\rvert}{(m-1)!} = \frac{\lvert\psi(m)\rvert}{\Gamma(m)}$
∞	0	0

In this connection it is worth noting that

$$\begin{aligned}&\lim_{n\to\infty}[\psi(k(n+1) - \psi(n+1)]\\ &\quad = \lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{k(n+1)-1}\right]\\ &\quad = 0_1+0_2+\cdots+0_{kn} = \log k. \qquad (25)\end{aligned}$$

Thus an infinite and completely null series may have a finite sum and, if $k \in \infty$, even an infinite sum since $\log\infty \in \infty$. But there is more than that: there can be *partially null* series whose sums change when the pattern or frequency of the null terms changes. Giving a minimal number of examples to illustrate the point, we begin with

$$H \equiv 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

the well-known harmonic series (so-called because each term after the first is the harmonic mean of the terms before and after it); and let H_e be the infinite series composed of all the even terms of H; and H_o, the series of all the odd terms of H. It is well known that $H_o - H_e = \ln 2$ (ln denotes the natural logarithm throughout this paper). But it can also be shown that $H_o + H_e = H + \ln 2$, whence $H_o = (1/2)H + \ln 2$ and $H = 2H_e = H_e + H_e$. It should be noted that the last binary operation means term-by-term addition; that is, it involves two identical but distinct series which are then added term-by-term to form the resultant series (in other words, the pace or the "beat" of the series must be respected). Thus, if we have $H_e(H_e)$, i.e. a series consisting of two interleaved H_e series

$$H_e(H_e) \equiv \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \cdots$$

it is then easy to show that $H_e(H_e) \neq 2H_e = H_e + H_e = H$. In fact, $H - H_e(H_e) = \ln 2$ as can readily be deduced, and hence $2H_e = H_e(H_e) + \ln 2$. We are now ready to address partially null series as mentioned above. Consider, for example, H_e interspersed with one and then two null terms. The two series follow:

$$H_e(0_1) \equiv \frac{1}{2} + 0 + \frac{1}{4} + 0 + \frac{1}{6} + 0 + \cdots,$$

$$H_e(0_2) \equiv \frac{1}{2} + 0 + 0 + \frac{1}{4} + 0 + 0 + \frac{1}{6} + 0 + 0 + \cdots$$

These two infinite series are not equal, nor is either equal to H_e. In fact, it may be shown that their difference is

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{3} \frac{1}{6n - 2k + 2} - \sum_{k=1}^{2} \frac{1}{4n - 2k + 2} \right) = \frac{1}{2} \ln 3 - \frac{1}{2} \ln 2, \quad \text{or} \quad H_e(0_1) - H_e(0_2) = \ln \sqrt{\frac{3}{2}} \tag{26}$$

and

$$H_e(0_1) + H_e(0_2) - H = \frac{1}{2} \ln 3 + \frac{1}{2} \ln 2 = \ln \sqrt{6}. \tag{27}$$

Analogous theorems result for $H_o(0_1) \pm H_o(0_2), \ldots$ Before leaving these instructive infinite series, we have space to note that $2H_o - H = \log 4$, which is a practically by-inspection corollary of a result already stated. We also add the remark that there are evident connections between frequency ratios of natural musical tones (i.e. not those of the tempered = tampered scale) and antilogarithms of squared differences among H_o, H_e family members and their kindred partially null series. Thus, the parent series H was well named "harmonic" and through Euler's constant γ – that so frequently occurring, finite companion of the infinite – it is connected with the discussions of the previous sections. Light on the vexed question of its structure is shed by our finding that $\gamma = \sum^{\infty}$ (the areas of an infinite series of right curvilinear triangles), with hyperbolic arcs as their hypotenuses and each containing a Euclidean right triangle and a hyperbolic segment or lune enclosed by an arc and its chord. The sum of the latter triangles is 1/2 the reciprocals of the triangular numbers or

Table 3
The finite differences between harmonic-type series

Pair-types of divergent infinite series whose differences are finite	Formula for these (finite) differences in terms of the sum of the nth block then summed form $n = 1$ to ∞	Simplified closed forms of those differences
(N and Z are respectively the numbers of Non-null & Zero terms. $N + Z =$ block length) Formula for H_o–H_o $(N, 0_Z)$ in terms of the generating function for the nth "block"	$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{N+Z}\frac{1}{2n(N+Z)-2k+1}-\sum_{k=1}^{N}\frac{1}{2nN-2k+1}\right)$	A *block* is the minimal group of difference terms whose structure is periodic througout the two interacting series. That structure in turn depends on N and Z.
These differences can be written very simply as functions of N and Z	Thus both H_o–H_o and H_e–H_e sum to:	$1/2\ln(1 + (Z/N))$
	And the H_o–H_e type sums to:	$1/2\ln 4(1 + (Z/N))$
Formula for H_e–$H_e(N, 0_Z)$ in terms of the generating function for the nth block	$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{N+Z}\frac{1}{2n(N+Z)-2k+2}-\sum_{k=1}^{N}\frac{1}{2nN-2k+2}\right)$	(Simplified closed form above)
Formula for H_o–$H_e(N, 0_Z)$ in terms of the generating function for the nth block	$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{N+Z}\frac{1}{2n(N+Z)-2k+1}-\sum_{k=1}^{N}\frac{1}{2nN-2k+2}\right)$	(Simplified closed form above)

$(1/2)\sum_{n=2}^{\infty} 2/(n(n+1)) = 1/2$. Hence the sum of the lunes is $\gamma - 1/2$ and since $\sum(lunes)$ is transcendental hence γ also is.

Now referring to Table 3, if we take $N = 1$, we obtain a new expression for $\ln r$ (r a real integer)

$$\sum_{n=1}^{\infty}\left(\left(\sum_{k=0}^{r-1}\frac{1}{nr-k}\right)-\frac{1}{n}\right) = \ln r \tag{28}$$

and we can investigate further via $\lim_{n\to\infty}(\sum_{k=1}^{n}(1/k) - \ln rn) = \ln r$.

Also, $\lim_{n\to\infty}\sum_{k=1}^{n}((1/k) - \lambda \ln n) = \infty, \gamma, -\infty$ according as $\lambda < 1$, $\lambda = 1$, or $\lambda > 1$. Note how critical a threshold is indicated by γ (when λ is exactly 1); that is, λ must equal unity to within an infinitesimal, and γ stands as the *Scheidepunkt* between $+\infty$ and $-\infty$.

Now addressing Table 3, each of the three basic types of finite differences generated by two infinite numbers (H_o–H_o, H_e–H_e, and H_o–H_e) has a distinct formula depending on the type, as shown explicitly in the table. From these fundamental expressions, many more may be derived, as we saw adumbrated in the preceding discussion of finite differences between two infinites like $H_e(0_1) - H_e(0_2)$ or, more explicitly $H_e(1, 0_1) - H_e(1, 0_2)$.

There is more to be said on partially null harmonic-type infinite series, but that is for another occasion. We close this section with a theorem whose corollaries we leave to the reader.

Theorem. *The general expression for the difference between two partially null harmonic series namely* $H_1 > H_2$, *both of same type* (*i.e.* H_e *or else* H_o), *with* $H_1 \equiv H(N_1, 0_{Z_1})$ *and* $H_2 \equiv H(N_2, 0_{Z_2})$, *is given by*

$$H(N_1, 0_{Z_1}) - H(N_2, 0_{Z_2}) = \frac{1}{2}\ln\frac{(N_2+Z_2)/N_2}{(N_1+Z_1)/N_1} = \frac{1}{2}\ln\frac{N_1(N_2+Z_2)}{N_2(N_1+Z_1)}.$$

Note that if $N_m/Z_m \equiv R_m$ *and if* $R_1 > R_2$, *then* $H_1 > H_2$ *and conversely*; *also, that if* $H_1 \equiv H_e$ (*or* H_o), *i.e.*, $N_1 = 1$ *and* $Z_1 = 0$, *the above expression for the difference reduces to the previous results given in Table* 3, *column* 3. *The corollary for the combination* H_o–H_e *is left to the reader*.

It is now time to have a closer look at one of the most intriguing functions in all mathematics, which has already played such a fundamental role in the Bernoulli numbers and in the function $\beta(s)$ that represents those numbers with their index made general and continuous, as previously discussed.

5. Riemann's zeta function

From the start, the number 1/2 is involved in this function, as the following little theorem shows: $\sum_{n=1}^{\infty}(\zeta(2n) - \zeta(2n+1)) = 1/2$, the proof being a straightforward yet interesting exercise (*hint*: *deploy the properties of* $\Psi(x) \equiv \mathrm{d}\ln(x!)/\mathrm{d}x \equiv (x!)'/x!$). This is a result of $\sum_{n=1}^{\infty}(\zeta(2n)-1)) = 3/4$; and if we change the argument from $2n$ to $2n+1$ the sum to infinity is $1/4$; hence again $\sum_{n=1}^{\infty}(\zeta(2n)-\zeta(2n+1)) = 1/2$. As before noted, Riemann surmised that the real part of all the complex roots of $\zeta(s)$ is also $1/2$, and he has been proved correct through literally millions of complex zeros. But number theory is subtle, and exceptions to a "rule" can begin sometimes only at astronomically huge numbers. Hence the sustained interest in what the mathematical reality actually is.

It is a testimony to the depth of Riemann's insight and expertise that he had found a far better computing tool for the complex zeros of his function than even the excellent Euler–Maclaurin summation – a far-reaching method now called the Riemann–Siegel asymptotic approximation formula since it was expertly retrieved from Riemann's manuscript notes by Carl Ludwig Siegel in the early 1930s. And before that, in 1926, the mathematician Bessel–Hagen found in Riemann's posthumous papers a remarkable representation of $\zeta(s)$ in terms of definite integrals, which the indefatigable Siegel also rescued from relative obscurity, commented on, and publicized to the mathematical community in 1932, duly crediting Bessel–Hagen. If Weierstrass (1815–97), a reigning doyen of the time, had not crushingly repressed Riemann's work for a spiteful decade, we might have heard from Riemann himself before he was untimely taken by tuberculosis in July 1866 before he had even reached the age of forty. Lucky for mathematics that Siegel came along, who could not only read Riemann's small and tight hand, but could understand his profoundly original ideas as well.

As an introduction to the definite integral representation we first note Riemann's important variant of $\zeta(s)$: the function $\xi(s) = ((1/2)s)!(s-1)\zeta(s)\pi^{-1/2}$, which has the same roots as $\zeta(s)$. Then the integral formula can assume the form $((1/2)s)^{-1}((1/2)s)!\zeta(s)\pi^{-1/2} = G(s) + \overline{G(1-\bar{s})}$, where G is $((1/2)s-1)!$ multiplied by a contour integral in exponentials of x and x^{-s}. The only fruitful simplification of this representation occurs when $s = (1/2)+t$, which affords a hint that what may be going on may deeply involve the number $1/2$. We now turn to a function usually called $Z(t)$ asymptotically approximated by the Riemann–Siegel formula and defined as $\exp \mathrm{i}\theta\zeta((1/2)+\mathrm{i}t)$, where $\theta = \mathrm{Im}((1/2)\mathrm{i}t-(3/4)) - (1/2)t\ln\pi$, which can be used to calculate roots of $\zeta(s)$. But the more interesting thing about $Z(t)$ is that there are endlessly many t-values that make its derivative zero such that $Z(t)$ is maximal at successively smaller distances d above the t-axis. This has worried some mathematicians into saying in effect that the Riemann hypothesis is "almost contradicted" – which itself is practically a contradiction in terms since "almost contradicted" means not contradicted.

Actually there is no cause for worry since $\lim_{t\to\infty} d = 0$ as the algebraic geometry of the situation indicates, and at no finite value of t is $d \leqslant 0$. At each such near-zero point where d is very small – for example, between the 6707th and 6708th zeros – there ensues a sharp dip in $Z(t)$ during which $Z'(t)$ becomes a large negative number, followed by an even sharper rise to a high positive maximum. The sharpness of such falls-and-then-rises increases the smaller d becomes (e.g. near the 13400000th zero where d is extremely small), so that if d were to be zero, $|Z'(t)|$ would be infinite and there would hence be a discontinuity in $Z(t)$ at that point. But it is not difficult to see that $Z(t)$ remains continuous for all finite values of t and that hence d could not become zero – hence turning Riemann's Vermutung into a theorem by an exercise, albeit an exotic one, in algebraic geometry.

There is an infinite difference in meaning between "any finitely positive number, however small" and "zero". Ignoring that difference lies at the core of the fallacies besetting Weierstrassian epsilontics which too often degenerate into epsil-*antics* and becloud the very rigour that the "proofs" so infected boast of having. There are functions that are continuous within an infinitesimal of a certain value, but become infinite and develop a discontinuity when that critical infinitesimal is crossed. In such cases, the behaviour of a function for any positive finite number, however great (or small), gives no clues as to, much less can be used as a guide to, the behaviour at "passing to the limit", and the two are not at all consistent or continuous with each other. Heaviside's unit function $H(t)$ is an example: being a square wave of infinite wavelength and unit amplitude, it remains unity for any positive value of t, but suddenly *drops* to zero at $t = \infty$; just as for any negative value of t, $H(t) = 0$, but for $t = -\infty$, there is a sudden *rise* to $H(t) = 1$. Or, for any real number r, however close to zero, $\Gamma(r)$ is finite, yet at $r = 0$, we have $\Gamma(r) = +\infty$ if r is on the positive side of zero, but $\Gamma(r) = -\infty$ if r is on the negative side of zero. Thus there can be a doubly infinite jump from $+\infty$ to $-\infty$ as r traverses the infinitesimal distance from $+\alpha$ to $-\alpha$ as per the definition and deployment of α in Section 3.

Now let us look at the matter from the algebraic and analytic number-theoretic standpoint. Riemann's findings uncovered an interesting equation involving $\zeta(s)$ which is still an enigma as to why Riemann included it in his epochal paper of 1859. Thus a current authority of Riemann's work, Harold M. Edwards, was the first to note publicly an important fact. He writes

> Riemann states that this series representation of $\zeta(s)$ as an even function of $s - (1/2)$ 'converges very rapidly', but he gives no explicit estimates and he does not say what role this series plays in the assertions which he makes next [15].

There is indeed a lacuna and clear non-sequitur at the place Edwards mentions. The answer to his implied query is that it seems likely that something in the original manuscript that Riemann had intended for printing never reached the typesetter. But this mishap, most likely due to chronic fatigue of the author, was within seven years of Riemann's death from tuberculosis, hastened by protracted poverty, continued strain of intense work, and without doubt by what must have been the unkindest cut of all – the coldly hostile disregard and criticism from Weierstrass who saw in Riemann's brilliant global approach to complex variable theory a threat to Weierstrass' narrowly arithmetic and piecemeal approach through analytic continuation. Riemann's use of Dirichlet's principle was fully vindicated by David Hilbert – but only four decades later; and Karl Weierstrass', as well as more recent criticisms of Godefrey Hardy (to the effect that theorems Riemann had in fact proved were 'conjectures' – and this despite Hardy's later fine support of Ramanujan!) are now seen for the mistakes they were and are.

Now let us look at the equation that Riemann left hanging [16] to haunt us, apparently through miscarriage to the printer. It can be written as follows:

$$\zeta\left(\frac{1}{2}+\mathrm{i}t\right) = 4\int_1^\infty \frac{\mathrm{d}\left[x^{3/2}\left(\mathrm{d}\sum_{n=1}^\infty \mathrm{e}^{-\pi n^2 x}/\mathrm{d}x\right)\right]}{\mathrm{d}x} x^{-1/4}\cos\left(\frac{t}{2}\ln x\right)\mathrm{d}x.$$

That $\zeta((1/2)+\mathrm{i}t)$ can be represented as a series is really secondary to what must have been Riemann's prime intention in so ingeniously working it out. Of course the infinite series of increasingly negative exponential functions ensures speedy convergence, but the real point lies in the periodic cosine term, which Riemann obtained from a complex hyperbolic cosine which reduces to a real cosine when $Re(s)=1/2$. This term ensures an infinity of zeros for $\zeta(s)$ as x increases indefinitely, and hence an infinity of roots of $\zeta(s)$ on the critical line. That was Riemann's point in introducing this form of $\zeta(s)$ and that had to be the gist of the lacuna now in the text of this extraordinary paper. So Hardy's 1914 proof [17] of the infinity was not actually an "advance" on the Riemann Vermutung since it was clearly implicit in the equation given 55 years earlier by Riemann, with an explanation now evidently lost, in his famous paper of 1859. The problem was, people were not reading Riemann himself enough or carefully enough. But the ensuing numerical investigations were wonderfully indefatigable and ingenious.

Subsequent work – and it was demanding and arduous – went into staking out the successive roots of $\zeta(s)$ as t increases indefinitely and demonstrating by actual calculation that they all lay on the critical line as foretold by Riemann. Gram [18] began this work (although Riemann himself had already calculated some of the first few roots), followed by Backlund [19] and by Hutchinson [20], who contributed some significant advances in method. Then, with the advent of machine computation, the investigation was pushed into astronomical numbers of zeros on the critical line: first by Lehmer [21], who formulated a machine-usuable version of the Riemann–Siegel formula, and then by

Haselgrove with Miller [22], Rosser et al. [23], van de Lune et al. [24], [3] and finally in a herculean effort (reaching the 10^{21}st complex zero!), by A.M. Odlyzko, [4] head of mathematics research at AT&T, still with no exception to Riemann's prophecy, but still without the complete assurance that had so long been sought.

Let us now have a closer look at Riemann's equations themselves, where so much mathematical treasure has been mined. In the functional equation, it is clear that the zeros for $\zeta(s)$ and $\zeta(1-s)$ are paired. But pairing arises also from the fact that $\zeta(s)$ is a member of the infinite set of analytic functions, for which we know that if $a+b\mathrm{i}$ is a root, then so is its conjugate. So in the case of $\zeta(s)$ there are two expressions for each pair of roots. Equating these expressions for any given root gives us $\rho = \sigma + \mathrm{i}t = s$, and then $\sigma - \mathrm{i}t = 1 - s = 1 - (\sigma + \mathrm{i}t)$, or $\sigma - \mathrm{i}t = 1 - \sigma - \mathrm{i}t$, whence *for any complex root* ρ of $\zeta(s)$,

$$2\sigma = 1 \tag{29}$$

and Riemann was right.

[3] van de Lune et al. computer-verified that the first billion and a half complex zeros of $\zeta(s)$ indeed were on the critical line.

[4] Odlyzko, in the introductory note (see [25]), says in part:

> Starting in the late 1970s, I carried out a series of computations [that] obtained accurate values ... if blocks of consecutive zeros high up on the critical strip [in order] to come closer to observing the true asymptotic behavior of the zeta function, which is often approached slowly. Those calculations stimulated the invention, jointly with Arnold Schonhage, [see Odlyzko's New analytic algorithms, in *number Theory*, pp. 466–475 of *Proceedings of the International Congress of Mathematics* 1986, published by the Amer Math. Soc. in 1987] of an improved algorithm for computing large sets of zeros [in Fast algorithms for multiple evaluations of the Riemann zeta function, *Transactions of the American Mathematical Society*, 309 (1988), pp. 797–809.]
>
> This author well concludes on pages 39 and 41 of his 1989 article in *Number Theory*: "While the progress in technology was rapid and contributed greatly to the computation of the zeros, a very significant contribution was made by advances in algorithms. This is a very common, and generally unappreciated, feature of many areas of computational science. However, it has to be acknowledged that technology is crucial to making the new algorithms work. To use the new methods effectively one needs large and fast machines."

He also wisely reminds us that the new algorithms cost more computer storage space, and are more efficient than the Riemann–Siegel formula only when the values of many zeros at close intervals are needed.

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