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Harmonic functions prestarlike in the unit disc [1]

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Abstract

A comprehensive class of complex-valued harmonic prestarlike univalent functions is introduced. Necessary and sufficient coefficient bounds are given for functions in this class to be starlike. Distortion bounds and extreme points are also obtained.

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1 Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain G if both u and v are real and harmonic in G. In any simply connected domain $D \subset G$ we can write $f = h+\overline{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and

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sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [1, 2]). Denote by $\mathcal{H}$ the family of functions $f = h + \overline{g}$ that are harmonic univalent and orientation preserving in the open unit disc $U = \{z : |z| < 1\}$ for which $f(0) = h(0) = 0 = f_z(0) - 1$. Thus for

$$f = h + \overline{g} \tag{1}$$

in $\mathcal{H}$ the functions h and g analytic $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \;\; g(z) = \sum_{m=1}^{\infty} b_m z^m \;\; (0 \le b_1 < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m}, \;\; (|b_1| < 1). \tag{2}$$

Note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to S, the class of normalized analytic univalent functions, if the co-analytic part of $f = h+\overline{g}$ is identically zero, that is $g \equiv 0$. Given two functions $\phi(z) = \sum\limits_{m=1}^{\infty} \phi_m z^m$ and $\psi(z) = \sum\limits_{m=1}^{\infty} \psi_m z^m$ in S, there Hadamard product or convolution $(\phi * \psi)(z)$ is defined by $(\phi * \psi)(z) = \phi(z) * \psi(z) = \sum\limits_{m=1}^{\infty} \phi_m \psi_m z^m$. Using the convolution ,in [6] Ruscheweyh introduced and studied the class of prestarlike function of order α, which are the function f such that $f * S_\alpha$ is a starlike function of order α, where

$$\mathcal{S}_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}}, \quad (z \in U, \, 0 \le \alpha < 1) \tag{3}$$

We also note that $\mathcal{S}_\alpha(z)$ can be written in the form

$$\mathcal{S}_\alpha(z) = z + \sum_{m=2}^{\infty} |C_m(\alpha)| z^n, \tag{4}$$

where

$$C_m(\alpha) = \frac{\prod\limits_{j=2}^{m} (j - 2\alpha)}{(m-1)!} \quad (m \in \mathbb{N} \setminus \{1\}, \mathbb{N} := \{1, 2, 3, \cdots\}). \tag{5}$$

We note that $C_n(\alpha)$ is decreasing in α and satisfies

$$\lim_{m\to\infty} C_m(\alpha) = \begin{cases} \infty & \text{if } \alpha < \frac{1}{2} \\ 1 & \text{if } \alpha = \frac{1}{2} \\ 0 & \text{if } \alpha > \frac{1}{2} \end{cases} . \tag{6}$$

For $f = h + \overline{g}$ given by (2) and $0 \le \alpha < 1$ we define the prestarlike harmonic function $f = h + \overline{g}$ in $\mathcal{H}$ by

$$S_\alpha(z) * f(z) = S_\alpha(z) * h(z) + \overline{S_\alpha(z) * g(z)} \tag{7}$$

where S_α is given by (4) and the operator $*$ stands for the Hadamard product or convolution product. Motivated by the earlier works of [3, 4, 5] on the subject of harmonic functions, we introduce here a new subclass $\mathcal{PS_H}(\alpha, \gamma)$ of $\mathcal{H}$.

For $0 \le \gamma < 1$, let $\mathcal{PS_H}(\alpha, \gamma)$ denote the subfamily of starlike harmonic functions $f \in \mathcal{H}$ of the form (1) such that

$$\frac{\partial}{\partial\theta}\left(arg S_\alpha(z) * f(z)\right) > \gamma \tag{8}$$

Equivalently

$$\operatorname{Re}\left\{\frac{z(S_\alpha(z) * h'(z)) - \overline{z(S_\alpha(z) * g'(z))}}{S_\alpha(z) * h(z) + \overline{(S_\alpha(z) * g(z))}}\right\} \ge \gamma \tag{9}$$

where $S_\alpha(z) * f(z)$ is given by (7) and $z \in U$.

We also let $\mathcal{PV_H}(\alpha, \gamma) = \mathcal{PS_H}(\alpha, \gamma) \bigcap V_{\mathcal{H}}$ where $V_{\mathcal{H}}$ the class of harmonic functions with varying arguments introduced by Jahangiri and Silverman [3], consisting of functions f of the form(1) in $\mathcal{H}$ for which there exists a real number ϕ such that

$$\eta_m + (m-1)\phi \equiv \pi \pmod{2\pi}, \ \delta_m + (m-1)\phi \equiv 0 \pmod{2\pi}, \ m \ge 2, \tag{10}$$

where $\eta_m = \arg(a_m)$ and $\delta_m = \arg(b_m)$.

In this paper we obtain a sufficient coefficient condition for functions f given by (2) to be in the class $\mathcal{PS_H}(\alpha,\gamma)$. It is shown that this coefficient condition is necessary also for functions belonging to the class $\mathcal{PV_H}(\alpha,\gamma)$. Further, distortion results and extreme points for functions in $\mathcal{PV_H}(\alpha,\gamma)$ are also obtained.

2 The class $\mathcal{PS_H}(\alpha,\gamma)$.

We begin deriving a sufficient coefficient condition for the functions belonging to the class $\mathcal{PS_H}(\alpha,\gamma)$.

Theorem 1 *Let $f = h + \overline{g}$ be given by (2). If*

$$\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}|a_m| + \frac{m+\gamma}{1-\gamma}|b_m|\right) C_m(\alpha) \le 1 - \frac{1+\gamma}{1-\gamma}b_1 \tag{11}$$

$0 \le \gamma < 1$, *then* $f \in \mathcal{PS_H}(\alpha,\gamma)$.

Proof. We first show that if the inequality (11) holds for the coefficients of $f = h + \overline{g}$, then the required condition (9) is satisfied.

Using (7) and (9), we can write

$$\text{Re}\left\{\frac{z(S_\alpha(z)*h(z))' - \overline{z(S_\alpha(z)*g(z))'}}{S_\alpha(z)*h(z) + \overline{S_\alpha(z)*g(z)}}\right\} = \text{Re}\,\frac{A(z)}{B(z)},$$

where

$$A(z) = [z(S_\alpha(z)*h(z))' - \overline{z(S_\alpha(z)*g(z))'}]$$

and

$$B(z) = S_\alpha(z)*h(z) + \overline{S_\alpha(z)*g(z)}.$$

In view of the simple assertion that Re $(w) \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it is sufficient to show that

$$|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| \geq 0. \tag{12}$$

Substituting for $A(z)$ and $B(z)$ the appropriate expressions in (12), we get

$$\begin{aligned}
&|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| \\
&\geq (2-\gamma)|z| - \sum_{m=2}^{\infty}(m+1-\gamma)C_m(\alpha)|a_m|\ |z|^m - \sum_{m=1}^{\infty}(m-1+\gamma)C_m(\alpha)|b_m|\ |\ |z|^m \\
&\quad -\gamma|z| - \sum_{m=2}^{\infty}(m-1-\gamma)C_m(\alpha)|a_m|\ |z|^m - \sum_{m=1}^{\infty}(m+1+\gamma)C_m(\alpha)|b_m|\ |z|^m. \\
&\geq 2(1-\gamma)|z|\left\{1 - \sum_{m=2}^{\infty}\frac{m-\gamma}{1-\gamma}C_m(\alpha)|a_m||z|^{m-1} - \sum_{m=1}^{\infty}\frac{m+\gamma}{1-\gamma}C_m(\alpha)|b_m||z|^{m-1}\right\}. \\
&\geq 2(1-\gamma)|z|\left\{1 - \frac{1+\gamma}{1-\gamma}b_1 - \left(\sum_{m=2}^{\infty}\left[\frac{m-\gamma}{1-\gamma}C_m(\alpha)|a_m| + \frac{m+\gamma}{1-\gamma}C_m(\alpha)|b_m|\right]\right)\right\} \geq 0
\end{aligned}$$

by virtue of the inequality (11). This implies that $f \in \mathcal{PS}_{\mathcal{H}}(\alpha,\gamma)$.

Now we obtain the necessary and sufficient condition for function $f = h+\overline{g}$ be given with condition (10):

Theorem 2 *Let $f = h+\overline{g}$ be given by (2). Then $f \in \mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$ if and only if*

$$\sum_{m=2}^{\infty}\left\{\frac{m-\gamma}{1-\gamma}|a_m| + \frac{m+\gamma}{1-\gamma}|b_m|\right\}C_m(\alpha) \leq 1 - \frac{1+\gamma}{1-\gamma}b_1 \tag{13}$$

$0 \leq \gamma < 1$.

Proof. Since $\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma) \subset \mathcal{PS}_{\mathcal{H}}(\alpha,\gamma)$, we only need to prove the necessary part of the theorem.

Assume that $f \in \mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$, then by virtue of (7) to (9), we obtain

$$\text{Re}\ \left\{\left[\frac{z(S_\alpha(z)*h(z))' - \overline{z(S_\alpha(z)*g(z))'}}{S_\alpha(z)*h(z) + \overline{S_\alpha(z)*g(z)}}\right] - \gamma\right\} \geq 0.$$

The above inequality is equivalent to

$$\mathrm{Re}\left\{\frac{z+\left(\sum\limits_{m=2}^{\infty}(m-\gamma)C_m(\alpha)|a_m|z^m-\sum\limits_{m=1}^{\infty}(m+\gamma)C_m(\alpha)|b_m|\overline{z}^m\right)}{z+\sum\limits_{m=2}^{\infty}C_m(\alpha)|a_m|z^m+\sum\limits_{m=1}^{\infty}C_m(\alpha)|b_m|\overline{z}^m}\right\}$$

$$= \mathrm{Re}\left\{\frac{(1-\gamma)+\sum\limits_{m=2}^{\infty}(m-\gamma)C_m(\alpha)|a_m|z^{m-1}-\frac{\overline{z}}{z}\sum\limits_{m=1}^{\infty}(m+\gamma)C_m(\alpha)|b_m|\overline{z}^{m-1}}{1+\sum\limits_{m=2}^{\infty}C_m(\alpha)|a_m|z^{m-1}+\frac{\overline{z}}{z}\sum\limits_{m=1}^{\infty}C_m(\alpha)|b_m|\overline{z}^{m-1}}\right\}\geq 0.$$

This condition must hold for all values of z, such that $|z|=r<1$. Upon choosing ϕ according to (10) we must have

$$\frac{(1-\gamma)-(1+\gamma)b_1-\left(\sum\limits_{m=2}^{\infty}(m-\gamma)C_m(\alpha)|a_m|r^{m-1}+(m+\gamma)C_m(\alpha)|b_m|r^{m-1}\right)}{1+|b_1|+\left(\sum\limits_{m=2}^{\infty}C_m(\alpha)|a_m|+\sum\limits_{m=1}^{\infty}C_m(\alpha)|b_m|\right)r^{m-1}}\geq 0. \tag{14}$$

If (13) does not hold, then the numerator in (14) is negative for r sufficiently close to 1. Therefore, there exists a point $z_0=r_0$ in (0,1) for which the quotient in (14) is negative. This contradicts our assumption that $f\in\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$. We thus conclude that it is both, necessary and sufficient, that the coefficient bound inequality (13) holds true when $f\in\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$. This completes the proof of Theorem 2.

If we put $\phi=2\pi/k$ in (10), then Theorem 2 gives the following corollary:

Corollary 1 *A necessary and sufficient condition for $f=h+\overline{g}$ satisfying (13) to be starlike is that*

$$\arg(a_m)=\pi-2(m-1)\pi/k,$$

and

$$\arg(b_m)=2\pi-2(m-1)\pi/k \quad ,(k=1,2,3,\dots).$$

3 Distortion and Extreme Points

In this section we obtain the distortion bounds for the functions $f\in\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$ that lead to a covering result for the family $\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$.

Theorem 3 *If $f \in \mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$ then*

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{C_2(\alpha_1)}\left(\frac{1-\gamma}{2-\gamma} - \frac{1+\gamma}{2-\gamma}|b_1|\right)r^2$$

and

$$|f(z)| \geq (1-|b_1|)r - \frac{1}{C_2(\alpha)}\left(\frac{1-\gamma}{2-\gamma} - \frac{1+\gamma}{2-\gamma}|b_1|\right)r^2.$$

Proof. We will only prove the right- hand inequality of the above theorem. The arguments for the left- hand inequality are similar and so we omit it.

Let $f \in \mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$. Taking the absolute value of f, we obtain

$$\begin{aligned} |f(z)| &\leq (1+|b_1|)r + \sum_{m=2}^{\infty}(|a_m|+|b_m|)r^m \\ &\leq (1+b_1)r + r^2\sum_{m=2}^{\infty}(|a_m|+|b_m|). \end{aligned}$$

This implies that

$$\begin{aligned} |f(z)| &\leq (1+|b_1|)r + \frac{1}{C_2(\alpha_1)}\left(\frac{1-\gamma}{2-\gamma}\right)\sum_{m=2}^{\infty}\left[\left(\frac{2-\gamma}{1-\gamma}\right)C_2(\alpha_1)|a_m| + \left(\frac{2-\gamma}{1-\gamma}\right)C_2(\alpha_1)|b_m|\right]r^2 \\ &\leq (1+|b_1|)r + \frac{1}{C_2(\alpha_1)}\left(\frac{1-\gamma}{2-\gamma}\right)\left[1 - \frac{1+\gamma}{1-\gamma}|b_1|\right]r^2 \\ &\leq (1+|b_1|)r + \frac{1}{C_2(\alpha_1)}\left(\frac{1-\gamma}{2-\gamma} - \frac{1+\gamma}{2-\gamma}|b_1|\right)r^2, \end{aligned}$$

which establishes the desired inequality.

As a consequence of the above theorem and Corollary (1), we state the following covering result:

Corollary 2 *Let $f = h + \overline{g}$ and of the form (2) be so that $f \in \mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$. Then*

$$\left\{w : |w| < \frac{2C_2(\alpha_1) - 1 - [C_2(\alpha_1)-1]\gamma}{(2-\gamma)C_2(\alpha_1)} - \frac{2C_2(\alpha_1) - 1 - [C_2(\alpha_1)+1]\gamma}{(2-\gamma)C_2(\alpha_1)}b_1\right\} \subset f(\mathcal{U}).$$

For a compact family, the maximum or minimum of the real part of any continuous linear functional occurs at one of the extreme points of the closed convex hull. Unlike many other classes, characterized by necessary and sufficient coefficient conditions, the family $\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$ is not a convex family . Nevertheless , we may still apply the coefficient characterization of the $\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$ to determine the extreme points.

Theorem 4 *The closed convex hull of $\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$ (denoted by $clco\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$) is*

$$\left\{f(z)=z+\sum_{m=2}^{\infty}|a_m|z^m+\overline{\sum_{m=1}^{\infty}|b_m|z^m},\ :\sum_{m=2}^{\infty}m[|a_m|+|b_m|]<1-b_1\right\}$$

By setting $\lambda_m=\frac{1-\gamma}{(m-\gamma)C_m(\alpha)}$ and $\mu_m=\frac{1-\gamma}{(m+\gamma)C_m(\alpha)}$, then for b_1 fixed, the extreme points for clco $\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$ are

$$\{z+\lambda_m xz^m+\overline{b_1 z}\}\cup\{z+\overline{b_1 z+\mu_m xz^m}\} \tag{15}$$

where $m\geq 2$ and $|x|=1-|b_1|$.

Proof. Any function f in $clco\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$ can be expressed as

$$f(z)=z+\sum_{m=2}^{\infty}|a_m|e^{i\eta_m}z^m+\overline{b_1 z}+\overline{\sum_{m=2}^{\infty}|b_m|e^{i\delta_m}z^m},$$

where the coefficients satisfy the inequality (11). Set

$$h_1(z)=z, g_1(z)=b_1 z,\ h_m(z)=z+\lambda_m e^{i\eta_m}z^m,\ g_m(z)=b_1 z+\mu_m e^{i\delta_m}z^m \text{ for } m=2,3,\ldots$$

Writing $X_m=\frac{|a_m|}{\lambda_m}$, $Y_m=\frac{|b_m|}{\mu_m}$, $m=2,3,\ldots$ and $X_1=1-\sum\limits_{m=2}^{\infty}X_m$; $Y_1=1-\sum\limits_{m=2}^{\infty}Y_m$, we get

$$f(z)=\sum_{m=1}^{\infty}(X_m h_m(z)+Y_m g_m(z)).$$

In particular, putting

$$f_1(z)=z+\overline{b_1 z} \text{ and } f_m(z)=z+\lambda_m xz^m+\overline{b_1 z}+\overline{\mu_m yz^m},\ (m\geq 2, |x|+|y|=1-|b_1|)$$

we see that extreme points of $clco\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)\subset\{f_m(z)\}$.
To see that $f_1(z)$ is not an extreme point, note that $f_1(z)$ may be written as

$$f_1(z)=\frac{1}{2}\{f_1(z)+\lambda_2(1-|b_1|)z^2\}+\frac{1}{2}\{f_1(z)-\lambda_2(1-|b_1|)z^2\},$$

a convex linear combination of functions in *clco* $\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$.

To see that f_m is not an extreme point if both $|x|\neq 0$ and $|y|\neq 0$, we will show that it can then also be expressed as a convex linear combinations of functions in *clco*

$\mathcal{PV_H}(\alpha,\gamma)$. Without loss of generality, assume $|x| \geq |y|$. Choose $\epsilon > 0$ small enough so that $\epsilon < \frac{|x|}{|y|}$. Set $A = 1+\epsilon$ and $B = 1 - |\frac{\epsilon x}{y}|$. We then see that both

$$t_1(z) = z + \lambda_m Axz^m + \overline{b_1 z + \mu_m yBz^m}$$

and

$$t_2(z) = z + \lambda_m (2-A)xz^m + \overline{b_1 z + \mu_m y(2-B)z^m}$$

are in *clco* $\mathcal{PV_H}(\alpha,\gamma)$, and that

$$f_m(z) = \frac{1}{2}\{t_1(z) + t_2(z)\}.$$

The extremal coefficient bounds show that functions of the form (15) are the extreme points for *clco* $\mathcal{PV_H}(\alpha,\gamma)$, and so the proof is complete.

4 Inclusion Relation

Following Avici and Zlotkiewicz [1] (see also Ruscheweyh [7]), we refer to the the $\delta-$neighborhood of the function $f(z)$ defined by (2) to be the set of functions F for which

$$(16)\qquad N_\delta(f) = \left\{F(z) = z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} \overline{B_m z^m},\right.$$
$$\left.\sum_{m=2}^{\infty} m(|a_m - A_m| + |b_m - B_m|) + |b_1 - B_1| \leq \delta\right\}.$$

In our case, let us define the generalized $\delta-$neighborhood of f to be the set

$$(17)\qquad N_\delta(f) = \left\{F : \sum_{m=2}^{\infty} C_m(\alpha)[(m-\gamma)|a_m - A_m| + (m+\gamma)|b_m - B_m|]\right.$$
$$\left. + \quad (1-\gamma)|b_1 - B_1| \leq (1-\gamma)\delta\right\}.$$

Theorem 5 *Let f be given by (2). If f satisfies the conditions*

$$(18)\qquad \sum_{m=2}^{\infty} m(m-\gamma)|a_m|C_m(\alpha) + \sum_{m=1}^{\infty} m(m+\gamma)|b_m|C_m(\alpha) \leq (1-\gamma),$$

$0 \leq \gamma < 1$ *and*

$$(19)\qquad \delta = \frac{1-\gamma}{2-\gamma}\left(1 - \frac{1+\gamma}{1-\gamma}|b_1|\right),$$

then $N_\delta(f) \subset \mathcal{PS}_{\mathcal{H}}(\alpha,\gamma)$.

Proof. Let f satisfy (18) and F(z) be given by

$$F(z) = z + \overline{B_1 z} + \sum_{m=2}^{\infty} \left(A_m z^m + \overline{B_m z^m}\right)$$

which belongs to $N(f)$. We obtain

$$(1+\gamma)|B_1| + \sum_{m=2}^{\infty} \left((m-\gamma)|A_m| + (m+\gamma)|B_m|\right) C_m(\alpha)$$

$$\le (1+\gamma)|B_1 - b_1| + (1+\gamma)|b_1| + \sum_{m=2}^{\infty} C_m(\alpha)\left[(m-\gamma)|A_m - a_m| + (m+\gamma)|B_m - b_m|\right]$$

$$+ \sum_{m=2}^{\infty} C_m(\alpha)\left[(m-\gamma)|a_m| + (m+\gamma)|b_m|\right]$$

$$\le (1-\gamma)\delta + (1+\gamma)|b_1| + \frac{1}{2-\gamma}\sum_{m=2}^{\infty} mC_m(\alpha)\left((m-\gamma)|a_m| + (m+\gamma)|b_m|\right)$$

$$\le (1-\gamma)\delta + (1+\gamma)|b_1| + \frac{1}{2-\gamma}\left[(1-\gamma) - (1+\gamma)|b_1|\right] \le 1-\gamma.$$

Hence for $\delta = \frac{1-\gamma}{2-\gamma}\left(1 - \frac{1+\gamma}{1-\gamma}|b_1|\right)$, we infer that $F(z) \in \mathcal{WS}_{\mathcal{H}}(\alpha,\gamma)$ which concludes the proof of Theorem 5.

Now, we will examine the closure properties of the class $\mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$ under the generalized Bernardi-Libera -Livingston integral operator$\mathcal{L}_c(f)$ which is defined by

$$\mathcal{L}_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt, c > -1.$$

Theorem 6 *Let* $f(z) \in \mathcal{PV}_H(\alpha,\gamma)$. *Then* $\mathcal{L}_c(f(z)) \in \mathcal{PV}_{\mathcal{H}}(\alpha,\gamma)$

Proof. From the representation of $\mathcal{L}_c(f(z))$, it follows that

$$\mathcal{L}_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1}\left[h(t) + \overline{g(t)}\right] dt.$$

$$= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t - \sum_{m=2}^{\infty} a_m t^n\right) dt + \overline{\int_0^z t^{c-1}\left(\sum_{m=1}^{\infty} b_m t^m\right) dt}\right)$$

$$= z - \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} B_m z^m$$

where

$$A_m = \frac{c+1}{c+m}\ a_m; B_m = \frac{c+1}{c+m}\ b_m.$$

Therefore,

$$\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1-\gamma}\frac{c+1}{c+m}|b_m|\right)C_m(\alpha)$$
$$\leq \sum_{m=1}^{\infty}\left(\frac{m-\gamma}{1-\gamma}|a_m| + \frac{m+\gamma}{1-\gamma}|b_m|\right)C_m(\alpha) \leq 2(1-\gamma).$$

Since $f(z) \in \mathcal{PV_H}(\alpha,\gamma)$, therefore by Theorem 2, $\mathcal{L}_c(f(z)) \in \mathcal{PV_H}(\alpha,\gamma)$.

Theorem 7 *For $0 \leq \delta \leq \gamma < 1$, let $f(z) \in \mathcal{PV_H}(\alpha,\gamma)$ and $F(z) \in \mathcal{PV_H}(\alpha,\delta)$. Then $f(z) * F(z) \in \mathcal{PV_H}(\alpha,\gamma) \subset \mathcal{PV_H}(\alpha,\delta)$.*

Proof. Let

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^n + \sum_{m=1}^{\infty} \overline{b}_m \overline{z}^m \in \mathcal{PV_H}(\alpha,\gamma)$$

and

$$F(z) = z - \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} \overline{B}_n \overline{z}^m \in \mathcal{PV_H}(\alpha,\delta).$$

Then

$$f(z) * F(z) = z - \sum_{m=2}^{\infty} a_m A_m z^m + \sum_{m=1}^{\infty} \overline{b}_m \overline{B}_m \overline{z}^m.$$

For $f(z) * F(z) \in \mathcal{PV_H}(\alpha,\delta)$ we note that $|A_m| \leq 1$ and $|B_m| \leq 1$. Now by Theorem 2 we have

$$\sum_{m=2}^{\infty}\frac{[m-\delta]C_m(\alpha)}{1-\delta}|a_m|\ \ |A_m| + \sum_{m=1}^{\infty}\frac{[m+\delta]C_m(\alpha)}{1-\delta}|b_m|\ \ |B_m|$$
$$\leq \sum_{m=2}^{\infty}\frac{[m-\delta]C_m(\alpha)}{1-\delta}|a_m| + \sum_{m=1}^{\infty}\frac{[m+\delta]C_m(\alpha)}{1-\delta}|b_m|$$

and since $0 \leq \delta \leq \gamma < 1$

$$\leq \sum_{m=2}^{\infty}\frac{[m-\gamma]C_m(\alpha)}{1-\gamma}|a_m| + \sum_{m=1}^{\infty}\frac{[m+\gamma]C_m(\alpha)}{1-\gamma}|b_m| \leq 1,$$

by Theorem 2 , $f(z) \in \mathcal{PV}H(\alpha,\gamma)$. Therefore $f(z) * F(z) \in \mathcal{PV}H(\alpha,\gamma) \subset \mathcal{PV}H(\alpha,\delta)$.

Concluding Remark: The various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler harmonic function classes (see [3, 4, 5]). The details involved in the derivations of such specializations of the results presented in this paper are fairly straight-forward.

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