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STARLIKENESS OF MULTIVALENT MEROMORPHIC HARMONIC FUNCTIONS

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Dedicated to my brother Er. G. Malmurugu

Abstract. We give sufficient coefficient conditions for starlikeness of a class of complex-valued multivalent meromorphic harmonic and orientation preserving functions in outside of the unit disc. These coefficient conditions are also shown to be necessary if the coefficients of the analytic part of the harmonic functions are positive and the coefficients of the co-analytic part of the harmonic functions are negative. We then determine the extreme points, distortion bounds, convolution and convex combination conditions for these functions.

1. Introduction

A continuous functions $f = u + iv$ is a complex valued harmonic function in a domain $D \subset \Omega$ if both u and v are real harmonic in D. In any simply connected domain we write $f = h + \overline{g}$ where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h^{'}| > |g^{'}|$ in D (sec [2]). Hengartner and Schober [3], investigated functions harmonic in the exterior of the unit disk $\widetilde{U} = \{z : |z| > 1\}$, among other things they showed that complex valued, harmonic, orientation preserving univalent mapping f, defined in $\widetilde{U}$ and satisfying $f(\infty) = \infty$, must admits the representation

$$f(z) = h(z) + g(z) + A \log |z|, \tag{1}$$

where

$$h(z) = \alpha z + \sum_{n=1}^{\infty} a_n \, z^{-n}; \; g(z) = \beta z + \sum_{n=1}^{\infty} b_n \, z^{-n} \tag{2}$$

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$0 \leq |\beta| < |\alpha|$, and $a(z) = \overline{f_{\overline{z}}}/f_z$ is analytic and satisfies $|a(z)| < 1$ for $z \in \widetilde{U}$. We call $h(z)$ the analytic part and g(z) the co-analytic part of the harmonic function $f(z)$. Recently Jahangiri [4], assumed $\alpha = 1$ and $\beta = 0$ and removed the logarithmic singularity by letting $A = 0$ in the representation of (1), and focused the study to the family of harmonic meromorphic functions.

For fixed positive integer m and $0 \leq \alpha < 1$, we let $M_H(m,\alpha)$ denote the family of multivalent meromorphic harmonic functions of the form

(3)
$$f(z) = h(z) + \overline{g(z)} = z^m + \sum_{n=1}^{\infty} a_{n+m-1}\, z^{-n-m+1} + \overline{\sum_{n=1}^{\infty} b_{n+m-1}\, z^{-n-m+1}},$$

$z \in \widetilde{U}$ and for some real ϕ, the function $f(z)$ satisfies the condition

$$\text{(4)} \qquad \text{Re}\left\{(1+e^{i\phi})\frac{zf'(z)}{z'f(z)} - me^{i\phi}\right\} \geq m\alpha,\ z \in \widetilde{U}.$$

For the harmonic functions $f = h + \overline{g}$ in (3), we define $f'(z) = \frac{\partial}{\partial\theta} f(z)$ and $z' = \frac{\partial}{\partial\theta}(z)$, with $z = re^{i\theta}$, $r > 1$, and θ is real. We further let $M_{\overline{H}}(m,\alpha)$ denote the subclass of $M_H(m,\alpha)$ consisting of multivalent meromorphic harmonic functions of the form

(5)
$$f(z) = h(z) + \overline{g(z)} = z^m + \sum_{n=1}^{\infty} a_{n+m-1}\, z^{-n-m+1} - \overline{\sum_{n=1}^{\infty} b_{n+m-1}\, z^{-n-m+1}},$$

$a_{n+m-1} \geq 0$, $b_{n+m-1} \geq 0$, and $z \in \widetilde{U}$.

Ahuja and Jahangiri [1] among other things proved that if $f = h + \overline{g}$ is of the form (3) and satisfies the coefficient condition

(6)
$$\sum_{n=1}^{\infty}\left\{\frac{n+m(1+\alpha)-1}{m(1-\alpha)}|a_{n+m-1}| + \frac{n+m(1-\alpha)-1}{m(1-\alpha)}|\overline{b_{n+m-1}}|\right\} \leq 1,$$

then f is orientation preserving, $m-$valent and starlike of order α in $\widetilde{U}$. They proved the condition (6) is also necessary for starlikeness of the functions $f = h + \overline{g}$ of the form (5). A multivalent meromorphic harmonic function of the form (3) is said to be starlike of order α in $\widetilde{U}$ if $\frac{\partial}{\partial\theta} \arg f(re^{i\theta}) \geq m\alpha, z \in \widetilde{U}$. In this note we look at certain subclass of $M_H(m,\alpha)$ and obtain sufficient condition for functions to be in the class and also we prove that the coefficient condition is necessary for functions in $M_{\overline{H}}(m,\alpha)$. Finally we characterize the extreme points for

$M_{\overline{H}}(m,\alpha)$ and prove closure properties under convolution and convex combinations.

2. Coefficients bounds

In this section we obtain coefficient bounds, distortion bounds and extreme points for functions in these classes.

Theorem 1. *Let $f = h + \overline{g}$ be given by* (3).

If $\sum\limits_{n=1}^{\infty}(n+m-1)\left(|a_{n+m-1}|+|b_{n+m-1}|\right) \le m$, where $m \ge 1$, then the harmonic functions f is sense preserving in $\widetilde{U}$ and $f \in M_H(m)$, the class of multivalent meromorphic harmonic functions.

Proof. To show that f is orientation preserving, we need to show that $|a(z)| = |\overline{f_{\overline{z}}}\ /f_z| < 1,\ z \in \widetilde{U}$.

We have $|a(z)| = |\overline{f_{\overline{z}}}\ /f_z| < \left|\frac{g'(z)}{h'(z)}\right| < 1$, hence

$$\begin{aligned}
|a(z)| &= \left|\frac{\sum\limits_{n=1}^{\infty}(-n-m+1)b_{n+m-1}\ z^{-n-m}}{mz^{m-1}+\sum\limits_{n=1}^{\infty}(-n-m+1)\ a_{n+m-1}z^{-n-m}}\right| \\
&= \left|\frac{-\sum\limits_{n=1}^{\infty}(n+m-1)b_{n+m-1}\ z^{-n-m}}{mz^{m-1}-\sum\limits_{n=1}^{\infty}(n+m-1)\ a_{n+m-1}z^{-n-m}}\right| \\
&\le \frac{\sum\limits_{n=1}^{\infty}(n+m-1)|b_{n+m-1|}}{m-\sum\limits_{n=1}^{\infty}(n+m-1)|a_{n+m-1|}} \le 1,
\end{aligned}$$

so f is orientation preserving in $\widetilde{U}$. □

In the next theorem, we give a sufficient coefficient condition for the functions of the form (3) to be in $M_H(m,\alpha)$.

Theorem 2. *Let $f = h + \overline{g}$ be of the form* (3), *and satisfies the condition*

$$\sum_{n=1}^{\infty}\left\{\frac{2n+m(3+\alpha)-2}{m(1-\alpha)}|a_{n+m-1}| + \frac{2n+m(1-\alpha)-2}{m(1-\alpha)}|b_{n+m-1}|\right\} \le 1, \tag{7}$$

then f is harmonic, orientation preserving and $f \in M_H(m,\alpha),\ 0 \le \alpha < 1,\ z \in \widetilde{U}$.

Proof. To prove $f \in M_H(m,\alpha)$, by definition, we only need to show that the condition (4) holds for f. Substituting $h+\overline{g}$ for f in (4), it suffices to show that

$$\text{(8)}\quad \operatorname{Re}\left\{\frac{(1+e^{i\phi})(zh^{'}(z)-\overline{zg^{'}(z)})-m(\alpha+e^{i\phi})(h(z)+\overline{g(z)})}{h(z)+\overline{g(z)}}\right\}\ge 0,$$

where $h^{'}(z)=\frac{\partial}{\partial z}h(z)$ and $g^{'}(z)=\frac{\partial}{\partial z}g(z)$. Substituting for $h,\ g,\ h^{'}$ and $g^{'}$ in (8), and dividing by $m(1-\alpha)z^m$, we obtain $\operatorname{Re}\{A(z)\ /\ B(z)\}\ \ge 0$, where

$$\begin{aligned}&A(z)\\ =\ & 1-\sum_{n=1}^{\infty}\frac{m(1+\alpha+2e^{i\phi})+(n-1)(1+e^{i\phi})}{m(1-\alpha)}a_{n+m-1}\ z^{-n-2m+1}\\ &+\sum_{n=1}^{\infty}\frac{m(1-\alpha)+(n-1)(1+e^{i\phi})}{m(1-\alpha)}\overline{b}_{n+m-1}\ \frac{(\overline{z})^{-n-m+1}}{z^m}\end{aligned}$$

and

$$\begin{aligned}&B(z)\\ =\ & 1+\sum_{n=1}^{\infty}a_{n+m-1}\ z^{-n-2m+1}+\sum_{n=1}^{\infty}\overline{b}_{n+m-1}\ \frac{(\overline{z})^{-n-m+1}}{z^m}.\end{aligned}$$

Using the fact $\operatorname{Re}(\omega)\ge 0$ if and only if $|1+\omega|\ge|1-\omega|$ in $\widetilde{U}$, substituting for $A(z)$ and $B(z)$ gives

$$\begin{aligned}&|A(z)+B(z)|-|A(z)-B(z)|\\ =\ & |2-\sum_{n=1}^{\infty}\frac{2m(\alpha+e^{i\phi})+(n-1)(1+e^{i\phi})}{m(1-\alpha)}a_{n+m-1}\ z^{-n-2m+1}\\ &+\sum_{n=1}^{\infty}\frac{2m(1-\alpha)+(n-1)(1+e^{i\phi})}{m(1-\alpha)}\overline{b}_{n+m-1}\ \frac{(\overline{z})^{-n-m+1}}{z^m}|\\ &-\left|-\sum_{n=1}^{\infty}\frac{(2m+n-1)(1+e^{i\phi})}{m(1-\alpha)}a_{n+m-1}\ z^{-n-2m+1}\right.\\ &\left.+\ \sum_{n=1}^{\infty}\frac{(n-1)(1+e^{i\phi})}{m(1-\alpha)}\overline{b}_{n+m-1}\ \frac{(\overline{z})^{-n-m+1}}{z^m}\right|\end{aligned}$$

$$\begin{aligned}
&= 2-\sum_{n=1}^{\infty}\frac{2m(\alpha+1)+2(n-1)}{m(1-\alpha)}|a_{n+m-1}|\ |z|^{-n-2m+1}\\
&\quad -\sum_{n=1}^{\infty}\frac{2m(1-\alpha)+2(n-1)}{m(1-\alpha)}|b_{n+m-1}|\ |z|^{-n-2m+1}\\
&\quad -\sum_{n=1}^{\infty}\frac{2(2m+n-1)}{m(1-\alpha)}|a_{n+m-1}|\ |z|^{-n-2m+1}\\
&\quad -\sum_{n=1}^{\infty}\frac{2(n-1)}{m(1-\alpha)}|b_{n+m-1}|\ |z|^{-n-2m+1}\\
&\geq 2\left\{1-\sum_{n=1}^{\infty}\frac{2n+m(3+\alpha)-2}{m(1-\alpha)}|a_{n+m-1}|\right.\\
&\quad \left.-\sum_{n=1}^{\infty}\frac{2n+m(1-\alpha)-2}{m(1-\alpha)}|b_{n+m-1}|\right\}.
\end{aligned}$$

The non-negativeness of the above expression follows from (7). □

In the following theorem we prove that the condition (7) is also necessary for functions to be in $M_{\overline{H}}(m,\alpha)$.

THEOREM 3. *Let $f=h+\overline{g}$ be given by (5). Then $f\in M_{\overline{H}}(m,\alpha)$ if and only if the inequality (7) holds for the coefficients of $f=h+\overline{g}$.*

Proof. In view of Theorem 2, we only need to prove the *only if* part of the theorem. Since $M_{\overline{H}}(m,\alpha)\subset M_H(m,\alpha)$, we wish to shows that $f\notin M_{\overline{H}}(m,\alpha)$ if the condition (7) does not hold. If $f\in M_{\overline{H}}(m,\alpha)$, then by (4) the condition (8) must be satisfied for all values of z in $\widetilde{U}$. Substituting for $h, g,\ h'$ and g' in (8) and choosing the values of z on the positive real axis where $z=r>1$, we are required to have $\mathrm{Re}\,\{P(z)/Q(z)\}\geq 0$, where

$$\begin{aligned}
P(r) = \quad m \quad &(1-\alpha)-\sum_{n=1}^{\infty}[n+m(1+\alpha)-1\\
+ \quad &e^{i\phi}(2m+n-1)]|a_{n+m-1}|r^{-n-2m+1}\\
- \quad &\sum_{n=1}^{\infty}[n+m(1-\alpha)-1+e^{i\phi}(n-1)]|b_{n+m-1}|r^{-n-2m+1}
\end{aligned}$$

and

$$Q(r) = \quad 1 \quad + \sum_{n=1}^{\infty} |a_{n+m-1}| r^{-n-2m+1} - \sum_{n=1}^{\infty} |b_{n+m-1}| r^{-n-2m+1}.$$

For Re $(e^{i\phi}) = |e^{i\phi}| = 1$ the required condition Re $\{P(r)/Q(r)\} \geq 0$ is equivalent to

(9)

$$\frac{1 - \sum\limits_{n=1}^{\infty} \frac{2n-2+m(3+\alpha)}{m(1-\alpha)} |a_{n+m-1}| r^{-n-2m+1} - \sum\limits_{n=1}^{\infty} \frac{2n-2+m(1-\alpha)}{m(1-\alpha)} |b_{n+m-1}| r^{-n-2m+1}}{1 + \sum\limits_{n=1}^{\infty} \left(|a_{n+m-1}| - |b_{n+m-1}|\right) r^{-n-2m+1}} \geq 0.$$

If the condition (9)does not hold, then P(r) is negative for r sufficiently close to 1. Thus there exists a $z_o = r_o > 1$ for which the quotient $P(r)/Q(r)$ is negative. This contradicts the required condition for $f \in M_{\overline{H}}(m, \alpha)$ and so proof is complete. □

Corollary 1. *The function in* $M_{\overline{H}}(m, \alpha)$ *are starlike of order* $\frac{1+\alpha}{2}$.

Proof. The proof is straight forward, substituting $\frac{1+\alpha}{2}$ for α in (6) to obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} [n - 1 + m\left(1 + \frac{1+\alpha}{2}\right)] |a_{n+m-1}| \\ &+ \sum_{n=1}^{\infty} [n - 1 + m\left(1 - \frac{1+\alpha}{2}\right)] |b_{n+m-1}| \\ \leq \quad &\left(1 - \frac{1+\alpha}{2}\right), \end{aligned}$$

which is equivalent to the required condition (7). □

In the following theorem we determine the distortion bounds for functions in the class $M_{\overline{H}}(m, \alpha)$.

Theorem 4. *If* $f(z) \in M_{\overline{H}}(m, \alpha)$ *for* $0 \leq \alpha < 1$ *and* $|z| = r > 1$, *then*

$$r^m - m(1-\alpha) r^{-m} \leq |f(z)| \leq r^m + m(1-\alpha) r^{-m}.$$

Proof. Let $f(z) \in M_{\overline{H}}(m,\alpha)$. Taking the absolute value of f we obtain

$$\begin{aligned} |f(z)| &= \left| z^m + \sum_{n=1}^{\infty} a_{n+m-1}\ z^{-n-m+1} - \sum_{n=1}^{\infty} b_{n+m-1}\ (\overline{z})^{-n-m+1} \right| \\ &\leq r^m + \sum_{n=1}^{\infty} (a_{n+m-1} + b_{n+m-1})\ r^{-n-m+1} \\ &\leq r^m + \sum_{n=1}^{\infty} (|a_{n+m-1}| + |b_{n+m-1}|)\ r^{-m} \\ &\leq r^m + r^{-m} \sum_{n=1}^{\infty} \{(2n + m(3+\alpha) - 2)|a_{n+m-1}| \\ &\quad + (2n + m(1-\alpha) - 2)|b_{n+m-1}|\} \\ &\leq r^m + m(1-\alpha) r^{-m}. \end{aligned}$$

The proof for the left inequality uses a similar argument and for the sake of brevity we omit it. □

3. Extreme points

We use the coefficients bounds obtained in section 2 to examine the extreme points for functions $M_{\overline{H}}(m,\alpha)$ and determine the extreme points of the closed convex hull of $M_{\overline{H}}(m,\alpha)$ denoted by *clco* $M_{\overline{H}}(m,\alpha)$.

THEOREM 5. *$f \in M_{\overline{H}}(m,\alpha)$ if and only if f can be expressed as*

$$f = \sum_{n=0}^{\infty} (X_{n+m-1} h_{n+m-1} + Y_{n+m-1} g_{n+m-1}), \tag{10}$$

where

$$\begin{aligned} & z \in \widetilde{U}, h_{m-1}(z) = z^m, \\ & h_{n+m-1}(z) = z^m + \frac{m(1-\alpha) z^{-n-m+1}}{2n + m(3+\alpha) - 2} (n = 1, 2, 3, \dots), \\ & g_{m-1}(z) = z^m, \\ & g_{n+m-1}(z) = z^m - \frac{m(1-\alpha) z^{-n-m+1}}{2n + m(1-\alpha) - 2} z^{-n-m+1} (n = 1, 2, 3 \dots), \\ & \sum_{n=0}^{\infty} (X_{n+m-1} + Y_{n+m-1}) = 1, X_{n+m-1} \geq 0 \text{ and } Y_{n+m-1} \geq 0. \end{aligned}$$

In particular, the extreme points of clco $M_{\overline{H}}(m,\alpha)$ *are* $\{h_{n+m-1}\}$, $\{g_{n+m-1}\}$, $(n = 0,1,2,\dots)$.

Proof. Note first that for functions f of the form (10) we may write

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty}(X_{n+m-1}h_{n+m-1}(z) + Y_{n+m-1}g_{n+m-1}(z)) \\
f(z) &= X_{m-1}h_{m-1}(z) + Y_{m-1}g_{m-1}(z)) \\
&\quad + \sum_{n=1}^{\infty} X_{n+m-1}\left(z^m + \frac{m(1-\alpha)z^{-n-m+1}}{2n+m(3+\alpha)-2}\right) \\
&\quad + \sum_{n=1}^{\infty} Y_{n+m-1}\left(z^m - \frac{m(1-\alpha)z^{-n-m+1}}{2n+m(1-\alpha)-2}\overline{(z)}^{-n-m+1}\right) \\
&= \sum_{n=0}^{\infty}(X_{n+m-1}Y_{n+m-1})z^m \\
&\quad + \sum_{n=1}^{\infty}\left\{\frac{m(1-\alpha)z^{-n-m+1}}{2n+m(3+\alpha)-2}X_{n+m-1}z^{-n-m+1}\right. \\
&\quad \left. - \frac{m(1-\alpha)z^{-n-m+1}}{2n+m(1-\alpha)-2}z^{-n-m+1}Y_{n+m-1}\overline{(z)}^{-n-m+1}\right\}.
\end{aligned}$$

Then by (7)

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{[2n+m(3+\alpha)-2]\frac{m(1-\alpha)X_{n+m-1}}{2n+m(3+\alpha)-2}\right. \\
&\qquad \left. +[2n+m(1-\alpha)-2]\frac{m(1-\alpha)Y_{n+m-1}}{2n+m(1-\alpha)-2}\right\} \\
&= m(1-\alpha)\sum_{n=1}^{\infty}(X_{n+m-1}+Y_{n+m-1}) \\
&= m(1-\alpha)[1 - X_{m-1} + Y_{m-1}] \le m(1-\alpha),
\end{aligned}$$

and so $f \in M_{\overline{H}}(m,\alpha)$.

Conversely, suppose that $f \in M_{\overline{H}}(m,\alpha)$. Then we write

$$f(z) = z^m + \sum_{n=1}^{\infty}\{a_{n+m-1}z^{-n-m+1} - b_{n+m-1}(\overline{z})^{-n-m+1}\}$$

where $a_{n+m-1} \geq 0,\ b_{n+m-1} \geq 0$, and

$$\sum_{n=1}^{\infty} \frac{m(1-\alpha)}{2n+m(3+\alpha)-2} a_{n+m-1} + \frac{m(1-\alpha)}{2n+m(1-\alpha)-2} b_{n+m-1} \leq 1.$$

Setting

$$X_{n+m-1} = \frac{2n+m(3+\alpha)-2}{m(1-\alpha)} \text{ and } Y_{n+m-1} = \frac{2n+m(1-\alpha)-2}{m(1-\alpha)},$$

$(n = 1, 2, \dots)$ $0 \leq X_{m-1} \leq 1$ and

$$Y_{m-1} = 1 - X_{m-1} - \sum_{n=1}^{\infty} (X_{m+n-1} + Y_{n+m-1}),$$

we get

$$f(z) = \sum_{n=1}^{\infty} (X_{n+m-1} h_{n+m-1}(z) + Y_{n+m-1} g_{n+m-1}(z))$$

as required. □

4. Convolution and convex combinations

In this section, we show that the class $M_{\overline{H}}(m, \alpha)$ is invariant under convolution and convex combinations of its members. For harmonic functions

$$f(z) = z^m + \sum_{n=1}^{\infty} a_{n+m-1} z^{-n-m+1} - \sum_{n=1}^{\infty} b_{n+m-1} (\overline{z})^{-n-m+1}$$

and

$$F(z) = z^m + \sum_{n=1}^{\infty} A_{n+m-1} z^{-n-m+1} - \sum_{n=1}^{\infty} B_{n+m-1} (\overline{z})^{-n-m+1},$$

we define the convolution of f and F as

$$\begin{aligned} (f * F)(z) &= f(z) * F(z) \\ &= z^m + \sum_{n=1}^{\infty} a_{n+m-1} A_{n+m-1} z^{-n-m+1} \\ &\quad - \sum_{n=1}^{\infty} b_{n+m-1} B_{n+m-1} (\overline{z})^{-n-m+1}. \end{aligned} \tag{11}$$

THEOREM 6. *If f and F belong to $M_{\overline{H}}(m, \alpha)$ so does the convolution function $f * F$.*

Proof. Let

$$f(z) = z^m + \sum_{n=1}^{\infty} a_{n+m-1} z^{-n-m+1} - \sum_{n=1}^{\infty} b_{n+m-1} (\overline{z})^{-n-m+1}$$

and

$$F(z) = z^m + \sum_{n=1}^{\infty} A_{n+m-1} z^{-n-m+1} - \sum_{n=1}^{\infty} B_{n+m-1} (\overline{z})^{-n-m+1},$$

be in $M_{\overline{H}}(m, \alpha)$. Then the convolution of f and F is given by (11). Note that $A_{n+m-1} \leq 1$ and $B_{n+m-1} \leq 1$, since $f \in M_{\overline{H}}(m, \alpha)$. Therefore we can write

$$\begin{aligned} & \sum_{n=1}^{\infty} [2n + m(3+\alpha) - 2] a_{n+m-1} A_{n+m-1} \\ & + \sum_{n=1}^{\infty} [2n + m(1-\alpha) - 2] b_{n+m-1} B_{n+m-1} \\ \leq \; & \sum_{n=1}^{\infty} [2n + m(3+\alpha) - 2] a_{n+m-1} + \sum_{n=1}^{\infty} [2n + m(1-\alpha) - 2] b_{n+m-1} \\ \leq \; & m(1-\alpha), \end{aligned}$$

hence $f * F \in M_{\overline{H}}(m, \alpha)$ by theorem (3). □

THEOREM 7. *The class $M_{\overline{H}}(m, \alpha)$ is closed under convex combination.*

Proof. For $i = 1, 2, \ldots$, suppose that $f_i(z) \in M_{\overline{H}}(m, \alpha)$, where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{n=1}^{\infty} a_{n+m-1,i} z^{-n-m+1} - \sum_{n=1}^{\infty} b_{n+m-1,i} (\overline{z})^{-n-m+1},$$
$$a_{n+m-1,i} \geq 0, \; b_{n+m-1,i} \geq 0.$$

Then by (7), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} (2n + m(3+\alpha) - 2) |a_{n+m-1,\, i}| \\ & \qquad + \sum_{n=1}^{\infty} (2n + m(1-\alpha) - 2) |b_{n+m-1,\, i}| \leq m(1-\alpha). \end{aligned}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1$, the convex combination of $f_i(z)$, may be written as

$$\begin{aligned}\sum_{i=1}^{\infty} t_i f_i(z) = z^m &+ \sum_{i=1}^{\infty} t_i \left(\sum_{n=1}^{\infty} a_{n+m-1,i}\right) z^{-n-m+1} \\ &- \sum_{i=1}^{\infty} t_i \left(\sum_{n=1}^{\infty} b_{n+m-1,i}\right) (\overline{z})^{-n-m+1}.\end{aligned}$$

Then by (12),

$$\begin{aligned}&\sum_{n=1}^{\infty}(2n + m(3+\alpha) - 2) \left(\sum_{i=1}^{\infty} t_i\ a_{n+m-1,\ i}\right) \\ &+\sum_{n=1}^{\infty}(2n + m(1-\alpha) - 2) \left(\sum_{i=1}^{\infty} t_i b_{n+m-1,\ i}\right) \\ = \ &\sum_{i=1}^{\infty} t_i \sum_{n=1}^{\infty} \Big[(2n + m(3+\alpha) - 2)|a_{n+m-1,\ i}| \\ &+(2n + m(1-\alpha) - 2)t_i|b_{n+m-1,\ i}|\Big] \\ \leq \ &\sum_{i=1}^{\infty} t_i m(1-\alpha) \leq m(1-\alpha).\end{aligned}$$

Thus $\sum\limits_{i=1}^{\infty} t_i f_i(z) \in M_H(m, \alpha)$, which completes the proof. □

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