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CERTAIN CLASSES OF HARMONIC FUNCTIONS PERTAINING TO SPECIAL FUNCTIONS

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Making use of generalized Dziok-Srivastava operator we introduced a new class of complex-valued harmonic functions which are orientation preserving, univalent and starlike in the unit disc. We investigate the coefficient bounds, distortion inequalities, extreme points and inclusion results for the generalized class of functions.

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С помощью обобщённого оператора Дзёка-Сриваставы вводится новый клас комплекснозначных гармонических функций, сохраняющих ориентацию, однолистность и звёздность. Устанавливаются оценки коэффициентов, неравенства искажения, экстремальные точки и доказываются результаты о включениях для обобщённого класса функций.

1. Introduction. A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply connected domain $D \subset \Omega$ we can write $f = h + \overline{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [1]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1}$$

which are harmonic univalent and orientation preserving in the open unit disc $U = \{z \in \mathbb{C}\colon |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, we may express

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \quad |b_1| < 1. \tag{2}$$

where the analytic functions h and g are in the forms

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad 0 \le b_1 < 1.$$

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We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class S of normalized univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero that is $g \equiv 0$. Due to Silverman [11] we denote by $\overline{\mathcal{H}}$ the subclass of $\mathcal{H}$ consists harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \ |b_1| < 1. \tag{3}$$

Let the Hadamard product (or convolution) of two power series $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ and $\psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n$ be defined by $(\phi * \psi)(z) = \phi(z) * \psi(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n$.

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ ($\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m$) the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$${}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \tag{4}$$
$$(l \leq m+1; l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in U)$$

where $\mathbb{N}$ denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0, \\ a(a+1)(a+2)\ldots(a+n-1), & n \in \mathbb{N}. \end{cases} \tag{5}$$

For positive real values of $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ ($\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m$), let

$$H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) \colon S \to S$$

be a linear operator defined by

$$[(H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m))(\phi)](z) := z {}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2 \ldots, \beta_m; z) * \phi(z) =$$
$$= z + \sum_{n=2}^{\infty} \omega_n(\alpha_1; l; m) \phi_n z^n \tag{6}$$

where

$$\omega_n(\alpha_1; l; m) = \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{1}{(n-1)!}. \tag{7}$$

For notational simplicity, we use a shorter notation $H_m^l[\alpha_1]$ for $H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)$ in the sequel. It follows from (6) that

$$H_0^1[1]\phi(z) = \phi(z), H_0^1[2]\phi(z) = z\phi'(z)$$

The linear operator $H_m^l[\alpha_1]$ is called Dziok-Srivastava operator (see [4]). It contains such well-known operators as the Hohlov linear operator, Saitho generalized linear operator, the Carlson-Shaffer linear operator, the Ruscheweyh derivative operator as well as its generalized versions, the Bernardi-Libera-Livingston operator, and the Srivastave-Owa fractional derivative operator. One may refer to [3], [4] and [12] for more details concerning these operators (see [2, 8, 9, 10]). By using the generalized hypergeometric function, we now define the linear operator $\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}$ as follows: $\mathcal{L}^{0}_{\mu,\alpha_1}\phi(z) = \phi(z)$,

$$\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}\phi(z) = z + \sum_{n=2}^{\infty} \omega_n^{\tau}(\alpha_1;\mu;l;m)\phi_n z^n \tag{8}$$

where

$$\omega_n^{\tau}(\alpha_1;\mu;l;m) = \left(\frac{(\alpha_1)_{n-1}\dots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_m)_{n-1}}\frac{[1+\mu(n-1)]}{(n-1)!}\right)^{\tau}, (n \in \mathbb{N}\setminus\{1\}, \tau \in \mathbb{N}_0; \mu \geq 0) \tag{9}$$

$\alpha_i > 0, (i = 1, 2, ...l), \beta_j > 0, (j = 1, 2, ...m), l \leq m+1; l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In view of the relationship (6) and the linear operator (8) for the harmonic function $f = h + \overline{g}$ given by (1), we define the operator

$$\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}f(z) = \mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}h(z) + \overline{\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z)}. \tag{10}$$

Motivated by Jahangiri et al. [5, 6], and Murugusundaramoorthy and Vijaya [7], we define a new subclass $\mathcal{FH}^{\tau}_{\mu}(\lambda,\gamma)$ of $\mathcal{H}$ in terms of the operator defined by (10).

For $0 \leq \gamma < 1$, we let $\mathcal{FH}^{\tau}_{\mu}(\lambda,\gamma)$ a subclass of $\mathcal{H}$ of the form $f = h + \overline{g}$ given by (2) and satisfying the analytic criteria

$$\operatorname{Re}\left\{(1-\lambda)\frac{\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}f(z)}{z} + \lambda\frac{\frac{\partial}{\partial\theta}\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}f(z)}{\frac{\partial}{\partial\theta}z}\right\} \geq \gamma, \tag{11}$$

where $\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}f(z)$ as given in (10), $0 \leq \lambda \leq 1$ and $z \in U$. We also let $\mathcal{F}_{\overline{\mathcal{H}}\mu}^{\tau}(\lambda,\gamma) = \mathcal{FH}^{\tau}_{\mu}(\lambda,\gamma)$ $\cap\overline{\mathcal{H}}$.

Note that as λ changes from 0 to 1, the family $\mathcal{FH}^{\tau}_{\mu}(\lambda,\gamma)$ produces a passage from the class of harmonic functions $\mathcal{PH}^{\tau}_{\mu}(\gamma) \equiv \mathcal{FH}^{\tau}_{\mu}(0,\gamma)$, consisting of functions f where

$$\operatorname{Re}\left\{\frac{\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}f(z)}{z}\right\} \geq \gamma, \ (0 \leq \gamma < 1)$$

to the class of harmonic functions $\mathcal{QH}^{\tau}_{\mu}(\gamma) \equiv \mathcal{FH}^{\tau}_{\mu}(1,\gamma)$ consisting of functions f where

$$\operatorname{Re}\left\{\frac{\frac{\partial}{\partial\theta}\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}f(z)}{\frac{\partial}{\partial\theta}z}\right\} > \gamma, \ (0 \leq \gamma < 1).$$

In this paper, we obtaine coefficient conditions for the classes $\mathcal{FH}^{\tau}_{\mu}(\lambda,\gamma)$ and $\mathcal{F}_{\overline{\mathcal{H}}\mu}^{\tau}(\lambda,\gamma)$. A representation theorem, inclusion properties and distortion bounds for the class $\mathcal{F}_{\overline{\mathcal{H}}\mu}^{\tau}(\lambda,\gamma)$ are also established.

2. The class $\mathcal{FH}^{\tau}_{\mu}(\lambda,\gamma)$. In this section we define a new class $\mathcal{FH}^{\tau}_{\mu}(\lambda,\gamma)$ and obtain necessary and sufficient conditions, a representation theorem, distortion bounds and inclusion properties for the functions to belong to the class $\mathcal{FH}^{\tau}_{\mu}(\lambda,\gamma)$.

In the following theorem we give the sufficient coefficient bound for $f \in \mathcal{FH}^{\tau}_{\mu}(\lambda,\gamma)$.

Theorem 1. *Let $f = h + \overline{g}$ be of the form (2). Then $f \in \mathcal{FH}^{\tau}_{\mu}(\lambda,\gamma)$ if*

$$\sum_{n=2}^{\infty}\omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n-1)+1||a_n| + \sum_{n=1}^{\infty}\omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n+1)-1||b_n| \leq 1-\gamma. \tag{12}$$

Proof. By taking

$$w(z) = (1-\lambda)\frac{\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}f(z)}{z} + \lambda\frac{\frac{\partial}{\partial\theta}\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}f(z)}{\frac{\partial}{\partial\theta}z},$$

it suffices to show that $|1-\gamma+w| \geq |1+\gamma-w|$. This is equivalent to show that if the condition (11) holds, then

$$\left|(1-\gamma)z + (1-\lambda)\left[H^l_m[\alpha_1]h(z) + \overline{\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}g(z)}\right] + \lambda\left[z\left(\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}h(z)\right)' - \overline{z\left(\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}g(z)\right)'}\right]\right| -$$
$$- \left|(1+\gamma)z - (1-\gamma)\left[\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}h(z) + \overline{\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}g(z)}\right] - \lambda\left[z\left(\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}h(z)\right)' - \overline{z\left(\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}g(z)\right)'}\right]\right| \geq 0.$$

Substituting for $\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}h(z)$ and $\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}g(z)$,$(\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}h(z))'$ and $(\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}g(z))'$ in the above condition we get

$$\left|(1-\gamma)z + \left\{\sum_{n=2}^{\infty}(\lambda(n-1)+1)\,a_n z^n - \sum_{n=1}^{\infty}(\lambda(n+1)-1)\,\bar{b}_n\bar{z}^n\right\}\omega^\tau_n(\alpha_1;\mu;l;m)\right| -$$
$$- \left|-\gamma z - \left\{\sum_{n=2}^{\infty}(\lambda(n-1)+1)\,a_n z^n - \sum_{n=1}^{\infty}(\lambda(n+1)-1)\,\bar{b}_n\bar{z}^n\right\}\omega^\tau_n(\alpha_1;\mu;l;m)\right| \geq$$
$$\geq 2\left[(1-\gamma) - \left(\sum_{n=2}^{\infty}|\lambda(n-1)+1|\,|a_n| + \sum_{n=1}^{\infty}|\lambda(n+1)-1|\,|b_n|\right)\omega^\tau_n(\alpha_1;\mu;l;m)\right].$$

The last expression is non-negative by the hypothesis and so the proof is complete. □

Our next theorem establishes that such coefficient bounds cannot be improved.

Theorem 2. *Let $f = h + \overline{g}$ be of the form (3).Then $f \in \mathcal{F}\overline{\mathcal{H}}^\tau_\mu(\lambda,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}\omega^\tau_n(\alpha_1;\mu;l;m)|\lambda(n-1)+1||a_n| + \sum_{n=1}^{\infty}\omega^\tau_n(\alpha_1;\mu;l;m)|\lambda(n+1)-1||b_n| \leq 1-\gamma. \quad (13)$$

Proof. The "if" part follows from Theorem 1 upon noting that $\mathcal{F}\overline{\mathcal{H}}^\tau_\mu(\lambda,\gamma) \subset \mathcal{F}\mathcal{H}^\tau_\mu(\lambda,\gamma)$. For the "only if" part we assume that $f \in \mathcal{F}\overline{\mathcal{H}}^\tau_\mu(\lambda,\gamma)$. Then for $z = re^{i\theta}$ in U we obtain,

$$\mathrm{Re}\left\{(1-\lambda)\frac{\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}f(z)}{z} + \lambda\frac{\frac{\partial}{\partial\theta}\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}f(z)}{\frac{\partial}{\partial\theta}z}\right\} =$$
$$= \mathrm{Re}\left\{(1-\lambda)\frac{\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}h(z) + \overline{\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}g(z)}}{z} + \lambda\frac{z(\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}h(z))' - \overline{z(\mathcal{L}^{\tau,\alpha_1}_{\mu,l,m}g(z))'}}{\frac{\partial}{\partial\theta}z}\right\} \geq$$
$$\geq 1 - \sum_{n=2}^{\infty}\omega^\tau_n(\alpha_1;\mu;l;m)|\lambda(n-1)+1||a_n|r^{n-1} - \sum_{n=1}^{\infty}\omega^\tau_n(\alpha_1;\mu;l;m)|\lambda(n+1)-1||b_n|r^{n-1} \geq \gamma.$$

The above inequality must hold for all $z \in U$. In particular, letting $z = r \to 1$, we get the required condition (13). □

As a special case of Theorem 2, we can state the following two corollaries.

Corollary 1. *$f = h + \overline{g} \in \mathcal{P}_{\overline{\mathcal{H}}_\mu}^{\tau}(\gamma) = \mathcal{P}_{\mathcal{H}_\mu}^{\tau}(\gamma) \cap \overline{\mathcal{H}}$, if and only if*

$$\sum_{n=2}^{\infty} \omega_n^{\tau}(\alpha_1;\mu;l;m)|a_n| + \sum_{n=1}^{\infty} \omega_n^{\tau}(\alpha_1;\mu;l;m)|b_n| \leq 1-\gamma.$$

Corollary 2. *$f = h + \overline{g} \in \mathcal{Q}_{\overline{\mathcal{H}}_\mu}^{\tau}(\gamma) = \mathcal{Q}_{\mathcal{H}_\mu}^{\tau}(\gamma) \cap \overline{\mathcal{H}}$, if and only if*

$$\sum_{n=2}^{\infty} n\omega_n^{\tau}(\alpha_1;\mu;l;m)|a_n| + \sum_{n=1}^{\infty} n\omega_n^{\tau}(\alpha_1;\mu;l;m)|b_n| \leq 1-\gamma.$$

Now, we give the inclusion relation between the classes $\mathcal{P}_{\overline{\mathcal{H}}_\mu}^{\tau}(\gamma)$ and $\mathcal{Q}_{\overline{\mathcal{H}}_\mu}^{\tau}(\gamma)$, $\mathcal{F}_{\overline{\mathcal{H}}_\mu}^{\tau}(\lambda,\gamma)$

Theorem 3. *For $m > -1$ and $0 \leq \gamma < 1$, we have*

i) $\mathcal{Q}_{\overline{\mathcal{H}}_\mu}^{\tau}(\gamma) \subset \mathcal{P}_{\overline{\mathcal{H}}_\mu}^{\tau}(\gamma)$,

ii) $\mathcal{Q}_{\overline{\mathcal{H}}_\mu}^{\tau}(\gamma) \subset \mathcal{F}_{\overline{\mathcal{H}}_\mu}^{\tau}(\lambda,\gamma)$, $0 \leq \lambda \leq 1$,

iii) $\mathcal{F}_{\overline{\mathcal{H}}_\mu}^{\tau}(\lambda,\gamma) \subset \mathcal{Q}_{\overline{\mathcal{H}}_\mu}^{\tau}(\gamma)$, $\lambda \geq 1$.

Proof. (i) In view of the Corollaries 1 and 2, since $f(z) \in \mathcal{P}_{\overline{\mathcal{H}}_\mu}^{\tau}(\gamma)$ we have

$$\sum_{n=2}^{\infty} \omega_n^{\tau}(\alpha_1;\mu;l;m)|a_n| + \sum_{n=1}^{\infty} \omega_n^{\tau}(\alpha_1;\mu;l;m)|b_n| \leq$$
$$\leq \sum_{n=2}^{\infty} n\omega_n^{\tau}(\alpha_1;\mu;l;m)|a_n| + \sum_{n=1}^{\infty} n\omega_n^{\tau}(\alpha_1;\mu;l;m)|b_n| \leq 1-\gamma,$$

so $f(z) \in \mathcal{Q}_{\overline{\mathcal{H}}_\mu}^{\tau}(\gamma)$, hence $\mathcal{Q}_{\overline{\mathcal{H}}_\mu}^{\tau}(\gamma) \subset \mathcal{P}_{\overline{\mathcal{H}}_\mu}^{\tau}(\gamma)$.

(ii) For $0 \leq \lambda < 1$, and for $f(z) \in \mathcal{F}_{\overline{\mathcal{H}}_\mu}^{\tau}(\lambda,\gamma)$, we observe that

$$\sum_{n=2}^{\infty} \omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n-1)+1||a_n| + \sum_{n=1}^{\infty} \omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n+1)-1||b_n| \leq$$
$$\leq \sum_{n=2}^{\infty} n\omega_n^{\tau}(\alpha_1;\mu;l;m)|a_n| + \sum_{n=1}^{\infty} n\omega_n^{\tau}(\alpha_1;\mu;l;m)|b_n| \leq 1-\gamma,$$

by Corollary 2 and so (ii) follows from Theorem 2.

(iii) Note that for $\lambda \geq 1$. Then by Theorem 2

$$\sum_{n=2}^{\infty} n\omega_n^{\tau}(\alpha_1;\mu;l;m)|a_n| + \sum_{n=1}^{\infty} n\omega_n^{\tau}(\alpha_1;\mu;l;m)|b_n| \leq$$
$$\leq \sum_{n=2}^{\infty} \omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n-1)+1||a_n| + \sum_{n=1}^{\infty} \omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n+1)-1||b_n| \leq 1-\gamma.$$

Therefore, the result (iii) follows from Corollary 2. □

Corollary 3. *For $0 \leq \gamma < 1$, we have $\mathcal{F}_{\overline{\mathcal{H}}_\mu}^{\tau}(\lambda,0) \subset \mathcal{Q}_{\overline{\mathcal{H}}_\mu}^{\tau}(0)$, if $0 \leq \lambda < 1$.*

Proof. If $f \in \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda, 0)$ and $0 \le \lambda < 1$, then by (13), we have

$$\sum_{n=2}^{\infty} \omega_n^\tau(\alpha_1; \mu; l; m)|\lambda(n-1)+1||a_n| + \sum_{n=1}^{\infty} \omega_n^\tau(\alpha_1; \mu; l; m)|\lambda(n+1)-1||b_n| \le$$
$$\le \sum_{n=2}^{\infty} \omega_n^\tau(\alpha_1; \mu; l; m)(\lambda(n-1)+1)|a_n| + \sum_{n=1}^{\infty} \omega_n^\tau(\alpha_1; \mu; l; m)(\lambda(n+1)-1)|b_n| \le$$
$$\le \sum_{n=2}^{\infty} n\omega_n^\tau(\alpha_1; \mu; l; m)|a_n| + \sum_{n=1}^{\infty} n\omega_n^\tau(\alpha_1; \mu; l; m)|b_n| \le 1,$$

and so $f \in \mathcal{Q}\overline{\mathcal{H}}_\mu^\tau(0)$. □

3. Distortion Bounds and Extreme Points. Now we obtain distortion bounds for functions in $\mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda, \gamma)$, which yields a covering result for the family.

Theorem 4. *If $\mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda, \gamma)$, and $|z| = r < 1$, then*

$$|f(z)| \le (1+|b_1|)r + \frac{1}{\omega_2^\tau(\alpha_1; \mu; l; m)} \left(\frac{1-\gamma}{1+\gamma} - \frac{|2\lambda - 1|}{1+\lambda}|b_1| \right) r^2,$$
$$|f(z)| \ge (1-|b_1|)r - \frac{1}{\omega_2^\tau(\alpha_1; \mu; l; m)} \left(\frac{1-\gamma}{1+\gamma} - \frac{|2\lambda - 1|}{1+\lambda}|b_1| \right) r^2.$$

Proof. We only prove the left hand inequality. The right hand inequality can be proved using similar arguments. Let $f(z) \in \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda, \gamma)$, then by Theorem 2, we obtain

$$|f(z)| = \left| z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n \right| \ge (1-|b_1|)r - \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n \ge$$
$$\ge (1-|b_1|)r - \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^2 \ge (1-|b_1|)r -$$
$$- \frac{1-\gamma}{\omega_2^\tau(\alpha_1; \mu; l; m)(1+\lambda)} \left[\sum_{n=2}^{\infty} \omega_n^\tau(\alpha_1; \mu; l; m) \left(\frac{|\lambda(n-1)+1|}{1-\gamma}|a_n| + \frac{|\lambda(n+1)-1|}{1-\gamma}|b_n| \right) \right] r^2 \ge$$
$$\ge (1-|b_1|)r - \frac{1}{\omega_2^\tau(\alpha_1; \mu; l; m)} \left[\frac{1-\gamma}{1+\lambda} - \frac{|2\lambda - 1|}{1+\lambda}|b_1| \right] r^2.$$ □

The covering result follows from the left hand inequality in Theorem 4.

Corollary 4. *If $f(z) \in \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda, \gamma)$, $\lambda \ge 1$ then*

$$\left\{ w \colon |w| < \frac{\omega_2^\tau(\alpha_1; \mu; l; m)(1+\lambda) - (1-\gamma)}{\omega_2^\tau(\alpha_1; \mu; l; m)(1+\lambda)} - \frac{\omega_2^\tau(\alpha_1; \mu; l; m)(1+\lambda) - |2\lambda - 1|)}{\omega_2^\tau(\alpha_1; \mu; l; m)(1+\lambda)} b_1 \right\} \subset f(U).$$

Proof. By the left-hand side inequality given in Theorem 4 and letting $r \to 1$, we prove that

$$(1-|b_1|) - \frac{1}{\omega_2^\tau(\alpha_1; \mu; l; m)} \left(\frac{1-\gamma}{1+\lambda} - \frac{|2\lambda - 1|}{1+\lambda} b_1 \right) =$$
$$= \frac{(1-b_1)\omega_2^\tau(\alpha_1; \mu; l; m)(1+\gamma) - (1-\gamma-|2\lambda-1|)b_1}{\omega_2^\tau(\alpha_1; \mu; l; m)(1+\lambda)} =$$
$$= \frac{2\omega_2^\tau(\alpha_1; \mu; l; m)(1+\lambda) - (1-\gamma)}{\omega_2^\tau(\alpha_1; \mu; l; m)(1+\lambda)} - \frac{[\omega_2^\tau(\alpha_1; \mu; l; m))(1+\lambda) - |2\lambda - 1|)]}{\omega_2^\tau(\alpha_1; \mu; l; m)(1+\lambda)} b_1 \subset f(U).$$ □

Theorem 5. *If* $f \in \mathcal{P}\overline{\mathcal{H}}^{\tau}_{\mu}(\gamma)$, *then*

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{\omega_2^{\tau}(\alpha_1;\mu;l;m)}\left(1-\alpha-|b_1|\right)r^2, \quad |z|=r<1,$$
$$|f(z)| \geq (1-|b_1|)r - \frac{1}{\omega_2^{\tau}(\alpha_1;\mu;l;m)}\left(1-\gamma-|b_1|\right)r^2, \quad |z|=r<1.$$

Proof. We prove the left-hand side inequality of the theorem. Taking the absolute value of $f(z)$ given by (3)

$$|f(z)| = \left|z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n\right| \geq (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \geq$$
$$\geq (1-|b_1|)r - \frac{1-\gamma}{\omega_2^{\tau}(\alpha_1;\mu;l;m)}\sum_{n=2}^{\infty}\left(\frac{1}{1-\gamma}|a_n| + \frac{1}{1-\gamma}|b_n|\right)\omega_n^{\tau}(\alpha_1;\mu;l;m)r^2 \geq$$
$$\geq (1-|b_1|)r - \frac{1-\gamma}{\omega_2^{\tau}(\alpha_1;\mu;l;m)}\left(1-\frac{1}{1-\gamma}|b_1|\right)r^2 \geq$$
$$\geq (1-|b_1|)r - \frac{1}{\omega_2^{\tau}(\alpha_1;\mu;l;m)}\left(1-\gamma-|b_1|\right)r^2.$$

□

On lines similar to the above theorem we can state the distortion bounds for functions in the class $\mathcal{Q}\overline{\mathcal{H}}^{\tau}_{\mu}(\gamma)$ without proof.

Theorem 6. *If* $f \in \mathcal{Q}\overline{\mathcal{H}}^{\tau}_{\mu}(\gamma)$, *then*

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{2\omega_2^{\tau}(\alpha_1;\mu;l;m)}\left(1-\gamma-|b_1|\right)r^2,$$
$$|f(z)| \geq (1-|b_1|)r - \frac{1}{2\omega_2^{\tau}(\alpha_1;\mu;l;m)}\left(1-\gamma-|b_1|\right)r^2, \quad |z|=r<1.$$

Next we determine a representation theorem for functions in $\mathcal{F}\overline{\mathcal{H}}^{\tau}_{\mu}(\lambda,\gamma)$.

Theorem 7. $f \in \mathrm{cl}\,\mathrm{co}\,\mathcal{F}\overline{\mathcal{H}}^{\tau}_{\mu}(\lambda,\gamma)$ *if and only if* f *can be expressed the form*

$$f(z) = \sum_{n=1}^{\infty}\left(X_n h_n(z) + Y_n g_n(z)\right)$$

where

$$h_1(z) = z, \;\; h_n(z) = z - \frac{1-\gamma}{\omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n-1)+1|}z^n, \quad n=2,3,\ldots$$

and

$$g_n(z) = z + \frac{1-\gamma}{\omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n+1)-1|}\overline{z}^n, \quad n=1,2,3\ldots$$

$\sum\limits_{n=1}^{\infty}(X_n+Y_n)=1$, $X_n \geq 0$ *and* $Y_n \geq 0$ *for* $n=2,3,\ldots$. *In particular, the extreme points of* $\mathcal{F}\overline{\mathcal{H}}^{\tau}_{\mu}(\lambda,\gamma)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. First we write

$$f(z)=\sum_{n=1}^{\infty}(X_nh_n(z)+Y_ng_n(z))=\sum_{n=1}^{\infty}(X_n+Y_n)z-\sum_{n=2}^{\infty}\frac{1-\gamma}{\omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n-1)+1|}X_nz^n+$$
$$+\sum_{n=2}^{\infty}\frac{1-\gamma}{\omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n+1)-1|}Y_n\overline{z}^n=$$
$$=z-\sum_{n=2}^{\infty}\frac{1-\gamma}{\omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n-1)+1|}X_nz^n+\sum_{n=2}^{\infty}\frac{1-\gamma}{\omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n+1)-1|}Y_n\overline{z}^n.$$

Then by Theorem 2, we have

$$\sum_{n=2}^{\infty}\frac{|\lambda(n-1)+1|\omega_n^{\tau}(\alpha_1;\mu;l;m)}{1-\gamma}|a_n|+\sum_{n=1}^{\infty}\frac{|\lambda(n+1)-1|\omega_n^{\tau}(\alpha_1;\mu;l;m)}{1-\gamma}|b_n|=$$
$$=\sum_{n=2}^{\infty}\frac{|\lambda(n-1)+1|\omega_n^{\tau}(\alpha_1;\mu;l;m)}{1-\gamma}\frac{1-\gamma}{\omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n-1)+1|}X_nz^n+$$
$$+\sum_{n=1}^{\infty}\frac{|\lambda(n+1)-1|\omega_n^{\tau}(\alpha_1;\mu;l;m)}{1-\gamma}\frac{1-\gamma}{\omega_n^{\tau}(\alpha_1;\mu;l;m)|\lambda(n+1)-1|}Y_n\overline{z}^n=$$
$$=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n=1-X_1\leq 1.$$

Hence $f(z)\in \operatorname{cl}\operatorname{co}\mathcal{F}\overline{\mathcal{H}}_{\mu}^{\tau}(\lambda,\gamma)$.
Conversely, suppose that $f(z)\in \operatorname{cl}\operatorname{co}\mathcal{F}\overline{\mathcal{H}}_{\mu}^{\tau}(\lambda,\gamma)$. Setting $X_n=\frac{|\lambda(n-1)+1|\omega_n^{\tau}(\alpha_1;\mu;l;m)}{1-\gamma}|a_n|$, $(n=2,3,\dots)$ and $Y_n=\frac{|\lambda(n+1)-1|\omega_n^{\tau}(\alpha_1;\mu;l;m)}{1-\gamma}|b_n|$, $(n=1,2,\dots)$ where $\sum_{n=1}^{\infty}(X_n+Y_n)=1$. Then

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\overline{z}^n=$$
$$=z-\sum_{n=2}^{\infty}\frac{|\lambda(n-1)+1|\omega_n^{\tau}(\alpha_1;\mu;l;m)}{1-\gamma}X_nz^n+\sum_{n=1}^{\infty}\frac{\lambda(n+1)-1|\omega_n^{\tau}(\alpha_1;\mu;l;m)}{1-\gamma}Y_n\overline{z}^n=$$
$$=z+\sum_{n=2}^{\infty}(h_n(z)-z)X_n+\sum_{n=1}^{\infty}(g_n(z)-z)Y_n=\sum_{n=1}^{\infty}(X_nh_n+Y_ng_n)(z).\qquad\square$$

4. Convolution results and closure properties. In the following theorem we prove that the class $\mathcal{F}\overline{\mathcal{H}}_{\mu}^{\tau}(\lambda,\gamma)$ is closed under the convolution.

Theorem 8. *For* $0\leq\delta<\gamma<1$, *suppose that* $f\in\mathcal{F}\overline{\mathcal{H}}_{\mu}^{\tau}(\lambda,\gamma)$. *Then* $f(z)*F(z)\in\mathcal{F}\overline{\mathcal{H}}_{\mu}^{\tau}(,\lambda,\gamma)\subset\mathcal{F}\overline{\mathcal{H}}_{\mu}^{\tau}(\lambda,\delta)$.

Proof. We can write

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\overline{z}^n\in\mathcal{F}\overline{\mathcal{H}}_{\mu}^{\tau}(\lambda,\gamma),$$
$$F(z)=z-\sum_{n=2}^{\infty}|A_n|z^n+\sum_{n=1}^{\infty}|B_n|\overline{z}^n\in\mathcal{F}\overline{\mathcal{H}}_{\mu}^{\tau}(\lambda,\delta).$$

Then the convolution of f and F is

$$f(z) * F(z) = z - \sum_{n=2}^{\infty} |a_n A_n| z^n + \sum_{n=1}^{\infty} |b_n B_n| \overline{z}^n.$$

Since $|A_n| \le 1, |B_n| \le 1$, if $f(z) * F(z) \in \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\delta)$ we can write

$$\sum_{n=2}^{\infty} \frac{|\lambda(n-1)+1|\omega_n^\tau(\alpha_1;\mu;l;m)}{1-\delta}|a_n||A_n| + \sum_{n=1}^{\infty} \frac{|\lambda(n+1)-1|\omega_n^\tau(\alpha_1;\mu;l;m)}{1-\delta}|b_n||B_n| \le$$
$$\le \sum_{n=2}^{\infty} \frac{|\lambda(n-1)+1|\omega_n^\tau(\alpha_1;\mu;l;m)}{1-\gamma}|a_n| + \sum_{n=1}^{\infty} \frac{|\lambda(n+1)-1|\omega_n^\tau(\alpha_1;\mu;l;m)}{1-\gamma}|b_n| \le 1,$$

because $0 \le \delta < \gamma < 1$ and $f \in \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\gamma)$. Therefore $f * F \in \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\gamma) \subset \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\delta)$. □

We determine the convex combination properties of the members of $\mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\gamma)$.

Theorem 9. *The class $\mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\gamma)$ is closed under convex combination.*

Proof. For $i = 1, 2, \dots$ suppose that $f_i(z) \in \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\alpha)$ where $f_i(z)$ is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n + \sum_{n=1}^{\infty} \overline{b}_{n,i} \overline{z}^n.$$

For $\sum\limits_{i=1}^{\infty} t_i = 1$, $0 \le t_i \le 1$, the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{n,i} \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i \overline{b}_{n,i} \right) \overline{z}^n.$$

By Theorem 2 we have

$$\sum_{n=2}^{\infty} \frac{\omega_n^\tau(\alpha_1;\mu;l;m)|\lambda(n-1)+1|}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i |a_{n,i}| \right) + \sum_{n=1}^{\infty} \frac{\omega_n^\tau(\alpha_1;\mu;l;m)|\lambda(n+1)-1|}{1-\gamma} \times$$
$$\times \left(\sum_{i=1}^{\infty} t_i |b_{n,i}| \right) = \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=2}^{\infty} \omega_n^\tau(\alpha_1;\mu;l;m) \frac{|\lambda(n-1)+1|}{1-\gamma}|a_{n,i}| + \right.$$
$$\left. + \sum_{n=1}^{\infty} \omega_n^\tau(\alpha_1;\mu;l;m) \frac{|\lambda(n+1)-1|}{1-\gamma}|b_{n,i}| \right\} \le \sum_{i=1}^{\infty} t_i = 1,$$

and so $\sum\limits_{i=1}^{\infty} t_i f_i \in \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\gamma)$. □

Now, we examine the closure properties of the class $\mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\gamma)$ under the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ which is defined by

$$L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt, \quad c > -1.$$

Theorem 10. *Let* $f(z) \in \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\gamma)$. *Then* $L_c(f(z)) \in \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\gamma)$

Proof. From the representation of $L_c(f(z))$, it follows that

$$L_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1}\left[h(t)+\overline{g(t)}\right]dt =$$

$$= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t-\sum_{n=2}^{\infty}a_nt^n\right)dt + \overline{\int_0^z t^{c-1}\left(\sum_{n=1}^{\infty}b_nt^n\right)dt}\right) = z-\sum_{n=2}^{\infty}A_nz^n+\sum_{n=1}^{\infty}B_nz^n$$

where $A_n = \dfrac{c+1}{c+n}a_n,\ B_n = \dfrac{c+1}{c+n}b_n$. Therefore,

$$\sum_{n=1}^{\infty}\left(\frac{|\lambda(n+1)-1|}{1-\gamma}\left[\frac{c+1}{c+n}|a_n|\right]+\frac{|\lambda(n-1)+1|}{1-\gamma}\left[\frac{c+1}{c+n}|b_n|\right]\right)\omega_n^\tau(\alpha_1;\mu;l;m) \le$$

$$\le \sum_{n=1}^{\infty}\left(\frac{|\lambda(n+1)-1|}{1-\gamma}|a_n|+\frac{|\lambda(n-1)+1|}{1-\gamma}|b_n|\right)\omega_n^\tau(\alpha_1;\mu;l;m) \le 1,$$

since $f(z) \in \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\gamma)$. Hence by Theorem 2, $L_c(f(z)) \in \mathcal{F}\overline{\mathcal{H}}_\mu^\tau(\lambda,\gamma)$. □

Concluding Remark. Since λ takes values from 0 to 1, we can state the above results for the function class $\mathcal{P}\overline{\mathcal{H}}_\mu^\tau(\gamma)$ and $\mathcal{Q}\overline{\mathcal{H}}_\mu^\tau(\gamma)$.

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